Orthomodular Algebraic Lattices and Combinatorial Posets

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15th Italian Conference on Theoretical Computer Science

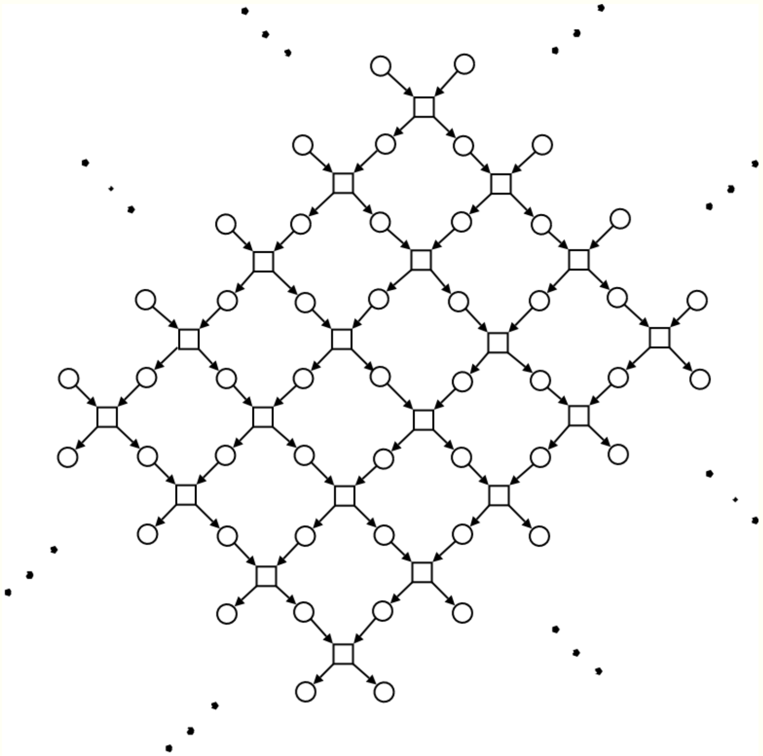
Partial orders and concurrent processes

$$P = (X, \leq)$$

$li = \leq \cup \geq \qquad co = (X \times X) \setminus li$

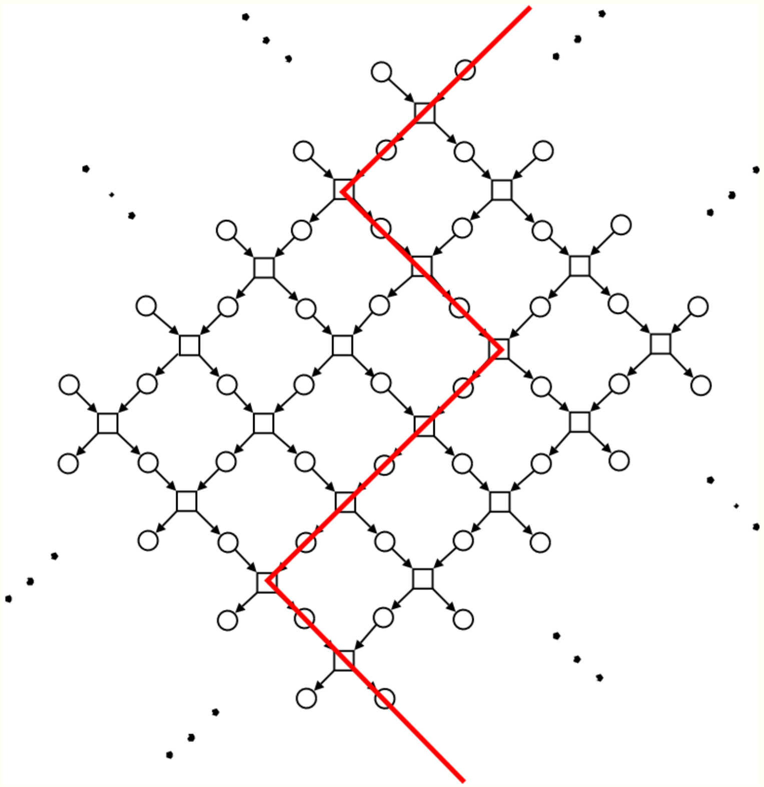
li and co are similarities
(symmetric, non-transitive relations)

