Approximations, integral transforms, convergence, error control

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I work, essentially, in four fields of mathematical analysis

- Approximation Theory;
- Partial Differential Equations;
- Analytic Number Theory;
- Special Functions and Analytic Combinatorics.

Theorem

*Let be $f \in L^2(\mathbb{R})$ (f with **finite energy**) such that:*

- *supp* $\widehat{f} \subseteq [-\pi W, \pi W]$, $W > 0$; *(f is **band limited**)*

Then

$$\sum_{k \in \mathbb{Z}} f\left(\frac{k}{W}\right) \cdot sinc(Wt - k) \;=\; f(t), \qquad \text{for every } t \in \mathbb{R}.$$

E.T. Whittaker - V.A. Kotelnikov - C.E. Shannon ('30 - '50)

Let $(K_w^\chi f)_{w>0}$ be the family of operators defined by

$$(K_w^\chi f)(x) := \sum_{k\in\mathbb{Z}} \chi(wx - t_k) \cdot \left[\frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\, du \right], \; x \in \mathbb{R}, w > 0,$$

where $f : \mathbb{R} \to \mathbb{R}$ is locally integrable functions, such that the above series is convergent for every $x \in \mathbb{R}$.

(C.Bardaro et al., Samp. Theory Signal Image Proc., 2007)

$\chi : \mathbb{R} \to \mathbb{R}$ is a **kernel** function if satisfies the following conditions:

- $(\chi 1)$ $\chi \in L^1(\mathbb{R})$ is locally bounded in $0 \in \mathbb{R}$;
- $(\chi 2)$ $\sum_{k \in \mathbb{Z}} \chi(u - t_k) = 1$ for every $u \in \mathbb{R}$;
- $(\chi 3)$ $\exists\ \beta > 0$: $\sup_{u \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |\chi(u - t_k)| \cdot |u - t_k|^\beta < +\infty$,

where $(t_k)_{k \in \mathbb{Z}}$, $t_k \in \mathbb{R}$ is a sequence such that

- $\lim_{k \to \pm\infty} t_k = \pm\infty$;
- $-\infty < t_k < t_{k+1} < +\infty$;
- $\Delta_k := t_{k+1} - t_k > 0$.

Fejer's kernel

$$F(x) := \frac{1}{2}\,\mathrm{sinc}^2\left(\frac{x}{2}\right), \qquad x \in \mathbb{R};$$

Jackson type kernels

$$J_k(x) := c_k\,\mathrm{sinc}^{2k}\left(\frac{x}{2k\pi\alpha}\right), \qquad x \in \mathbb{R},$$

where:

$$c_k := \left[\int_{\mathbb{R}} \mathrm{sinc}^{2k}\left(\frac{u}{2k\pi\alpha}\right)\,du\right]^{-1};$$

Central B-spline of order $k \in \mathbb{N}$

$$M_k(x) \;:=\; \frac{1}{(k-1)!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \left(\frac{k}{2} + x - i \right)_+^{k-1}, \qquad x \in \mathbb{R};$$

Multivariate Product Kernels

$$\chi(\underline{x}) \;:=\; \prod_{i=1}^{n} \chi_i(x_i), \qquad \underline{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

where any χ_i, $i = 1, \ldots, n$, is a kernel of one-variable.

Radial Kernels: Bochner-Riesz kernels

$$r_n(\underline{x}) \;:=\; \frac{2^n}{\sqrt{2\pi}} \Gamma(n+1) \, \|\underline{x}\|_2^{-n-1/2} \, J_{n+1/2}(\|\underline{x}\|_2), \qquad \underline{x} \in \mathbb{R}^n,$$

J_λ is the Bessel function of order λ and Γ is the Euler gamma function.

Pointwise and Uniform Convergence

Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable and bounded function. Then

$$\lim_{w \to +\infty} (K_w^{\chi} f)(x) = f(x)$$

at any point x of continuity of f . Moreover, if f is uniformly continuous and bounded, we have

$$\lim_{w \to +\infty} \|K_w^{\chi} f - f\|_{\infty} = 0.$$

(C.Bardaro et al., Samp. Theory Signal Image Proc., 2007) These results have also been proved in $L^p(\mathbb{R})$ (ibidem), in $\mathbb{R}^n$, $n \geq 1$ (D. Costarelli et al., Bollettino U.M.I., 2011) and in other general spaces.

Define:

$$Lip_\infty(\nu) \;:=\; \{f \in C(\mathbb{R}) : \; \|f(\cdot) - f(\cdot + t)\|_\infty = O(|t|^\nu), \text{ as } t \to 0\},$$

for $0 < \nu \leq 1$. Hence:

Order of Approximation

Let $f \in Lip_\infty(\nu)$, $0 < \nu \leq 1$ be fixed. Then

$$\|K_w^\chi f - f\|_\infty \;=\; O(w^{-\varepsilon}), \qquad as \qquad w \to +\infty,$$

where $\varepsilon := \min\{\nu, \; \beta\}$ and $\beta > 0$ is the constants of conditions $(\chi 3)$.

