International Workshop on Partial Differential Equations for the 80th birthday of James Serrin

Perugia, June 25-26, 2008
Hotel Giò

Abstracts

$$div(\mathscr{A}(x,u,\nabla u))=\mathscr{B}(x,u,\nabla u)$$
$$Min \int_{\Omega} F(x,u,\nabla u)$$

$$(-\Delta)^K u=\lambda u+g(u)$$
$$u_t + u\cdot\nabla u = \nu\Delta u - \nabla p + f$$

Speakers

A. Ambrosetti (S.I.S.S.A Trieste)
M. F. Bidaut Veron (Univ. Tours)
L. Boccardo (Univ. Roma "La Sapienza")
H. Brezis (Univ. Paris VI, Rutgers and Technion)
A. Farina (Univ. Picardie)
F. Gazzola (Pol. Milano)
H.-C. Grunau (Univ. Magdeburg)
J. Mawhin (Univ. Cat. Louvain)
J. B. McLeod (Univ. Pittsburg)
E. Mitidieri (Univ. Trieste)
F. Pacella (Univ. Roma "La Sapienza")
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On systems of Schrödiger equations.

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Abstract. Some recent results dealing with the existence of bound states for systems of coupled Schrödiger equations on $\mathbb{R}^n$ will be discussed.

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Quasilinear elliptic equations with source terms of order 0 or 1

in collaboration with Haydar ABDELHAMID

Abstract

Here we consider a quasilinear equation of the form

$$\begin{cases} -\Delta_p v = \lambda f(x)(1+g(v))^{p-1} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $p > 1$, $f > 0, \lambda > 0$, and g is nondecreasing, $g(0) = 0$. We show a precise link with a second problem presenting a source gradient term with a natural growth:

$$\begin{cases} -\Delta_p u = \beta(u) \left|\nabla u\right|^p + \lambda f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\beta(u) \geqq 0$, defined on an interval of $\mathbb{R}$. We deduce new results of existence and multiplicity concerning the two problems, and the existence of extremal solutions, in particular when g is convex and $\lim_{t\longrightarrow\infty} g(t)/t = \infty$.

DIRICHLET PROBLEMS WITH GRADIENT QUADRATIC LOWER ORDER TERMS

LUCIO BOCCARDO

1. Introduction and main results

It is classic (Morrey, Serrin) that integral functionals like

$$\Phi(v)=\frac{1}{2}\int_\Omega a(x,v)|Dv|^2-\int_\Omega g(x)v$$

(with Ω a bounded, open subset of $\mathbb{R}^N$, $N>2$, $0<\alpha\le a(x,s)\le\beta$) have a minimum $u\in W_0^{1,2}(\Omega)$. Moreover u satisfies

$$\begin{cases} -\mathrm{div}(a(x,u)Du)+\frac{1}{2}a_s(x,u)|Du|^2=g(x) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

in the sense

$$\int_\Omega a(x,u)DuD\phi+\frac{1}{2}\int_\Omega a_s(x,u)|Du|^2\phi=\int_\Omega g(x)\phi,\quad \forall\ \phi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$$

If we try to solve directly the previous boundary value problem, we readily see that unfortunately (fortunately?) the differential operator $-\mathrm{div}(a(x,u)Du)+\frac{1}{2}a_s(x,u)|Du|^2$ is not pseudomonotone in the meaning of Brezis; moreover $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ is a good framework space.

In the "linear" case, it is known that it is possible to find bounded solutions of the Dirichlet problem (Stampacchia, Serrin)

$$\begin{cases} -\mathrm{div}(a(x,u)Du)=g(x) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

if $g\in L^m(\Omega)$, $m>N/2$.

Then, in [B-Murat-Puel, SIAM 1992], it is proved that there exists a solution $u\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ of the boundary value problem

$$\begin{cases} -\mathrm{div}(a(x,u)Du)+u=H(x,u,Du)+f(x) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

under the assumptions $f\in L^m(\Omega)$, $m>N/2$, $|H(x,s,\xi)|\le\gamma|\xi|^2$.

Existence results of (unbounded) solutions if $f\in L^{N/2}(\Omega)$ are due to Dall'Aglio-Giachetti-Puel.

In a paper dedicated to Haim Brezis for his 60th birthday ([B-2005]), I proved the existence of solutions in $W_0^{1,2}(\Omega)$ for the previous boundary

Date: Perugia, 26.6.2008.

value problem under the assumptions $0 \leq f(x) \leq \frac{A}{|x|^2}$, $A < \frac{\alpha^2 \mathcal{H}}{\gamma}$, where $\mathcal{H}$ is the Hardy constant. A nonexistence theorem says that these assumptions are sharp.