Combinatorial: $\leq = (\lessdot)^*$

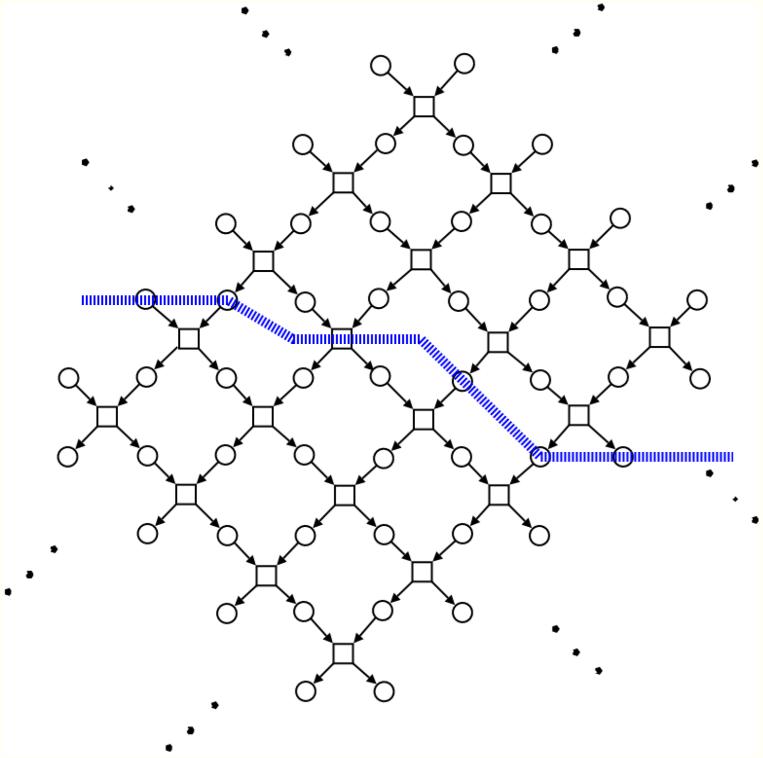


Lines and cuts

line: maximal clique of li



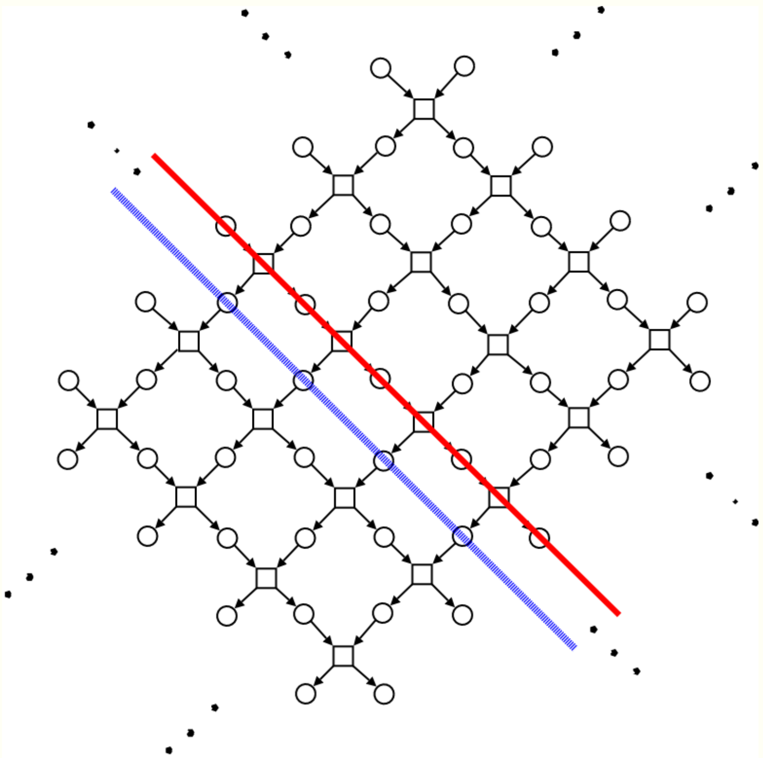
cut: maximal clique of $co \cup id_X$



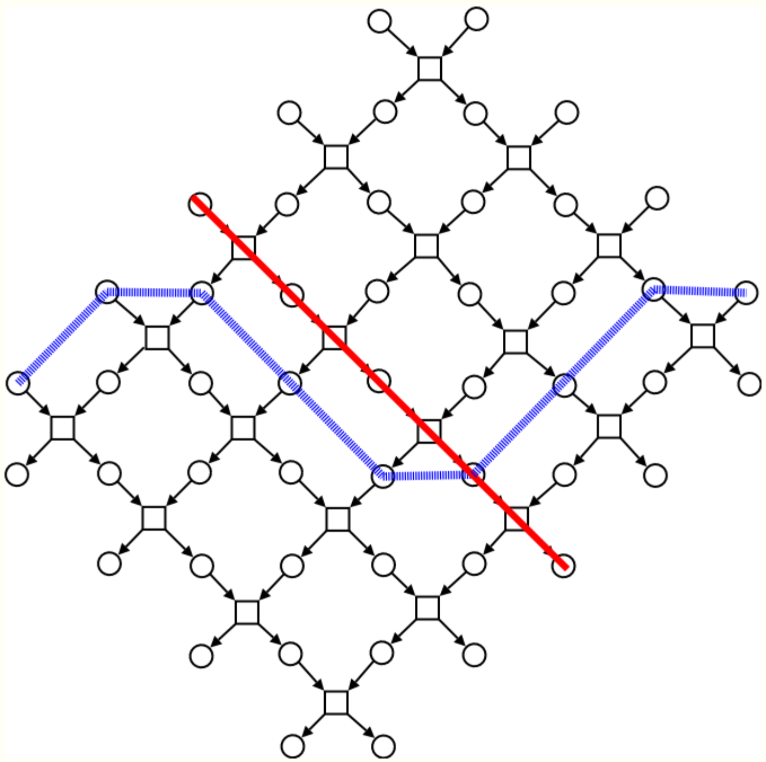
K-density and N-density

K-density: every line crosses every cut

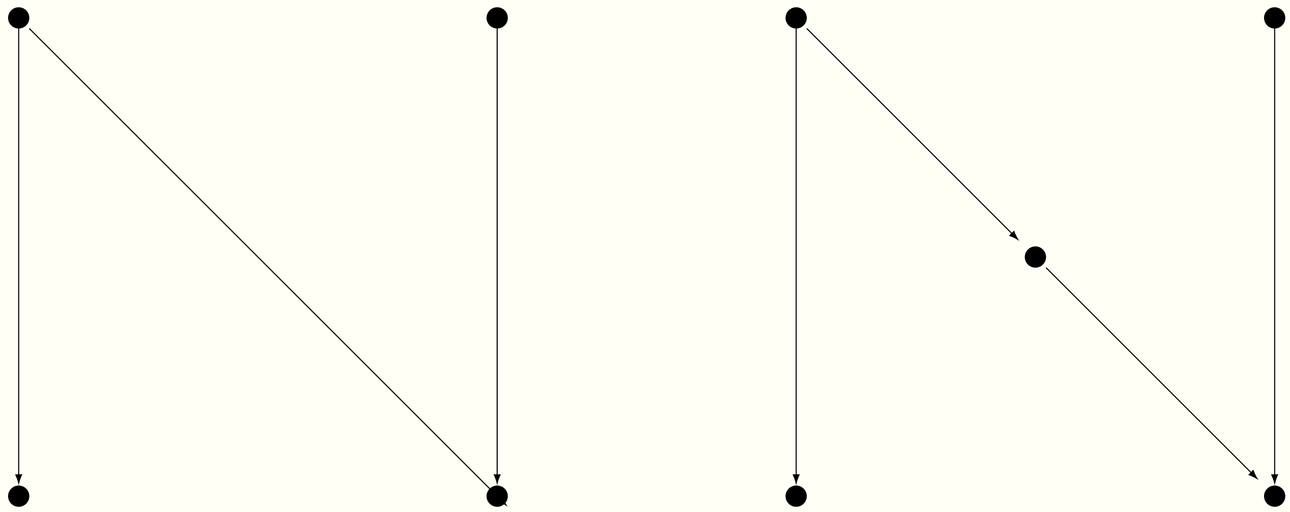
P is not K-dense



