Einstein Pseudo-Riemannian metrics on Solvable Lie Groups

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08 February 2023

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Outline

Introduction

M differentiable manifold. Look for "best" (special) metric on M

Einstein metric $\quad \mathrm{Ric}_g = \lambda \mathrm{Id} \quad$ for some $\lambda \in \mathbb{R}$ $\quad$ (E)

Ricci Soliton metric $\quad \mathrm{ric}_g = \lambda g - \frac{1}{2}\mathcal{L}_X g \quad$ for some $\lambda \in \mathbb{R}$, $X \in \mathfrak{X}(M)$

- If $\lambda = 0$ the metric is *Ricci-flat*
- Ricci-flat metrics are related to string theory, fixed point of Ricci-flow

Einstein Riemannian Homogeneous case:

1. Homogeneous Einstein Ricci-flat ⇒ are flat [Alekseevsky-Kimel′fel′d]
2. Positive scalar curvature ⇒ compact homogeneous spaces G/K [Myers]
3. Negative scalar curvature ⇒ non-compact [Bochner]

Alekseevsky Conjecture - Theorem [Böhm, Lafuente]

All Riemannian homogeneous Einstein manifolds of negative scalar curvature are solvmanifolds

Solvmanifold: simply connected solvable Lie group with a left-invariant metric

The Setting

Lie group G	$\leftrightsquigarrow$	Lie algebra $\mathfrak{g}$
left-invariant (pseudo-)Riemannian metric g	$\leftrightsquigarrow$	(in)definite (non degenerate) bilinear form g on $\mathfrak{g}$
Levi Civita Connection, Ricci, Riemann curvature	$\leftrightsquigarrow$	tensors on $\mathfrak{g}$

$\mathfrak{g}$ *Nilpotent*: $\mathfrak{g} \supset \mathfrak{g}' \supset \cdots \supset \mathfrak{g}^s = \{0\}$ s-step $\mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}]$

$\mathfrak{g}$ *Solvable*: $\mathfrak{a}_i = [\mathfrak{a}_{i-1}, \mathfrak{a}_{i-1}]$ $\mathfrak{a}_0 = \mathfrak{g}$ $\mathfrak{a}_0 = \mathfrak{g} \supset \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_r = \{0\}$

$\mathfrak{g}$ *Unimodular*: $\operatorname{tr}\operatorname{ad}(v) = 0$

$\mathfrak{g}$ nilpotent Lie algebra, g left-invariant metric is *Algebraic Ricci soliton* if

$$\operatorname{Ric} = \lambda \operatorname{Id} + D \qquad D \in \operatorname{Der}(\mathfrak{g}),\ \lambda \in \mathbb{R} \quad (\textit{nilsoliton})$$

$\mathfrak{s}$ solvable Lie algebra is *standard* (Riemannian) if $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}] \oplus^\perp \mathfrak{a}$ and $[\mathfrak{a}, \mathfrak{a}] = 0$ (i.e. $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}] \rtimes \mathbb{R}^k$)

The Riemannian Case

$(\mathfrak{s}, g)$ Einstein Riemannian solvmanifold implies:

- The following are equivalent:
  1. $\mathfrak{s}$ is *unimodular* ($\mathrm{tr}\,\mathrm{ad}(v) = 0$)
  2. $\mathfrak{s}$ is *Ricci-flat*
  3. $\mathfrak{s}$ is *flat*
- $\mathfrak{s}$ is *standard*, i.e. $\mathfrak{s} = \mathfrak{n} \rtimes \mathfrak{a}$ (orthogonal sum) [Lauret 10]
- $\mathfrak{s}$ contains an Einstein *rank one* standard extension $\mathfrak{n} \rtimes \mathbb{R}$ [Heber 98]
- restricted metric on $\mathfrak{n}$ is *nilsoliton*, i.e. $\mathrm{Ric}\,|_{\mathfrak{n}} = \lambda \mathrm{Id} + D$, $D \in \mathrm{Der}(\mathfrak{n})$ [Heber 98]
- D is diagonalizable and satisfies: $\mathrm{tr}(D \circ X) = \mathrm{tr}(X)$, $\forall X \in \mathrm{Der}(\mathfrak{n})$
- D above is *Nikolayevsky* (or *pre-Einstein*) *derivation*, exists on any Lie algebra, unique up to automorphisms [Nikolayevsky 11]
- D has positive and rational eigenvalues [Heber 98, Nikolayevsky 11]
- the metric on $\mathfrak{s}$ is determined by metric on $\mathfrak{n}$ [Heber 98]