(D. Costarelli et al., J. Int. Eq. Appl., 2014a, 2014b)

If $\chi(x)$ is a bounded kernel then $\chi_s(x) := \chi(x+s)$, $s \in \mathbb{R}$ is a kernel. Also, if $\chi(x)$ is continuous, then

$$\overline{\chi}_s(t) := \frac{1}{s}\int_0^s \chi(u+t)\, du,\ s > 0$$

is also a kernel. Define

$$Lip_\infty(1) \ := \ \{f \in C(\mathbb{R}) : \ \|f(\cdot) - f(\cdot + t)\|_\infty = O(|t|), \ \text{as } t \to 0\},$$

Theorem

Let $f \in C(\mathbb{R})$ and $\chi \in C(\mathbb{R})$ be fixed. Assume that

$$\|K_w^\chi f - f\|_\infty = O\left(w^{-1}\right)$$

as $w \to +\infty$. Then, if we consider the kernel $\overline{\chi}_s(t)$, for some fixed $s \geq 1$, we have:

$$\left\|\left(K_w^{\overline{\chi}_s} f\right)'\right\|_\infty = O(1), \ \text{as } w \to +\infty \qquad \Longleftrightarrow \qquad f \in \mathrm{Lip}_\infty(1).$$

(M. Cantarini et al., Dolomites research notes on approx., 2020)

Theorem

Let $f \in C(\mathbb{R})$ *and* $\chi \in C(\mathbb{R})$ *be a kernel. Assume that*

$$\|K_w^{\chi} f - f\|_\infty = O\left(w^{-1}\right)$$

and there exists $s \geq 1$ *such that*

$$\|K_w^{\chi_s} f - f\|_\infty = O\left(w^{-1}\right), \; w \to +\infty.$$

Then $f \in \mathrm{Lip}_\infty(1)$.

(M. Cantarini et al., Dolomites research notes on approx., 2020)

If $\chi(x) \in C^1(\mathbb{R})$, then, for every measurable and locally integrable function on $\mathbb{R}$, we have $\left(K_w^{\chi} f\right)'(x) = \left(w K_w^{\chi'} f\right)(x)$.

Theorem

Let χ be a kernel and let $f \in L^{\infty}(\mathbb{R})$ be a function.

(i) *If $f'(x)$ exists at some point $x \in \mathbb{R}$, then:*

$$\lim_{w \to +\infty} \left(w K_w^{\chi'} f\right)(x) = f'(x).$$

(ii) *If $f \in C^{(1)}(\mathbb{R})$ with f' uniformly continuous on a certain interval $I \subset \mathbb{R}$, then the family $\left(w K_w^{\chi'} f\right)$ converges uniformly to f' on I. In particular, if $f \in C^1(\mathbb{R})$ the family $\left(w K_w^{\chi'} f\right)$ converges uniformly to f' on the whole $\mathbb{R}$.*

The previous result, can be generalized to to functions with higher derivatives. (M. Cantarini et al., J. Math. Anal. Appl., 2022)

Theorem (Part 1)

Let χ *be a kernel such that* $\chi'(x) = 0$ *for every* $x \in [0,1)$, *and let* $f \in L^\infty(\mathbb{R})$ *be a function. Suppose in addition that* f *is not differentiable at* $t \in \mathbb{R}$, *and both* $f'_-(t), f'_+(t)$ *exist and are finite, with* $f'_-(t) \neq f'_+(t)$. *Then the following assertions are equivalent:*

- *there exists* $\alpha \in \mathbb{R}$ *such that*

$$\lim_{w \to +\infty} \left(w K_w^{\chi'} f \right)(t) \; = \; f'_+(t)\,\alpha \, + \, f'_-(t)\,(1-\alpha)\,.$$

- $\forall x \in [0,1)$ *we have* $\Psi_1^-[\chi'](x) + \frac{\Psi_0^-[\chi'](x)}{2} = \alpha$.
- $\forall x \in [0,1)$ *we have* $\Psi_1^+[\chi'](x) + \frac{\Psi_0^+[\chi'](x)}{2} = 1 - \alpha$.

$$\Psi_1^{\pm}\left[\chi'\right](x) := \sum_{k \lessgtr x} \chi'(x-k)\,(k-x)\,,\; \Psi_0^{\pm}\left[\chi'\right](x) := \sum_{k \lessgtr x} \chi'(x-k)\,.$$

(M. Cantarini et al., J. Math. Anal. Appl., 2022)

Theorem (Part 2)

- $\int_{-\infty}^{0} \left[\frac{\chi(u)'}{2} - u\chi'(u)\right] e^{-2\pi iuk} du = \begin{cases} \alpha, & k = 0 \\ 0 & k \in \mathbb{Z} \setminus \{0\} \end{cases}$
- $\int_{0}^{+\infty} \left[\frac{\chi(u)'}{2} - u\chi'(u)\right] e^{-2\pi iuk} du = \begin{cases} 1-\alpha, & k = 0 \\ 0 & k \in \mathbb{Z} \setminus \{0\} \end{cases}.$

(M. Cantarini et al., J. Math. Anal. Appl., 2022)

Some examples:

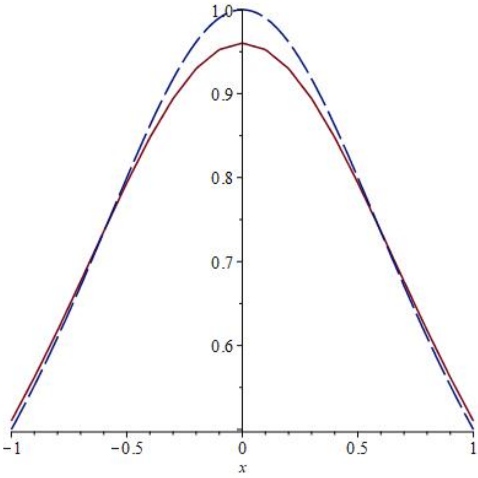


Figura: $f_1(x) := \arctan(x)$; the function f_1' (blue line) and its approximation by the sampling Kantorovich operator with $w = 10$ (red dashed line).

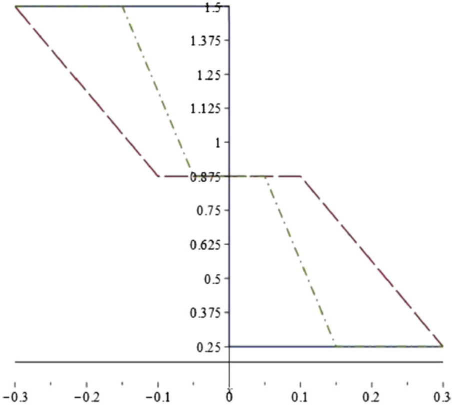


Figura: $f_2(x) := \begin{cases} 1, & x > 1 \\ \frac{x}{4}, & 0 \leq x \leq 1 \\ \frac{3}{2}x, & -1 \leq x < 0 \\ -\frac{3}{2}, & x < -1. \end{cases}$; the function f_2' (blue line) and its approximations by the sampling Kantorovich operator with $w = 10$ (red dashed line) and with $w = 20$ (green dashed/dotted line).

$$\sum_{k=1}^{N} \alpha_k \, \sigma(\underline{w}_k \cdot \underline{x} - \theta_k),$$

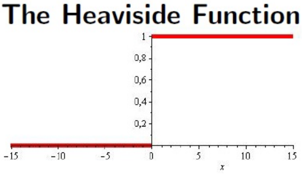
$\underline{x} \in \mathbb{R}$ are the inputs;

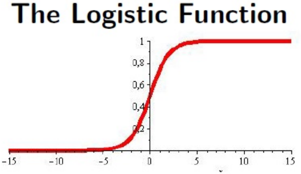
$\underline{w}_k \in \mathbb{R}$ are the weights;

$\theta_k \in \mathbb{R}$ are the thresholds (or bias);

$\sigma : \mathbb{R} \to \mathbb{R}$ is the activation function.

where
$\underline{w}_k \cdot \underline{x} := \sum_{i=1}^{n} w_{k_i} x_i$





Definition

A measurable function $\sigma : \mathbb{R} \to \mathbb{R}$ *is called a* **sigmoidal function** *if*

$$\lim_{x \to -\infty} \sigma(x) = 0 \quad \textit{and} \quad \lim_{x \to +\infty} \sigma(x) = 1.$$

Neural Networks activated by Sigmoidal functions are very ductile tools and are widely used in Approximation Theory (Cybenko, Approximation by Superpositions of a Sigmoidal Function, 1989, Mhaskar, Micchelli,1992, Chen, Chen, Liu, 1992, Light, 1993). We consider non-decreasing sigmoidal functions σ satisfying all the following assumptions:

- $(\Sigma 1)$ $g_\sigma(x) := \sigma(x) - 1/2$ is an odd function;
- $(\Sigma 2)$ $\sigma \in C^2(\mathbb{R})$ is concave for $x \geq 0$;
- $(\Sigma 3)$ $\sigma(x) = O(|x|^{-1-\alpha})$ as $x \to -\infty$, for some $\alpha > 0$.

The Density Functions

$$\phi_\sigma(x) \;:=\; \frac{1}{2}[\sigma(x+1) - \sigma(x-1)], \qquad x \in \mathbb{R}.$$

Lemma

(*i*) $\phi_\sigma(x) \geq 0$, $x \in \mathbb{R}$, *with* $\phi_\sigma(1) > 0$, *and* $\lim_{x \to \pm\infty} \phi_\sigma(x) = 0$;

(*ii*) $\phi_\sigma(x)$ *is an even function;*

(*iii*) *for every* $x \in \mathbb{R}$, $\sum_{k \in \mathbb{Z}} \phi_\sigma(x - k) = 1$;

(*iv*) $\phi_\sigma(x)$ *is non-decreasing for* $x < 0$ *and non-increasing for* $x \geq 0$;

(*v*) *for* $x \in [-1, 1] \subset \mathbb{R}$, $n \in \mathbb{N}^+$, *then*

$$\sum_{k=-n}^{n} \phi_\sigma(nx - k) \geq \phi_\sigma(1) > 0;$$

(*vi*) $\phi_\sigma(x) = O(|x|^{-1-\alpha})$ *as* $x \to \pm\infty$;

(*vii*) $\sum_{k \in \mathbb{Z}} \phi_\sigma(x - k)$ *converges uniformly on the compacts of* $\mathbb{R}$.