P is K-dense



N-density: a local form of K-density



Non N-dense N-dense

Closure operators

Closure operator on a set X

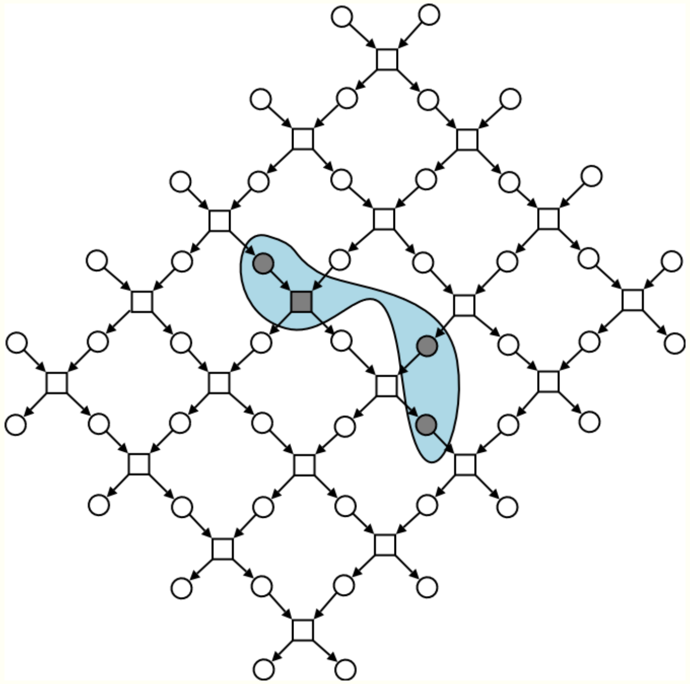
$$C : \mathbb{P}(X) \to \mathbb{P}(X)$$

- $A \subseteq X \quad A \subseteq C(A)$
- $A, B \subseteq X \quad A \subseteq B \Rightarrow C(A) \subseteq C(B)$
- $A \subseteq X \quad C(C(A)) = C(A)$