Starting with a Riemannian Nilsoliton $(\mathfrak{n}, g)$ we have:

- Any *nilsoliton* can be extended to an Einstein solvmanifold (S, g')
- The nilsoliton is unique up to isometry [Lauret 01]
- the metric on $\mathfrak{s}$ is determined by metric on $\mathfrak{n}$ [Heber 98]

Hence:

$$\begin{array}{c} \text{Classification of} \\ \text{Einstein Solvmanifolds} \end{array} \longleftrightarrow \begin{array}{c} \text{Classification of} \\ \text{Nilsolitons} \end{array}$$

Einstein Indefinite Case

Examples of Ricci-flat metrics

- that are not flat
- on compact manifolds
- on non-unimodular manifolds

Some results on nilpotent Lie group:

- [Milnor 76] A nilpotent Lie group cannot have a Riemannian Einstein metric (unless it is abelian)
- Nilpotent Lie groups with bi-invariant metrics are necessarily Ricci-flat
- Examples of Einstein indefinite metric ($s \neq 0$) on non-abelian nilpotent Lie algebras [Conti, — 19a], [Conti, — 20]
- Many examples of Ricci-flat metrics [Conti, del Barco, — 21]
- Obstruction [Conti, — 19a], [Tibssirte 22]:
  $\mathfrak{g}$ unimodular (with Killing form zero). If $\mathfrak{g}$ has Einstein metric with $s \neq 0$. Then $\mathrm{Der}(\mathfrak{g}) \subset \mathfrak{sl}(\mathfrak{g})$

Example

Example ([Conti, — 19a])

$$(0,0,0,0,e^{12}+e^{34},e^{14}-e^{23},-e^{24}+e^{35}+e^{16},-e^{13}+e^{26}+e^{45})$$

Admits two Einstein metric with scalar curvature $s=\frac{56}{15}$, with signature $(6,2)$ and $(3,5)$:

$$e^1\otimes e^1+e^2\otimes e^2\pm(e^3\otimes e^3+e^4\otimes e^4)-\frac{7}{3}e^5\otimes e^5\mp\frac{7}{3}e^6\otimes e^6\pm\frac{98}{15}(e^7\otimes e^7+e^8\otimes e^8)$$

$\{e_1,\dots,e_8\}\subset\mathfrak{g} \quad \rightsquigarrow \quad \{e^1,\dots,e^8\}\subset\mathfrak{g}^*$ s.t.
$de^1=de^2=de^3=de^4=0, \quad de^5=e^1\wedge e^2+e^3\wedge e^4=e^{12}+e^{34},$
$de^6=e^{14}-e^{23}, \quad de^7=-e^{24}+e^{35}+e^{16}, de^8=-e^{13}+e^{26}+e^{45}$

Theorem (Conti, — 19a)

There exist Einstein pseudo-Riemannian metrics on 8-dimensional nilpotent Lie groups with $s\neq 0$ of any signature.

Nice Lie Algebras

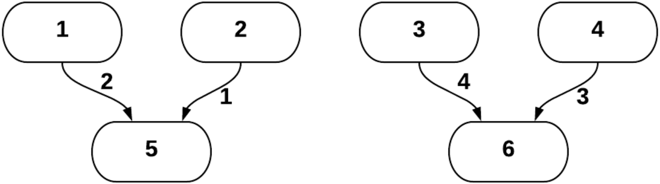
Nice nilpotent Lie algebra is a pair $(\mathfrak{g}, \mathcal{B})$: $\mathfrak{g}$ nilpotent, $\mathcal{B} = \{e_1, \dots, e_n\}$ a basis with structure constants c_{ij}^k s.t.:

1. $\forall i < j$ there exists at most one k s.t. $c_{ij}^k \neq 0$
2. $c_{ij}^k, c_{lm}^k \neq 0$ implies $\{i, j\} = \{l, m\}$ or $\{i, j\} \cap \{l, m\} = \emptyset$