Let f be a sufficiently regular function then the family of neural network operators of the Kantorovich type are

$$K_n(f,x) = \frac{\displaystyle\sum_{k=-n}^{n-1} n \int_{k/n}^{(k+1)/n} f(u)\, du\, \phi_\sigma(nx-k)}{\displaystyle\sum_{k=-n}^{n-1} \phi_\sigma(nx-k)},$$

where $n \in \mathbb{N}^+$, $x \in [-1,1]$ (Costarelli et al., J. of App. Th., 2014). In particular, I have focused on studying asymptotic formulas for these operators, used in order to prove a Voronovskaja-type theorem, that is, the calculation of limits of the form

$$\lim_{n\to+\infty} n\left[K_n(f,x) - f(x)\right].$$

Theorem (Sketch)

Let σ be a sigmoidal function with suitable hypotheses. If $f \in C^r([-1,1])$, $r \in \mathbb{N}^+$ we have

$$K_n(f,x) = f(x) + \sum_{\nu=1}^{r} \frac{f^{(\nu)}(x)}{\nu! n^\nu} F_\nu(n,x,\sigma) + o\left(n^{-r}\right)$$

as $n \to \infty$, where $x \in [-1,1]$, $n \in \mathbb{N}$ and F is a suitable and computable function. Furthermore, we have

$$\lim_{n \to +\infty} n\left[K_n(f,x) - f(x)\right] = Cf'(x)$$

where C is a computable constant depending on σ.

(M. Cantarini et al., Mediterranean J. of Math., 2021)

Future works: We want to study further properties of the family of the sampling Kantorovich operators. In particular

- Convergence in Soboelv spaces;
- Convergence in classical fractional Sobolev spaces

$$\widehat{W}^{s,p}(\mathbb{R}) := \left\{ u \in L^p(\mathbb{R}) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{1}{p}+s}} \in L^p(\mathbb{R} \times \mathbb{R}) \right\}$$

with $s \in (0,1), p \in [1, +\infty)$ and/or in symmetric fractional Sobolev spaces

$$\widetilde{W}^{s,p}(\mathbb{R}) := {}^{+}W^{s,p}(\mathbb{R}) \cap {}^{-}W^{s,p}(\mathbb{R})$$

where

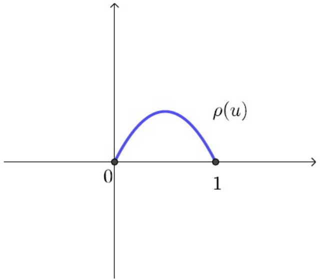
$${}^{\pm}W^{s,p}(\mathbb{R}) := \left\{ f \in W^{m,p} : {}^{\pm}\mathcal{D}^s f \in L^p(\mathbb{R}) \right\}$$

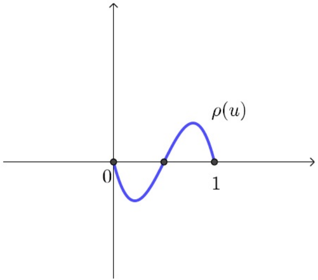
and where ${}^{\pm}\mathcal{D}^s f$ is the left/right weak fractional derivative of f of order s, $s > 0$, $m = \lfloor s \rfloor$, $1 \leq p \leq \infty$.

Simpest form of reaction-diffusion equation

$$v_\tau = v_{xx} + \rho(v), \quad v(\tau, x) \in [0, 1]$$

where $v(\tau, x)$ is the density in x at time τ and the reaction term $\rho \in C([0, 1])$, with $\rho(0) = \rho(1) = 0$ (0 and 1 are stationary solutions of the problem) can be strictly positive in $(0, 1)$, or have a change of sign.





Reaction-diffusion-convection (RDC) equations are a generalization of the previous equations:

$$v_\tau + f(v)v_x = (D(v)v_x)_x + \rho(v), \quad v(\tau, x) \in [0,1]$$

- introducing a density-dependent diffusion term D;
- introducing a convective term f;

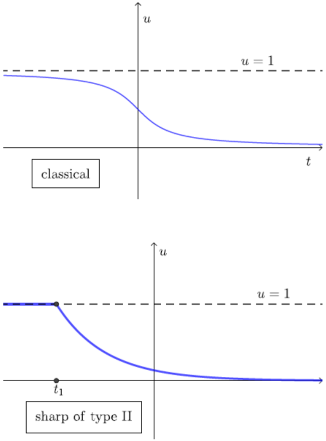
where $f \in C([0,1])$ is a generic function, $D \in C[0,1] \cap C^1(0,1)$ is positive in $(0,1)$, possibly vanishing at one or both the equilibria $0, 1$.