A closure operator based on concurrency

$P = (X, \leq)$

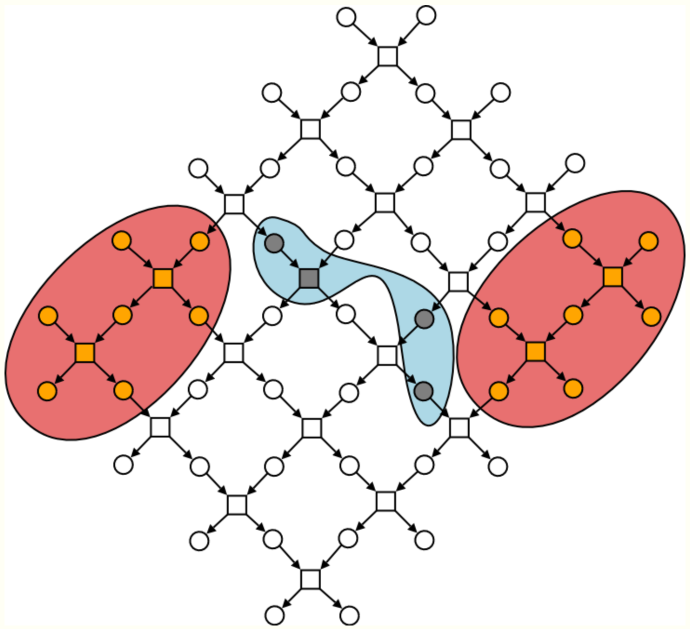
$S \subseteq X$



A closure operator based on concurrency

$P = (X, \leq)$ $\qquad$ $S \subseteq X$

$$S^{\perp} \;=\; \{x \in X \mid \forall y \in S : (x, y) \in co\}$$



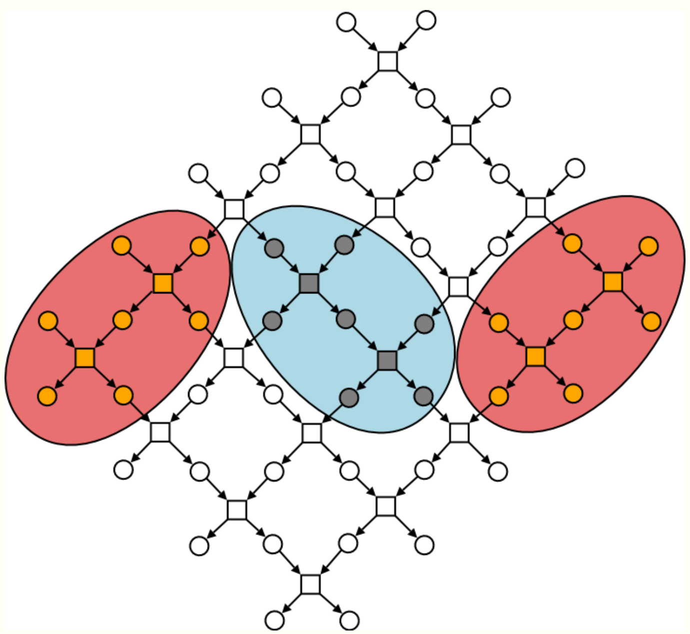