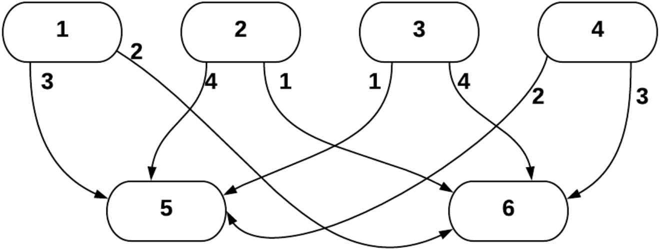
(i.e. $[e_i, e_j]$ and $e_i \lrcorner \, de^j$ are a basis element up to constant)

Two nice nilpotent Lie algebras $(\mathfrak{g}, \mathcal{B})$, $(\mathfrak{g}', \mathcal{B}')$ are considered *equivalent* if there is a Lie algebra isomorphism f that maps basis elements to multiples of basis elements
$(\mathfrak{g}, \mathcal{B})$ is unique up to equivalence, i.e. up to the action of $\Sigma_n \ltimes D_n$.

- Introduced and studied by [Lauret, Will 13], useful in the study of nilsoliton and Ricci Flow (e.g. [Nikolayevsky 11], [Payne 10], Lauret)
- Explicit left-invariant Einstein pseudo-Riemannian metrics on nilpotent Lie group with $s \neq 0$ [Conti, — 20] and $s = 0$ [Conti, — 19b]



$$(0, 0, 0, 0, e^{12}, e^{34})$$



$$(0, 0, 0, 0, e^{13} + e^{24}, e^{12} + e^{34})$$

Link with Diagram

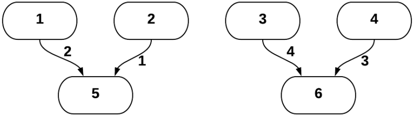
Nice nilpotent $(\mathfrak{g}, \mathcal{B}) \rightsquigarrow$ diagram Δ:

- $N(\Delta) = \mathcal{B}$
- an arrow with label $i \xrightarrow{j} k$ if $[e_i, e_j] = c_{ij}^k e_k$ with $c_{ij}^k \neq 0$

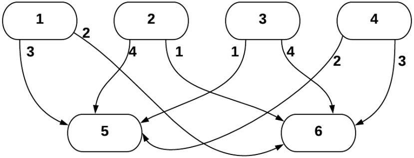
Direct, acyclic, with no multiple arrows with same surce and destination.

$$M_\Delta = \begin{pmatrix} -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{pmatrix}$$

M_Δ *root matrix*



$(0, 0, 0, 0, e^{12}, e^{34})$



$(0, 0, 0, 0, e^{13} + e^{24}, e^{12} + e^{34})$

Algorithm to construct Nice diagram

Δ has a natural filtration on nodes $N(\Delta) = N_0 \supset N_1 \supset \cdots \supset N_s$, where N_{i+1} contains all the nodes that are reached by at least one arrow in N_i.
A diagram has *type* $(a_1, \dots, a_s)$ if

$$(|N|, |N_1|, \dots, |N_s|) = (a_1 + \cdots + a_s, a_2 + \cdots + a_s, \dots, a_s).$$

Step 1. *Classify diagrams of type* $(a_1, \dots, a_s)$.

Proposition

Let $s > 1$, $n = a_2 + \cdots + a_s$. *Up to equivalence, any diagram of type* $(a_1, \dots, a_s)$ *can be obtained from a diagram of type* $(a_2, \dots, a_s)$ *by the following procedure:*

- *add* a_1 *nodes labeled* $n, \dots, n + a_1$.
- *choose appropriate subsets* $A_1 \leq A_2 \leq \cdots \leq A_{a_1}$ *of* $\{1, \dots, n\}$ *such that* $A_1 \cup \cdots \cup A_{a_1} = \{1, \dots, n\}$.
- *for each* $1 \leq i \leq a_1$, *add an arrow* $(n + i) \to j$ *whenever* $j \in A_i$.