Definition

A function $u \in C^1(a,b)$*, with* $-\infty \leq a < b \leq +\infty$*, is said to be a decreasing t.w.s. for equation RDC if* $D(u)u' \in C^1(a,b)$*,*

- $(D(u)u')' + (c - f(u))u' + \rho(u) = 0 \quad$ *for every* $t \in (a,b)$
- $u(a^+) = 1,\ u(b^-) = 0$
- $\lim_{t \to a^+} D(u(t))u'(t) = \lim_{t \to b^-} D(u(t))u'(t) = 0$

(the last condition is trivial when $a = -\infty$ *and/or* $b = +\infty$*).*


u
u = 1
t
classical
u
u = 1
t_0
sharp of type I
u
u = 1
t_1
sharp of type II
u
u = 1
t_1
t_0
sharp of type III

In the monostable case, the existence of decreasing t.w.s. is equivalent to the solvability of the following singular boundary value problem

$$\begin{cases} \dot{z} = f(u) - c - \dfrac{D(u)\rho(u)}{z(u)} \\ z(0^+) = z(1^-) = 0 \\ z(u) < 0 \quad \text{in } (0,1). \end{cases}$$

There exists a threshold wave speed c^* such that equations RDC

admits t.w.s. with speed c if and only if $c \geq c^*$, that is the set of admissible wave speeds

$$\Gamma := \{c \ : \ \text{there exists t.w.s. with speed } c\}$$

is a halfline: $\Gamma = [c^*, +\infty)$. Moreover, the solution is unique, up to shifts.

Some known estimates for RCD equations:

- $$f(0) + 2\sqrt{(D\rho)'(0)} \leq c^* \leq \max_{u\in[0,1]} f(u) + 2\sqrt{\sup_{u\in(0,1]} \frac{D(u)\rho(u)}{u}}.$$

- $$c^* \leq \sup_{u\in(0,1]} \fint_0^u f(s)\mathrm{d}s + 2\sqrt{\sup_{u\in(0,1]} \fint_0^u \frac{D(s)\rho(s)}{s}\mathrm{d}s}.$$

where $\fint_a^b f(x)\,dx := \frac{1}{b-a}\int_a^b f(x)\,dx$.
(Malaguti et al., Math. Nachr, 2002), (Marcelli et al., El. Jou. of Qual. Th. of Diff.Eq., 2018),

I studied

$$f(v)\, v_x + g(v)\, v_\tau = (D(v) v_x)_x + \rho(v)\,,\; v(\tau, x) \in [\alpha, \beta]$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$ and the accumulation term g is a generic continuous function, not necessarily positive.

Second order ODE:

$$(D(u)u')' + (cg(u) - f(u))u' + \rho(u) = 0$$

First order singular b.v.p.:

$$\begin{cases} \dot{z} = f(u) - cg(u) - \dfrac{D(u)\rho(u)}{z(u)} \\ z(\alpha^+) = z(\beta^-) = 0 \\ z(u) < 0 \;\; \text{in } (\alpha, \beta). \end{cases}$$

If g assumes also negative sign, then the monotonicity properties of the solution z_c with respect to c is lost.

Theorem

Assume that the function $u \mapsto D(u)\rho(u)$ *is differentiable at* α. *If*

$$\inf_{u\in(\alpha,\beta)} \int_\alpha^u \frac{cg(s) - f(s)}{u-\alpha} ds > 2\sqrt{\sup_{u\in(\alpha,\beta)} \frac{1}{u-\alpha} \int_\alpha^u \frac{D(s)\,\rho(s)}{s-\alpha} \mathrm{d}s}$$

then there exists a t.w.s. having speed c. *Moreover, the t.w.s. is unique, up to shift. Instead, if*

$$cg(\alpha) - f(\alpha) < 2\sqrt{\frac{\mathrm{d}(\mathrm{D}\rho)}{\mathrm{d}u}(\alpha)}$$

then no t.w.s. exists with speed c.

(M. Cantarini et al., J. of Diff. Eq., 2022)

Theorem

Let $\Gamma := \{c \ : \ \text{there exists a t.w.s. with speed } c\}$. *Assume that the function* $u \mapsto D(u)\rho(u)$ *is differentiable at* α. *If* $g(\alpha) > 0$ *then* Γ *is nonempty and there exists a minimal wave speed* $c^* := \min \Gamma$, *satisfying*

$$c^* g(\alpha) - f(\alpha) \geq 2\sqrt{\frac{\mathrm{d}(\mathrm{D}\rho)}{\mathrm{d}u}(\alpha)}$$

and

$$\inf_{u \in (\alpha,\beta)} \int_\alpha^u \frac{c^* g(s) - f(s)}{u - \alpha} ds \leq 2\sqrt{\sup_{u \in (\alpha,\beta)} \frac{1}{u-\alpha} \int_\alpha^u \frac{D(s)\rho(s)}{s-\alpha} \mathrm{ds}}$$

Moreover, if $\int_\alpha^u g(s)\, \mathrm{ds} \geq 0$ *for every* $u \in (\alpha, \beta)$, *then* $\Gamma = [c^*, +\infty)$.