A closure operator based on concurrency

$P = (X, \leq)$ $\qquad S \subseteq X$

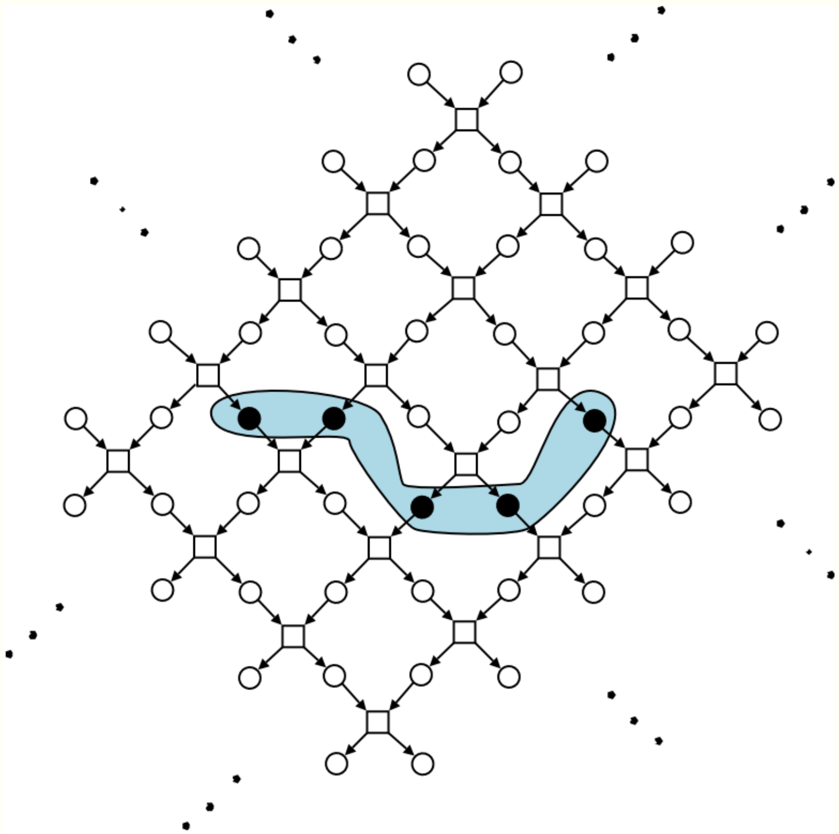
$S^{\perp} \;=\; \{x \in X \mid \forall y \in S : (x, y) \in co\}$

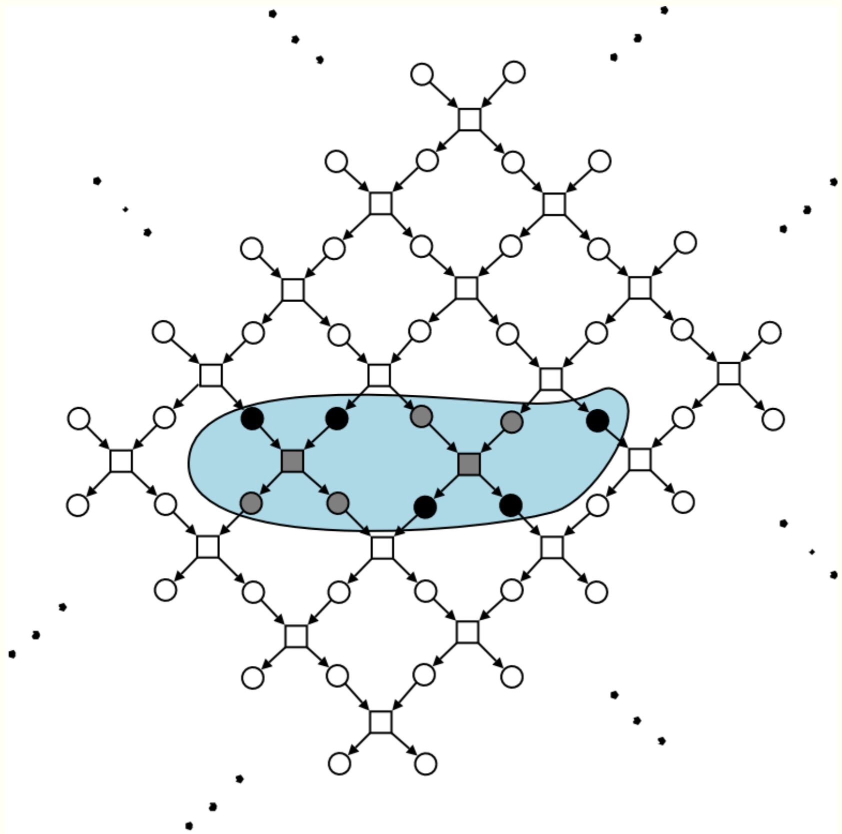
$(.)^{\perp\perp} : \mathbb{P}(X) \rightarrow \mathbb{P}(X)$