Step 2. *Remove the diagrams where some nodes have an odd number of incoming arrows.*

Step 3. *Eliminate isomorphic diagrams*.
Δ, Δ' are isomorphic $\Rightarrow$ the resulting (families of) nice Lie algebras will be equivalent.
We introduce an appropriate hash function: by construction, Δ and Δ' isomorphic only if $\#(\Delta) = \#(\Delta')$
Step 4. *For each diagram, compute the possible labelings*.
Adding labels in pairs, iteratively, until the diagram is fully labeled.
Step 5. *Eliminate equivalent diagrams*.
As in Step 3 to eliminate duplicates, with the difference that the bijections are isomorphisms of nice diagrams.
Step 6. *Eliminate diagrams for which Condition* 4 *is violated*.

Classifying nice Lie algebras

Step A. Choose $\mathcal{J}_{\Delta,2}$ a maximal set of $\mathbb{Z}_2$-linearly independent rows of $M_{\Delta,2}$, and choose $\mathcal{J}_\Delta \supset \mathcal{J}_{\Delta,2}$ parametrizing a maximal set of $\mathbb{R}$-linearly independent rows of M_Δ.

Proposition

$\mathcal{J}_{\Delta,2} \subset \mathcal{J}_\Delta \subset \mathcal{I}_\Delta$ as above. Set $\mathring{V}_\Delta = \left\{\sum c_I E_I \mid c_I \neq 0 \ \forall I\right\}$ *and*

$$W = \left\{\sum c_I E_I \mid c_I = 1 \ \forall I \in \mathcal{J}_{\Delta,2};\ c_I = \pm 1 \ \forall I \in \mathcal{J}_\Delta \setminus \mathcal{J}_{\Delta,2}\right\}.$$

Then $\mathring{W} = W \cap \mathring{V}_\Delta$ is a fundamental domain in $\mathring{V}_\Delta$ for the action of D_n.

We are reduced to the set $\mathring{W}$ of elements $\sum c_I E_I$ where

$$c_I = \begin{cases} 1 & I \in \mathcal{J}_{\Delta,2} \\ \pm 1 & I \in \mathcal{J}_\Delta \setminus \mathcal{J}_{\Delta,2} \\ \text{a nonzero constant} & \text{otherwise.} \end{cases}$$

Step B. Determine the action of $\mathrm{Aut}(\Delta)$ on the set of connected components of $\mathring{W}$, and choose connected components $W_1, \dots, W_k$ of $\mathring{W}$, one for each orbit.
Step C. On each component W_j, impose the Jacobi identity

$$\sum_{I,J} c_I c_J \mathfrak{a}(E_{IJ}) = 0;$$

Neglecting quadratic equations for the moment, determine the subspace $L_j \subset W$ defined by linear equations and inequalities.
Step D. For each nonempty L_j, consider the corresponding family of Lie algebras obtained by imposing the quadratic constraints from the Jacobi identity.

Theorem

Let Δ be a nice diagram , $\mathring{W} \supset W_1, \dots, W_k$ as before. Let $B_j \subset W_j$ be the subset defined by the Jacobi equations. Then:
- *each element of B_j defines a nice Lie algebra with diagram Δ;*
- *up to equivalence, any nice Lie algebra with diagram Δ is obtained in this way.*

Moreover, if $j \neq k$, elements of B_j and B_k determine inequivalent nice Lie algebras.

We obtain [Conti, — 19b]:

- Algorithm to construct nice diagrams Δ
- Classifying nice Lie algebras (up to equivalence)
- Up to dim 7, "most" nilpotent Lie algebras are nice (by comparison with classification of nilpotent Lie algebras [Gong 98])
- Nice nilpotent Lie algebras classified up to dimension 9

dim	NLA	NLA with nice basis	nice NLA
3	2	2	2
4	3	3	3
5	9	9	9
6	34	33	36
7	175 + 9 families	$141 + 4\frac{1}{2}$ families	152 + 41 families
8	?	?	917 + 45 families
9	?	?	6386 + 501 families

Construction of Einstein Nice Nilpotent Lie Algebras

Lemma (Conti, — 19b)

Let $\mathfrak{g}$ be a nice Lie algebra with diagram Δ.
Then all derivations of $\mathfrak{g}$ are traceless if and only if $(1, \dots, 1) \in \mathbb{R}^n$ is in the space spanned by the rows of M_Δ, i.e.

$$ {}^tM_\Delta X = (1, \dots, 1) $$

Proposition (Conti, — 19b)