The example

$$\begin{cases} -\Delta u = \gamma |\nabla u|^2 + \frac{A}{|x|^2} & \text{in } B(0,1) \qquad (\gamma,\, A \in \mathbb{R}) \\ u = 0 & \text{on } \partial B(0,1). \end{cases}$$

shows that there exists 2 solutions $u_i(r) = B_i \log(r)$, with

$$\begin{cases} B_1 = \frac{-(N-2)-\sqrt{(N-2)^2-4A\gamma}}{2\gamma} \\ B_2 = \frac{-(N-2)+\sqrt{(N-2)^2-4A\gamma}}{2\gamma} \end{cases}$$

if $A < \frac{\mathcal{H}}{\gamma}$, but I am not able to comment on possible relationships with the well-known nonuniqueness counterexample of Serrin.

One more example of integral functional:

$$J(v) = \frac{1}{2} \int\limits_{\Omega} (1 + |v|^r)|Dv|^2 - \int\limits_{\Omega} f(x)v(x), \quad \boxed{r > 1},$$

whose Euler-Lagrange equation is

$$u \in W_0^{1,2}(\Omega) : -\text{div}((1 + |u|^r)Du) + \frac{r}{2} u|u|^{r-2}|Du|^2 = f.$$

Note that $[u|u|^{r-2}|Du|^2] \cdot u \geq 0$. In general, we consider Dirichlet problems of the type

$$u \in W_0^{1,2}(\Omega) : \; -\text{div}(a(x,u)Du) + b(x,u)|Du|^2 = f(x),$$

under the "quite natural" assumption on $b(x,s)$: $b(x,s)s \geq 0$.

Moreover the existence theorem proved in [B-Gallouet, Nonlinear Anal. 1992] shows that the existence of finite energy weak solutions can be proved only with the weaker assumption $f \in L^1(\Omega)$, thanks to the presence of the lower order term with quadratic dependence with respect to the gradient and $b(x,s)\,\text{sign}(s) \geq \sigma > 0$. This result is somewhat surprising because it is not true in the linear case! In this framework, recall also the results of removable singularities and nonexistence proved by H. Brezis-L. Nirenberg and B-Gallouet-Orsina.

Consider then the functional

$$J(v) = \frac{1}{2} \int\limits_{\Omega} (1 + |v|^\theta)|Dv|^2 - \int\limits_{\Omega} f(x)v(x), \quad \boxed{0 < \theta < 1},$$

whose Euler-Lagrange equation $\boxed{\text{is}}$

$$u \in W_0^{1,2}(\Omega) : -\text{div}((1 + |u|^\theta)Du) + \frac{\theta}{2} u|u|^{\theta-2}|Du|^2 = f$$

$f \geq 0 \Rightarrow u \geq 0$

$$u \in W_0^{1,2}(\Omega) : -\text{div}((1 + u^\theta)Du) + \frac{\theta}{2} \frac{|Du|^2}{u^{1-\theta}} = f$$

The kernel of the talk will be devoted to the presentation of recent results ([B, ESAIM 2008]) concerning Dirichlet problems with lower order terms having quadratic growth with respect to Du and singular with respect to u as in the above example.

In general we will present existence results on the "Arcoya problem":

$$\int_\Omega M(x)DuD\phi + b\int_\Omega \frac{|Du|^2}{u}\phi = \int_\Omega f\phi, \quad \forall\, \phi \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$$

with $0 \le f \in L^m(\Omega)$, $\quad m \ge \frac{2N}{N+2}$, $\quad f \not\equiv 0$, $\alpha > 2b \Rightarrow$.

We will prove that there exists $\quad u \in W_0^{1,2}(\Omega)$, $u > 0$ in Ω, with $\frac{|Du|^2}{u} \in L^1(\Omega)$, weak solution of the above singular-quadratic Dirichlet problem.

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Returning to the critical exponent in 3-d

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Abstract. In 1983 L. Nirenberg and myself discovered that some nonlinear elliptic equations involving the critical exponent have a special behaviour in 3d. Our work prompted further research, including a remarkable extension by P. Pucci and J. Serrin to polyharmonic operators. I will present some results we obtained recently with M. Willem on this subject.

Liouville theorems for Lane-Emden-Fowler equations

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Abstract. I will present some sharp Liouville theorems for solutions (possibly unbounded and sign-changing) of semilinear and quasilinear Lane-Emden-Fowler equations on unbounded domains of $\mathbb{R}^N$. Part of the results are obtained in collaboration with L. Damascelli, B. Sciunzi and E. Valdinoci.

Partially overdetermined elliptic boundary value problems

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In a celebrated paper, J. Serrin studied elliptic equations in bounded domains overdetermined with both Dirichlet and Neumann boundary conditions and (under suitable assumptions) he showed that if a solution exists, then the domain is necessarily a ball. In this talk we address the problem of *partially* overdetermined boundary value problems. We consider semilinear elliptic Dirichlet problems in bounded domains, overdetermined with a Neumann condition on a proper part of the boundary. We also consider Neumann problems overdetermined with a Dirichlet condition on a proper part of the boundary and the case of partially overdetermined problems on exterior domains. Under different kinds of assumptions, we show that these problems admit a solution only if the domain is a ball (this is joint work with I. Fragalà). When these assumptions are not fulfilled, we exhibit counterexamples (this is joint work with I. Fragalà, J. Lamboley, M. Pierre).