If $g(\alpha) < 0$ a "symmetric" result holds.
(M. Cantarini et al., J. of Diff. Eq., 2022)

When $g(\alpha) = 0$, if $f(\alpha) + 2\sqrt{(D\rho)'(\alpha)} > 0$, then no t.w.s. exists, for any $c \in \mathbb{R}$. An example is

$$vv_\tau = v_{xx} + v(1 - v).$$

(with $D \equiv 1$ and $f \equiv 0$).

The highly degenerate case

$$g(\alpha) = f(\alpha) + 2\sqrt{(D\rho)'(\alpha)} = 0$$

has to be investigated.

c	$D(\alpha)$	$\dot{D}(\alpha)$	$f(\alpha) - cg(\alpha)$	$u'(b^-)$	t.w.s
any	$\neq 0$	any	any	0	smooth
any	0	$+\infty$	any	0	smooth
$c > c^*$	0	$\neq 0, \neq +\infty$	any	0	smooth
$c = c^*$	0	$\neq 0, \neq +\infty$	0	0	smooth
$c = c^*$	0	$\neq 0, \neq +\infty$	$\neq 0$	< 0	sharp
$c > c^*$	0	0	$\neq 0$	0	smooth
$c = c^*$	0	0	$\neq 0$	$-\infty$	sharp

c	$D(\beta)$	$\dot{D}(\beta)$	$f(\beta) - cg(\beta)$	$u'(a^+)$	t.w.s.
any	$\neq 0$	any	any	0	smooth
any	0	$-\infty$	any	0	smooth
any	0	$\neq 0, -\infty$	≤ 0	0	smooth
any	0	$\neq 0, -\infty$	> 0	< 0	sharp
any	0	0	< 0	0	smooth
any	0	0	> 0	$-\infty$	sharp

Future works: We want to study the wavefront solution of partial differential equations of the type

$$g(u)u_\tau + f(u)u_x = (D(u)u_x)_x + \rho(u),\ \tau \geq 0,\ x \in \mathbb{R}$$

where the set

$$\{u \in [0,1] :\ g(u) \neq 0\} = \bigcup_{j=1}^{m} G_j,\ m \in \mathbb{N}$$

is made up of a finite number of disjoint intervals G_j , in each of them the function g does not vanish, and $D \in C^1(0,1)$ is a function such that it may have at most a finite number of zeros.

Let $f(n)$ be a function that counts the number of representation of n as sum of elements that belong to some fixed subsets $P_1, \ldots, P_\ell \subset \mathbb{N}$, $\ell \in \mathbb{N}$. Is $f(n)$ a positive function? Many important mathematicians have studied these type of problems over the past three centuries: from Euler to Hardy, Littlewood, Ramanujan, Landau, Erdos up to the present day with, among the many, two recent fields medals Tao and Maynard.

The case $P_1 = P_2 = \mathbb{P}$, where $\mathbb{P}$ is the set of primes, is still unsolved and out to be reached, for now.

Theorem

Let $\Lambda(n) = \begin{cases} \log(p), & n = p^m,\ m \in \mathbb{N}^+,\ p \in \mathbb{P} \\ 0, & otherwise \end{cases}$ *and* $R_3(N) := \sum_{m_1+m_2+m_3=N} \Lambda(m_1)\Lambda(m_2)\Lambda(m_3)$. *Then for every fixed* $A > 0$ *we have*

$$R_3(N) = \frac{1}{2}\mathfrak{S}_3(N)N^2 + O\left(N^2(\log N)^{-A}\right)$$

where $\mathfrak{S}_3(N) = \left(\prod_{p|N}\left(1 - \frac{1}{(p-1)^2}\right)\right)\left(\prod_{p\nmid N}\left(1 + \frac{1}{(p-1)^3}\right)\right)$.

(Vinogradov, 1937)

Consider the power series

$$S(z) := \sum_{n\geq 0} f(n) e^{-nz},\ z \in \mathbb{C},\ \mathrm{Re}(z) > 0$$

Then, fixing $\mathrm{Re}(z) = 1/N$, $N \in \mathbb{N}^{+}$ and tacking $\mathrm{Im}(z) = 2\pi i\alpha$, $\alpha \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, we have $S(z) = S(\alpha)$ and we can prove that

$$f(N) = \int_{-1/2}^{1/2} S(\alpha) e^{-2\pi i N\alpha} d\alpha$$

we have therefore transformed an arithmetic problem into a purely analytical one.

Sometimes it can be useful to study weighted averages of the counting functions. If $\mathrm{Re}(z) = 1/N$, $N \in \mathbb{N}^+$ is fixed and $\mathrm{Im}(z) = y$, $y \in \mathbb{R}$ and we know that $w(x)$ is a compact support (for semplicity $[0,1]$) and

$$w(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} W(z)\, e^{xz} dz$$

then

$$\sum_{n \leq N} f(n)\, w(N-n) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} W(z)\, S(z)\, e^{Nz} dz$$

for suitable $a > 0$. The choice $w(x) = \begin{cases} \frac{x^k}{\Gamma(k+1)}, & 0 \leq x < 1 \\ 0, & x \geq 1 \end{cases}$ $k \in \mathbb{R}^+$ leads to the so-called Cesaro-averages. (A. Languasco et al., J. Math. Anal. Appl., 2013),(A. Languasco et al., Forum Math., 2015)