$\mathfrak{g}$ nice, g diagonal metric. Then:

$$ \mathrm{Ric} = \frac{1}{2}\left({}^tM_\Delta X\right)^D $$

$$ X = (x_{ijk}) \quad \text{s.t.} \quad \frac{x_{ijk}}{c_{ijk}^2} = \frac{g_k}{g_i g_j} $$

Theorem (Conti, — 19b)

Let $\mathfrak{g}$ be a nice nilpotent Lie algebra with root matrix M_Δ. Then $\mathfrak{g}$ admits a diagonal metric of signature δ satisfying $\operatorname{Ric} = \frac{1}{2}k\mathrm{Id}$ *if and only if for some X:*

- *(K)* ${}^tM_\Delta X = [k]$
- *(H) each component x_I is not zero;*
- *(L)* $(\operatorname{logsign} x_I) = M_{\Delta,2}\delta$;
- *(P) for a basis $\alpha_1, \ldots, \alpha_j$ of* $\ker {}^tM_\Delta$ *we have:*

$$|X|^{\alpha_i} = (c^2)^{\alpha_i} \qquad i = 1, \ldots, j$$

σ involution of order two

A metric on $\mathfrak{g}$ is *σ-diagonal* if the metric tensor has the form

$$g = g_i e^i \otimes e^{\sigma_i} \qquad 0 \neq g_i \in \mathbb{R}$$

Similar Theorem for $\sigma \in \operatorname{Aut} \Delta$

- Construction and classification of diagonal and σ-diagonal Einstein metrics [Conti, — 19b; Conti, — 20]

dim of nNLA	$\leq$**5**	**6**	**7**	**8**	**9** + M_Δ **surj.**
diagonal Ricci-flat	-	**1**	**3**	**9**	-
σ-**diag RF**	-	-	**1**	**4**	-
diag. Einstein $s \neq 0$	-	-	-	**8**	**48**
σ-**diag. E** $s \neq 0$	-	-	-	**4**	**7**

- We obtain a one-parameter family of non-isometric Ricci-flat metrics

Theorem (Conti, — 20)

For each $n \geq 8$, there exist n-dimensional nice nilpotent Lie algebras with an Einstein diagonal metric with $s \neq 0$

Diagram Involutions and homogeneous Ricci-flat metrics

Definition

Δ a nice diagram, a permutation σ of nodes will be called an *arrow-breaking involution* if it has order two and:

$$i \xrightarrow{j} k \quad \Longrightarrow \quad \begin{cases} \nexists\ \sigma(i) \xrightarrow{\bullet} \sigma(k) \\ \nexists\ \sigma(i) \xrightarrow{\sigma(j)} \bullet \end{cases}$$

Proposition (Conti, Del Barco, — 21)

$\mathfrak{g}$ nice nilpotent Lie algebra with diagram Δ, and σ an arrow-breaking involution. Then any σ-diagonal metric is Ricci-flat.

- Arrow breaking condition can be rewritten using polinomials P_Δ, Q_Δ

Results

We obtain [Conti, Del Barco, — 21]:

- Existence of arrow-braking involutions for nilpotent Lie algebras with "large center" $(n - \dim \mathfrak{z} \leq \dim \mathfrak{z} + 3)$
- Any 2-step nilpotent Lie algebra attached to a graph has an arrow-breaking involution
- Every nice nilpotent Lie algebra of dimension ≤ 7 has a Ricci-flat metric
- Every 6-dimensional nonabelian nilpotent Lie algebra has a nonflat Ricci-flat metric
- Sufficient conditions to have Ricci-flat nonflat metrics
- Construction of infinite families of Ricci-flat metrics associated to nilradicals of parabolic subalgebras of split simple Lie algebras $(A_n,\ B_n,\ C_n,\ G_2)$