$S = S^{\perp\perp} \Leftrightarrow S$ is closed

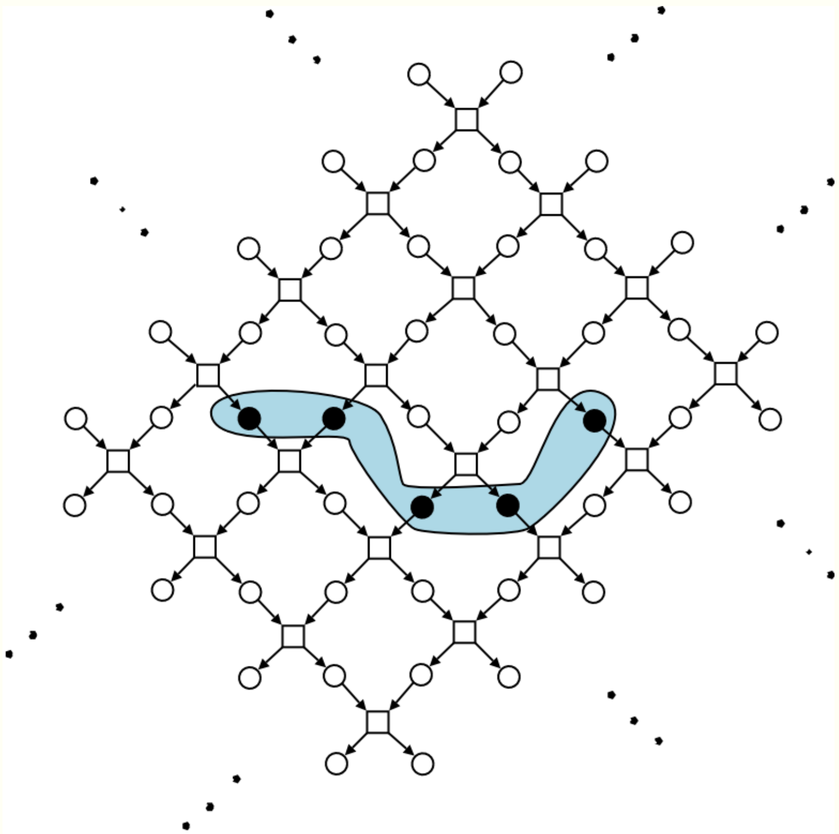


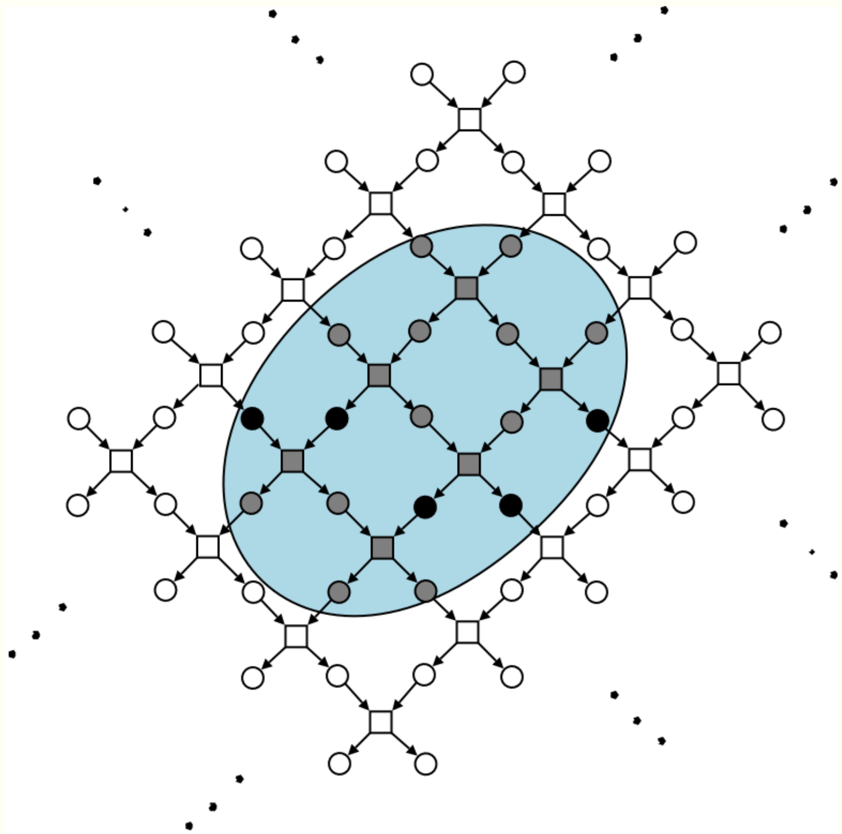
Causally closed subprocesses





Causally closed subprocesses





Lattices generated by concurrency

$L(P)$ collection of closed sets

$(.)^{\perp}$ applied to elements of $L(P)$ is an orthocomplement

$$\boxed{\begin{array}{c}\mathcal{L}(P) = (L(P), \subseteq, \emptyset, X, (.)^{\perp}) \\ \text{is an } \textbf{orthocomplete} \text{ lattice}\end{array}}$$

Lattices generated by concurrency

$L(P)$ collection of closed sets

$(.)^{\perp}$ applied to elements of $L(P)$ is an orthocomplement

$$\mathcal{L}(P) = (L(P), \subseteq, \emptyset, X, (.)^{\perp})$$