Theorem (Part 1)

Let N be a sufficiently large integer, $k \in \mathbb{R}$, $k > 3/2$ and define

$$M_1(N,k) = \frac{N^{k+2}\pi}{4\Gamma(k+3)}$$

$$M_2(N,k) = \frac{N^{k+1}\pi}{4}\sum_{\rho}\frac{N^{\rho}\Gamma(\rho)}{\Gamma(k+2+\rho)} - \frac{N^{k+3/2}\sqrt{\pi}}{2}\sum_{\rho}N^{\rho/2}\frac{\Gamma(\rho/2)}{\Gamma(k+5/2+\rho/2)}$$

$$M_3(N,k) = \frac{N^{k+1/2}\sqrt{\pi}}{2}\sum_{\rho_1}\sum_{\rho_2}N^{\rho_1+\rho_2/2}\frac{\Gamma(\rho_1)\Gamma\left(\frac{\rho_2}{2}\right)}{\Gamma(k+3/2+\rho_1+\rho_2/2)} + \frac{N^{k+1}}{4}\sum_{\rho_1}\sum_{\rho_2}N^{\rho_1/2+\rho_2/2}\frac{\Gamma\left(\frac{\rho_1}{2}\right)\Gamma\left(\frac{\rho_2}{2}\right)}{\Gamma(k+2+\rho_1/2+\rho_2/2)}$$

$$M_4(N,k) = \frac{N^{k}}{4}\sum_{\rho_1}\sum_{\rho_2}\sum_{\rho_3}\frac{N^{\rho_1+\rho_2/2+\rho_3/2}\Gamma(\rho_1)\Gamma\left(\frac{\rho_2}{2}\right)\Gamma\left(\frac{\rho_3}{2}\right)}{\Gamma(k+\rho_1+\rho_2/2+\rho_3/2)}.$$

Theorem (Part 2)

Let $r_{SP}(n) := \sum_{m_1+m_2^2+m_3^2=n} \Lambda(m_1)\Lambda(m_2)\Lambda(m_3)$. *Then*

$$\sum_{n\leq N} r_{SP}(n)\frac{(N-n)^k}{\Gamma(k+1)} = M_1(N,k)+M_2(N,k) + M_3(N,k)+M_4(N,k)+O\left(N^{k+1}\right)$$

where $\rho=\beta+i\gamma$, *with or without subscripts, runs over the non-trivial zeros of the Riemann zeta function* $\zeta(s)$.

(M. Cantarini, Journal of Num. Th., 2018)

A "variant" of the circle method. Let

$$R(n;k) := \sum_{n=m_1^k+\cdots+m_{k+1}^k} \Lambda(m_1)\cdots\Lambda(m_{k+1})$$

where k is a sufficient large integer. We can prove, tacking $N \in \mathbb{N}^+$ and a suitable $H = H(N) \in \mathbb{N}^+$ that

$$\sum_{n=N}^{N+H} e^{-n/N} R(n;k) = \int_{-1/2}^{1/2} \widetilde{S}_k(\alpha)^{k+1} e^{-2\pi i N\alpha} U(-\alpha, H)\, d\alpha$$

where

$$\widetilde{S}_k(\alpha) = \sum_{n\geq 1} \Lambda(n)\, e^{-n^k(1/N-2\pi i\alpha)}, \quad U(-\alpha, H) = \sum_{m=1}^{H} e^{-2\pi i m\alpha}.$$

Theorem

If $k \geq 2$ be a fixed integer. For every $\varepsilon > 0$ there exists a constant $C = C(\varepsilon)$ such that

$$\sum_{n=N+1}^{N+H} R(n;k) = \Gamma\left(1+\frac{1}{k}\right)^k N^{1/k} H + O_k\left(HN^{1/k}\exp\left(-C\left(\frac{\log(N)}{\log(\log(N))}\right)^{1/3}\right)\right)$$

as $N \to +\infty$, uniformly for $N^{1-5/(6k)+\varepsilon} < H < N^{1-\varepsilon}$.

(M. Cantarini et al., Proc. of Am. Math. Soc., 2020)

Theorem

If $k \geq 2$ be a fixed integer and assume that the Riemann Hypothesis holds. Then

$$\sum_{n=N+1}^{N+H} R(n;k) = \Gamma\left(1+\frac{1}{k}\right)^{k} N^{1/k} H + O_k\left(\Phi(N,H)\right)$$

uniformly for $H = \infty\left(N^{1-1/k}L^{3}\right)$ with $H = o(N)$, where $f = \infty(g)$ means that $g = o(f)$, $L := \log(N)$ and

$$\Phi(N,H) := NL^{3} + H^{2}N^{1/k-1} + H^{1/2}N^{1/2+1/(2k)}L + HN^{1/(2k)}L^{3/2}.$$

(M. Cantarini et al., Proc. of Am. Math. Soc., 2020)

Future works: We want to study the connection between the general average

$$\sum_{n\leq\lambda b}\sum_{m\leq\lambda b-n} g_2(n)\, g_1(m)\, f\left(\frac{n+m}{\lambda}\right)$$

where $\lambda > 0$, $a, b \in \mathbb{R}$, $a < b$ and the convolution

$$(G_1 * G_2)(\lambda w) := \int_0^{\lambda w} G_1(\lambda w - s)\, G_2(s)\, ds$$

where

$$G_j(x) := \begin{cases} \sum_{n\leq x} g_j(n), & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

with $j = 1, 2$ (actually, we think that we can generalize the main results for $j = 1, \ldots, N$, $N \geq 2$). This approach could be used also from theory of distribution point of view.