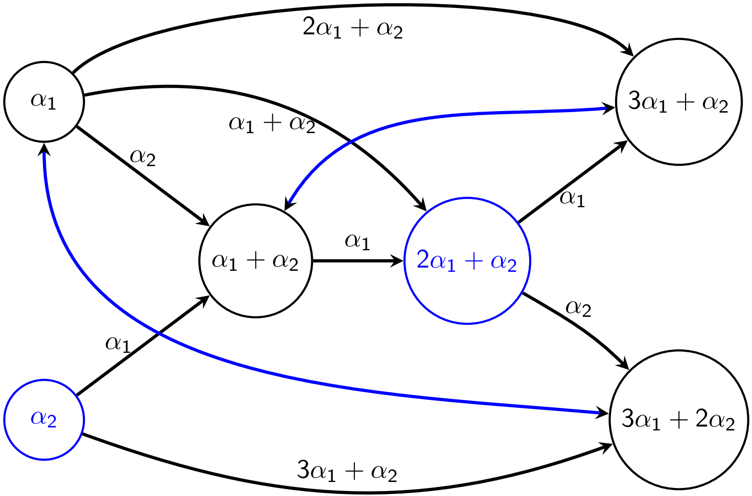


Figure: Diagram of Nilradical of parabolic subalgebra of $\mathfrak{g}_2$

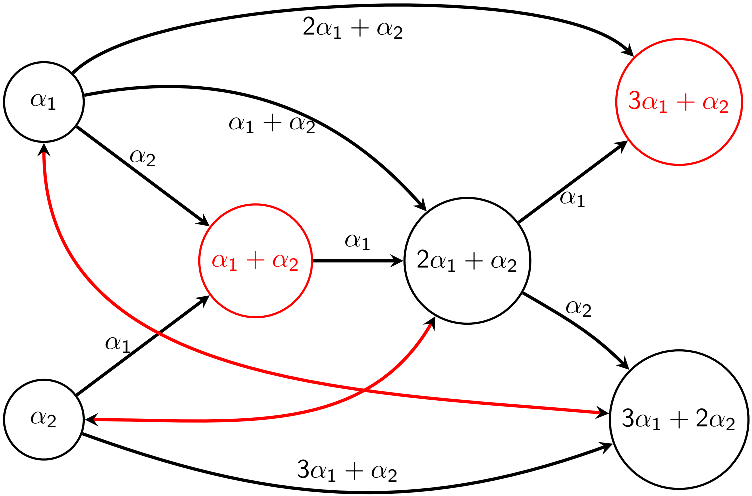


Figure: Diagram of Nilradical of parabolic subalgebra of $\mathfrak{g}_2$

Pseudo-Riemannian Nilsolitons

The pair $(\mathfrak{g}, g)$ is a *(Algebraic) nilsoliton* if
$\mathfrak{g}$ nilpotent Lie algebra, g a pseudo-Riemannian metric and

$$\mathrm{Ric} = \lambda \mathrm{Id} + D, \quad \lambda \in \mathbb{R}, D \in \mathrm{Der}\,\mathfrak{g}$$

- There exists Ricci soliton which are not algebraic [Batat, Onda 17]

Theorem ([Conti, — 22])

g be a nilsoliton metric on a nilpotent Lie algebra $\mathfrak{g}$. Then either

1. *$\lambda = 0$ and D is a nilpotent derivation in the null space of $\mathrm{Der}\,\mathfrak{g}$; or*
2. *$\lambda \neq 0$ and setting $\tilde{D} = -\frac{1}{\lambda} D$, we have*

$$\mathrm{tr}(X) = \mathrm{tr}(\tilde{D} \circ X), \quad X \in \mathrm{Der}\,\mathfrak{g};$$

relative to the Jordan decomposition $\tilde{D} = \tilde{D}_s + \tilde{D}_n$, $\tilde{D}_s$ is a Nikolayevsky derivation and $\tilde{D}_n$ a nilpotent derivation in the null space of $\mathrm{Der}\,\mathfrak{g}$.

