PROGRAM OF THE ALGEBRAIC GEOMETRY COURSE

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Brief summary.

Algebraic Geometry is a huge area of mathematics, which went through several phases: Hilbert's fundamental paper from 1899, sheaves and cohomology introduced by Serre in the 1950's, Grothendieck's theory of schemes in the 1960's and so on.

In this course we will cover the state of affair after Hilbert's paper. We will give the proof of his important theorems with an emphasis on computational methods. In particular we will use Gröbner bases systematically.

The highlights of the course are the Nullstellensatz, Gröbner bases, Hilbert's syzygy theorem and the Hilbert function, Bézout's theorem, Mora division, semi-continuity of the fiber dimension, Bertini's theorem, Cremona resolution of plane curves and finally parametrization of rational curves. We will also take a glance at the Hilbert scheme and interpret lead term ideals as limits under one-parameter subgroups in $\mathrm{PGL}(n+1)$.

Weekly program

1st week. Hilbert's Nullstellensatz and the ideal membership problem, Gröbner bases and Buchberger's criterion, the projection theorem, the algebra-geometry dictionary.
2nd week. Component decomposition, noetherian rings and primary decomposition, localization, associated primes, the rational function field, dimension and transcendence degree, a Gröbner basis dimension criterium, the lying-over theorem, Krull dimension, constructive ideal and module theory.
3rd week. $\mathbb{P}^n$, graded rings and the homogeneous coordinate ring, the syzygy theorem and the Hilbert polynomial, intersection multiplicities, multiplicity of points on plane curves, Bézout's theorem, local rings and the Lemma of Nakayama, completions and the ring of formal power series, Grauert division and the Weierstrass preparation theorem, Mora division, tangent space and tangent cone, Segre products, morphisms, linear projections, a dimension bound.
4th week. Veronese embeddings, the fundamental theorem of elimination, projective morphisms, semi-continuity of the fiber dimension, the blow-up, resolution of singularities, Cremona transformations, linear systems of plane curves, Grassmannians, the Hilbert scheme, initial ideals and one-parameter subgroups, Bertini's theorem and the geometric interpretation of the degree, the dual variety, dynamical intersection numbers, a bound on the number of singular points of plane curves, rational curves, the geometric genus.

Detailed notes are provided by the slides for this online course, which are provided in separate documents.

Date: Saarbrücken, May 2021.

Further literature

- William Fulton: *Algebraic Curves: An Introduction to Algebraic Geometry*, Benjamin 1989, available online at CurveBook.pdf
- David A Cox; John Little; Donal O'Shea: *Ideals, Varieties, and Algorithms*, Springer 1990
- David A Cox; John Little; Donal O'Shea: *Using algebraic geometry*, GTM 185, Springer 1998
- M.F. Atiyah; I.G. Macdonald: *Introduction to Commutative Algebra*, Addison-Wesley 1969
- David Eisenbud: *Commutative Algebra with a View Toward Algebraic Geometry*, GTM 150, Springer 1995

These books gave inspiration for this course.

Prerequisite. I assume that the students are familiar with linear algebra and some basic notions from algebra: Groups, rings and fields, algebraic and transcendental field extension.

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