is an **orthocomplete** lattice

Theorem If $\mathcal{P} = (P, \leq)$ is combinatorial, then $\mathcal{L}(P)$ is orthomodular if and only if $\mathcal{P} = (P, \leq)$ is N-dense.

Orthomodular lattices

An orthocomplete lattice

$$(L, \leq, 0, 1, (.)')$$

is **orthomodular** if

$$S_1 \leq S_2 \text{ implies } S_2 = S_1 \vee (S_2 \wedge S_1')$$

Any orthomodular lattice can be seen as a family of partially overlapping Boolean algebras.

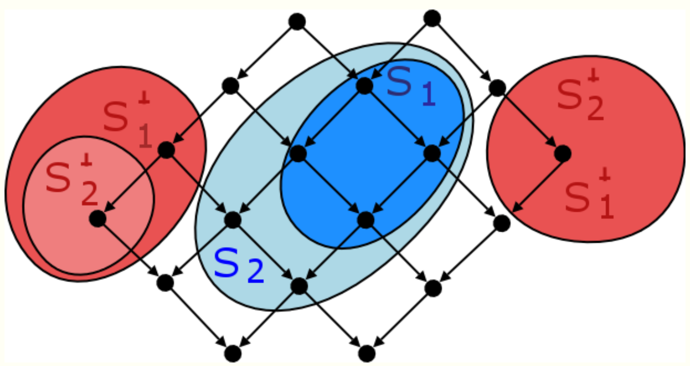
Lattices generated by concurrency

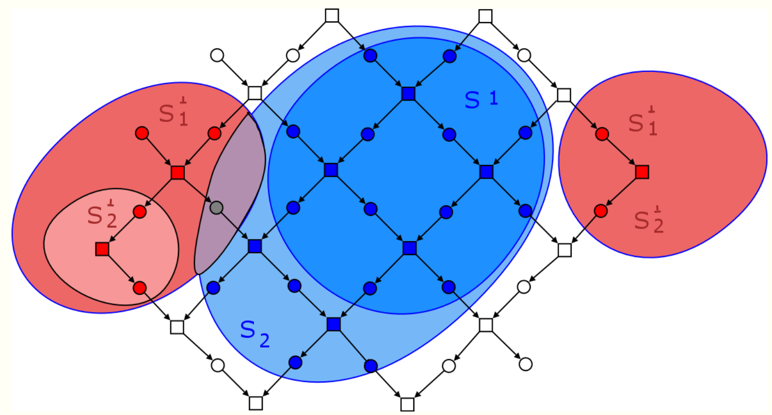
Theorem The closure operator $(.)^{\perp\perp}$ on a poset $\mathcal{P}$ is algebraic if, and only if, $\mathcal{P}$ is K-dense.