In either case, the eigenvalues of D are rational

Types of pseudo-Riemannian Nilsolitons

1. *(Nil1)* $\lambda = 0,\ D = 0$
   This is the Ricci-flat case, examples of which exist in abundance (see e.g.[Conti, Del Barco, — 21] or [Conti, — 19b])

2. *(Nil2)* $\lambda = 0,\ D \neq 0$
   D nilpotent (nonsemisimple), D is not Nikolayevsky Derivation

3. *(Nil3)* $\lambda \neq 0,\ D = 0$
   - This is the Einstein case, with nonzero scalar curvature:
     First example in [Conti, — 19a], more constructions in [Conti, — 20]
   - Obstruction: if $\mathfrak{n}$ has a Einstein metric with $\lambda \neq 0$, then $\operatorname{Der} \mathfrak{n} \subset \mathfrak{sl}(\mathfrak{n})$ [Conti, — 19a], [Tibssirte 22]
   - no counterpart in the Riemannian case [Milnor 76]

4. *(Nil4)* $\lambda \neq 0,\ D \neq 0$
   - Similar to the Riemannian situation
   - If D is diagonalizable, it is a multiple of a Nikolayevsky derivation

Standard and Pseudo-Iwasawa Lie Algebras

Definition

A *standard decomposition* of a metric Lie algebra $\tilde{\mathfrak{g}}$ is a decomposition $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus^{\perp} \mathfrak{a}$, where $\mathfrak{g}$ is a nilpotent ideal and $\mathfrak{a}$ is an abelian subalgebra

- Standard Riemannian solvmanifolds are also standard for this definition
- $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] = \tilde{\mathfrak{g}}' \subset \mathfrak{g} \subset \mathfrak{n}$
- Excluding cases $\mathfrak{g}$ with degenerate metric or with $\mathfrak{g}^{\perp}$ not abelian

Definition

A standard decomposition $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus^{\perp} \mathfrak{a}$ is *pseudo-Iwasawa* if $\operatorname{ad} X = (\operatorname{ad} X)^*$, $X \in \mathfrak{a}$

- Solv. Einstein Examples with $(\operatorname{ad} X)^*$ not a derivation
- Solv. Einstein Examples with $[\operatorname{ad} X, (\operatorname{ad} X)^*] \neq 0$
- $\tilde{\mathfrak{g}}$ Solv. Einstein with Standard decomposition s.t. $\forall X \in \mathfrak{a}$ $(\operatorname{ad} X)^*$ is a derivation and $[\operatorname{ad} \mathfrak{a}, (\operatorname{ad} X)^*] = 0$. Then $\tilde{\mathfrak{g}}$ is isometric to a Pseudo-Iwasawa

From Einstein Solvmanifolds to Nilsolitons

Theorem ([Conti, — 22])

Let $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus^{\perp} \mathfrak{a}$ *be a pseudo-Iwasawa decomposition. Then the Einstein equation* $\widetilde{\mathrm{Ric}} = \lambda \mathrm{Id}$ *on* $\tilde{\mathfrak{g}}$ *is satisfied if and only if*

1. $\mathfrak{g}$ *with the induced metric satisfies the nilsoliton equation*
$$\mathrm{Ric} = \lambda \mathrm{Id} + D, \qquad D = \mathrm{ad}\, H$$
2. $\langle \mathrm{ad}\, X, \mathrm{ad}\, Y \rangle_{\mathrm{tr}} = -\lambda \langle X, Y \rangle$ *for all* $X, Y \in \mathfrak{a}$.

In this case, then
$$\mathrm{tr}\, D^2 = -\lambda\, \mathrm{tr}\, D$$

Where:

- $\langle X, Y \rangle_{\mathrm{tr}} := \mathrm{tr}(X \circ Y)$
- H is the metric dual of $v \mapsto \mathrm{tr}\, \widetilde{\mathrm{ad}} v$, i.e. $g(H, v) = \mathrm{tr}\, \widetilde{\mathrm{ad}} v,\ v \in \tilde{\mathfrak{g}}$

Einstein Solvmanifold $\lambda \neq 0$

Corollary ([Conti, — 22])

Given a pseudo-Iwasawa solvable Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus^{\perp} \mathfrak{a}$ *satisfying* $\widetilde{\mathrm{Ric}} = \lambda \mathrm{Id}$ *for some nonzero* λ*, then either:*

1. $\tilde{\mathfrak{g}}$ *is unimodular,* $H = 0$ *and* $\mathfrak{g}$ *is a nilsoliton of type (Nil3), with* $\mathrm{Ric} = \lambda \mathrm{Id}$*; or*
2. $\tilde{\mathfrak{g}}$ *is not unimodular,* $\langle H, H \rangle \neq 0$, $\mathfrak{g} \oplus \mathrm{Span}\{H\}$ *is also Einstein with a pseudo-Iwasawa decomposition, and* $\mathfrak{g}$ *is a nilsoliton of type (Nil4), with* $\mathrm{Ric} = \lambda \mathrm{Id} + D$ *and* $\mathrm{tr}\, D \neq 0$