Boundary value problems for symmetric Willmore surfaces

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June 25/26, 2008

The Willmore functional is the integral of the square of the mean curvature over the unknown surface and is to be minimised among all surfaces which obey suitable boundary conditions or, in the case of closed surfaces, constraints of topological or geometrical type. The Willmore equation as the corresponding Euler-Lagrange equation may be considered as frame invariant counterpart of the clamped plate equation and is quasilinear and of fourth order. This equation is of interest not only in mechanics and membrane physics but in particular in differential geometry. Quite far reaching results were achieved concerning closed surfaces. Concerning boundary value problems, by far less is known. These will be discussed in symmetric situations.

The lecture is based on joint work with A. Dall'Acqua, K. Deckelnick (Magdeburg) and S. Fröhlich (Free University of Berlin).

The coexistence of a maximum and a uniform anti-maximum principle

Jean Mawhin, Université Catholique de Louvain

The maximum principle for some linear differential operators has a long history going back to the early days of potential theory. It is still a very lively and fruitful research area, with many applications to linear and nonlinear boundary value problems, as shown by Pucci-Serrin's recent monograph *The Maximum Principle.* Clément-Peletier's anti-maximum principle is a more recent very illuminating tool for the study of differential equations.

We exhibit a class of linear operators for which a maximum principle always coexist with a uniform anti-maximum principle. Examples are given, belonging to ordinary or partial differential equations, as well as to difference equations. This is a joint work with J. Campos and R. Ortega of the University of Granada.

A problem in dislocation theory

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A problem in dislocation theory leads to an infinite set of simultaneous nonlinear equations. The lecture will discuss the derivation of the equations and the existence and asymptotic properties of a solution.

Liouville theorems for some classes of quasilinear problems

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Abstract. We present some recent Liouville theorems for quasilinear elliptic equations and systems of integral equations on R^n.

Symmetry and Liouville type theorems for semilinear elliptic equations

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Abstract. The talk will focus on recent results about symmetry of solutions with low Morse index of semilinear elliptic equations in spherically symmetric, bounded or unbounded domains. The results hold both for positive and sign changing solutions under some convexity assumptions on the nonlinearity. In the case when the domain is the whole space or the exterior of a ball some nonexistence theorems are derived.

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- F.Gladiali, F.Pacella and T.Weth , Symmetry and nonexistence of low Morse index solutions to semilinear elliptic equations in unbounded domains, (in preparation).

Results for parabolic problems arising in some physical models, having natural growth in the gradient

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2000 Mathematics Subject Classification. 35K55, 35K65, 35B05, 35B40.

In this talk we study the problem

$$\begin{cases} u_t - \Delta u &= \beta(u)|\nabla u|^2 + f(x,t) & \text{in } Q \equiv \Omega \times (0,+\infty) \\ u(x,t) &= 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) &= u_0(x) & \text{in } \Omega, \end{cases}$$

where Ω is a bounded regular domain, β is a positive nondecreasing function and f, u_0 are positive functions satisfying some hypotheses of summability. Among others contribution the main one is to prove a *wild* non-uniqueness result.

This is a joint work with

- Boumediene Abdellaoui, *Département de Mathématiques, Université Aboubekr Belkaïd, Tlemcen, Tlemcen 13000, Algeria.*
- Andrea Dall'Aglio, *Dipartamento de Matemática, U. Roma I La Sapienza, I-00185 Roma, Italy.*

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Journal de Mathmatiques Pures et Appliques (2008), doi: 10.1016/j.matpur.2008.04.004.

Boundary singularities of solutions of non-monotone nonlinear elliptic equations

Laurent Véron

Abstract

Let $\Omega \subset \mathbf{R^N}$ be a smooth domain, $x_0 \in \partial\Omega$ and $q \geq (N+1)/(N-1)$. We study the behavior near x_0 of any positive solution of (E) $-\Delta u = u^q$ in Ω which coincides with some $\zeta \in C^2(\partial\Omega)$ on $\partial\Omega \setminus \{x_0\}$. We prove that, if $(N+1)/(N-1) < q < (N+2)/(N-2)$, u satisfies $u(x) \leq C|x-x_0|^{-2/(q-1)}$ and we give the limit of $|x-x_0|^{2/(q-1)}u(x)$ as $x \to x_0$. In the case where $q = (N+1)/(N-1)$, u satisfies $u(x) \leq C|x-x_0|^{1-N}\,(\ln(1/|x|))^{(1-N)/2}$ and a corresponding precise asymptotic is obtained. We also study some existence and uniqueness questions for related equations on spherical domains.