Lattices generated by concurrency on event structures

Elementary event structure $(E, \sqsubseteq)$

$\boxed{\mathcal{L}(E) \text{ is not orthomodular}}$



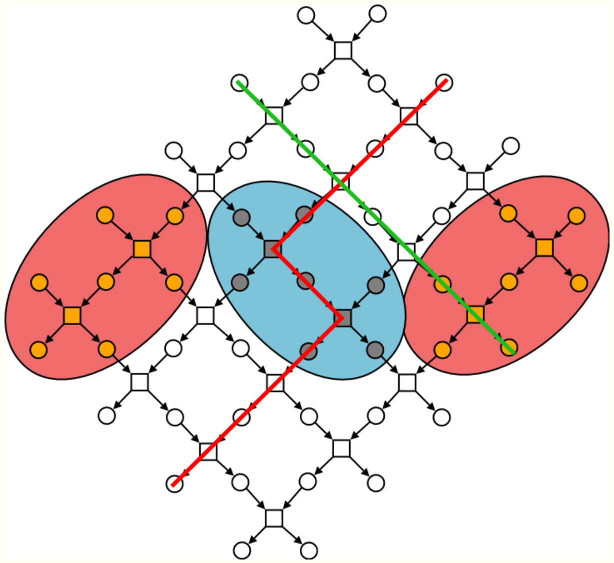


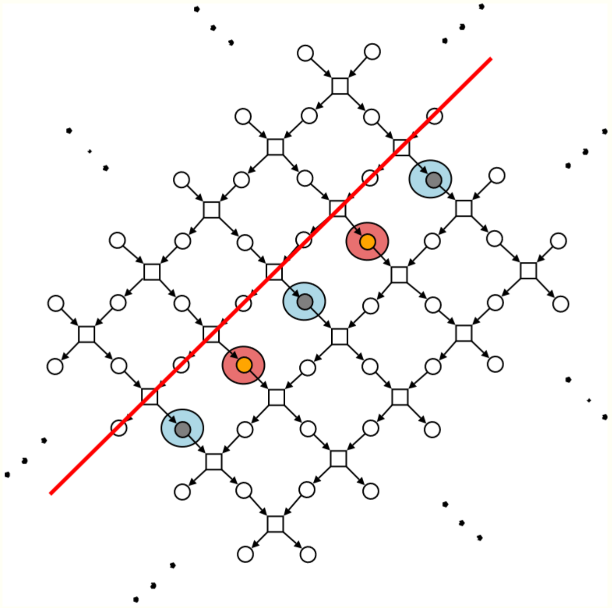
$S_1 \subseteq S_2$ but $S_2 \neq S_1 \vee (S_2 \wedge S_1^\perp)$

Properties of closed sets

If P is K-dense, then

S a closed set, λ a line: $\lambda \cap S \neq \emptyset$ or $\lambda \cap S^{\perp} \neq \emptyset$





Lines and two-valued states

$\langle L, \leq, 0, 1, (.)' \rangle$ orthomodular lattice

Two valued state on $\mathbf{L}$ $\quad s : L \to \{0, 1\}$

- $s(1) = 1$
- $s\Big(\bigvee_i A_i\Big) = \sum_i s(A_i) \quad A_i \in X$, $(i \neq j \Rightarrow A_i \leq A_j')$

Lines and two-valued states

Let λ a line and $S \in L(P)$ a closed set

Define $\delta_\lambda(S) = 1$ if $S \cap \lambda \neq \emptyset$, and $\delta_\lambda(S) = 0$ otherwise

$\boxed{\textbf{Theorem}\quad \text{The map } \delta_\lambda \text{ is a two-valued state of } L(P)}$

Conclusions

Ongoing work Closure operators defined on occurrence nets with conflicts (general unfoldings). Closure operators derived from the dependence relation, reflexive and irreflexive, closed sets as propositions.

Future developments Cyclic nets (cycloids).