- if $\lambda \neq 0$, then $\mathfrak{g}$ is the nilradical

From Nilsolitons to Einstein Solvmanifolds

Type *(Nil4)* and *(Nil3)* nilsolitons

Theorem ([Conti, — 22])

Let $\mathfrak{g}$ be a nilsoliton, $\mathrm{Ric} = \lambda \mathrm{Id} + D$*,* $\lambda \neq 0$*. Let $\mathfrak{a} \subset \mathrm{Der}\,\mathfrak{g}$ be a subalgebra of self-adjoint derivations containing D, and assume that $\langle , \rangle_{\mathrm{tr}}$ is nondegenerate on $\mathfrak{a}$.*

Then the metric $\langle , \rangle_{\mathfrak{g}} - \frac{1}{\lambda}\langle , \rangle_{\mathrm{tr}}$ on $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \mathfrak{a}$ is Einstein with $\widetilde{\mathrm{Ric}} = \lambda \mathrm{Id}$ and the decomposition $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus^{\perp} \mathfrak{a}$ is pseudo-Iwasawa

Called *Pseudo-Iwasawa Extension*

- Let $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus^{\perp} \mathfrak{a}$ be an Einstein solvable Lie algebra of pseudo-Iwasawa type of nonzero scalar curvature. Up to isometric isomorphisms, $\tilde{\mathfrak{g}}$ is a pseudo-Iwasawa extension of $\mathfrak{g}$
- Recover the 1-dimensional extensions of [Yan 20] and [Yan, Deng 21]

Summary

$\tilde{\mathfrak{g}} = \mathfrak{g} \oplus^{\perp} \mathfrak{a} \in \mathcal{S}$	$\longleftrightarrow$	$\mathfrak{g} \in \mathcal{N}$
Nonunimodular Pseudo-Iwasawa Einstein Solvmanifold $s \neq 0$	$\longleftrightarrow$	Nilsoliton $\lambda \neq 0$, tr $D \neq 0$ *(Nil4)*
Unimodular Pseudo-Iwasawa Einstein Solvmanifold $s \neq 0$	$\longleftrightarrow$	$\mathfrak{g}$ Einstein metric $\lambda \neq 0$ *(Nil3)*
Nonunimodular Pseudo-Iwasawa Ricci-Flat Solvmanifold	$\longleftrightarrow$ $\leftarrow$ - -	Nilsoliton Ric $= D$ *(Nil1)* or *(Nil2)*
Unimodular Pseudo-Iwasawa Ricci-Flat Solvmanifold	$\longleftrightarrow$	Ricci-Flat metric *(Nil1)*

Conclusions

- Strong relation between diagrams and geometric objects
- The pseudo-Riemannian case is richer:
  - Einstein nilpotent metrics
  - We describe the pseudo-Iwasawa Einstein solvmanifolds
  - We construct many nilsolitons [Conti, — 21], uniqueness missing
  - We use these results to construct Einstein solvmanifolds with other compatible structures (pseudo-Kähler, para-Kähler [— 22], pseudo-Sasaki [Conti, —, Segnan Dalmasso 22])
- There is a “generalized nilsoliton” equation to construct Einstein Solvmanifold that are not pseudo-Iwasawa [Conti, —, Segnan Dalmasso 22]

Open Problems

- Construction and classification of non-nilpotent nice Lie algebras
  Construction and classification nilpotent nice Lie algebras over $\mathbb{F}_p$
- Can a nilpotent Lie algebra admit two inequivalent nice bases with the same diagram?
- Existence of nNLA with infinite many inequivalent basis?
- Does every nilpotent Lie algebra admit a Ricci-flat indefinite metric?
- Relation between diagram and ad-invariant metric (building blocks)
- $\mathrm{Der}(\mathfrak{g}) \subset \mathfrak{sl}(\mathfrak{g}) \implies$ Existence of Einstein metric with $s \neq 0$?
- Understand better the Einstein pseudo-Riemannian Solvmanifolds

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Thank you

for your kind attention