The study of properties and characterizations of hypergeometric functions, in particular those of the type

$$ {}_{p+1}F_p\left(a_1, \ldots, a_{p+1}; b_1, \ldots, b_p; z\right), \; p \in \mathbb{N} $$

is classical and is related to names such as Euler, Gauss, Riemann, Ramanujan, Kummer, Schwarz.

I studied hypergeometric functions of the form

$$\sum_{n\geq0}\binom{2n}{n}^{k}\frac{1}{4^{nk}}\frac{p(n)}{q(n)}H_{f(n)}^{(j)}$$

where k is an integer, $p(n), q(n)$ are polynomials fulfilling suitable hypotheses and $H_{f(n)}^{(j)}$, $j, f(n) \in \mathbb{R}$ are some classes of generalized harmonic numbers.

Useful tools:

1) the Fourier-Legendre (FL) expansion: if f is a function that verifies some suitable conditions and

$$\frac{a_n}{2n+1} = \int_0^1 f(x)\, P_n(2x-1)\, dx \text{ then } \sum_{n\geq 0} a_n P_n(2x-1) = f(x)$$

where $P_n(x)$, $n \in \mathbb{N}$ are the Legendre polynomials, that is, orthogonal polynomials solutions of the Legendre's differential equation. (P. Levrie, Ramanujan J., 2010), (J.M. Campbell et al., J. Mat. An. App., 2019), (M. Cantarini et al., Bollettino U.M.I., 2019)

2) The Riemann-Liouville fractional operators

$$(D^{-1/2}f)(x) = \int_0^x \frac{f(t)}{\sqrt{\pi(x-t)}}\, dt,$$

$$(D^{1/2}f)(x) = \frac{f(x)}{\sqrt{\pi x}} + \frac{1}{2\sqrt{\pi x}} \int_0^1 \frac{f(x)-f(xy)}{(1-y)^{3/2}}\, dy.$$

(M. Cantarini et al., Int. Trans. Spec. Funct, 2022)

3) The modified Abel lemma

$$\sum_{n=1}^{\infty} B_n \nabla A_n = \left(\lim_{m \to \infty} A_m B_{m+1} \right) - A_0 B_1 + \sum_{n=1}^{\infty} A_n \overline{\Delta} B_n$$

where $A_n = A_n(x)$, $B_n = B_n(x)$ are sequences that may depend on a real parameter $x \in [0,1]$ and $\nabla \tau_n = \tau_n - \tau_{n-1}$, $\overline{\Delta} \tau_n = \tau_n - \tau_{n+1}$.
(M. Cantarini et al., Turkish J. of Math., 2022)
Some special functions: $K(x)$, $E(x)$, that is, the complete elliptic integral of the first kind and second kind, respectively; $\mathcal{J}(x)$, that is, the generalized complete elliptic integral; $Li_n(s)$, $n \in \mathbb{N}$, that is, the polylogarithm functions.

Some examples:

$$\sum_{n\geq 0}\left[\frac{1}{4^{n}}\binom{2n}{n}\right]^{3}\frac{(-1)^{n}(4n+1)}{p(n)},\ \sum_{n\geq 0}\left[\frac{1}{4^{n}}\binom{2n}{n}\right]^{4}\frac{(4n+1)}{p(n)^{2}}$$

$$\sum_{n\geq 0}\left[\frac{1}{4^{n}}\binom{2n}{n}\right]^{2}\frac{1}{(2n+1)^{3}},\ \sum_{n\geq 0}\left[\frac{1}{4^{n}}\binom{2n}{n}\right]^{2}\frac{H_{2n}}{2^{n}(n+1)}$$

$$\sum_{n\geq 0}\left[\frac{1}{4^{n}}\binom{2n}{n}\right]^{5}(-1)^{n}(4n+1)\left(\psi^{(1)}\left(n+\frac{1}{2}\right)-\psi^{(1)}(n+1)\right)$$

where $\psi^{(1)}(z)$ is the trigamma function. (Cantarini et al., Bollettino U.M.I., 2019, Cantarini Ramanujan Jou., 2021, Cantarini et al., Int. Trans. Spec. Func., 2022, Cantarini et al. Turkish J. of Math., 2022).

Future works: We want to study the link between the so-called lemniscate-like constants, that is, series of the form

$$L_1^{f_n} := \sum_{n\geq 0} \binom{2n}{n} \frac{1}{4^n} \frac{f_n}{4n+1}, \; L_2^{f_n} := \sum_{n\geq 0} \binom{2n}{n} \frac{1}{4^n} \frac{f_n}{4n+3}$$

where $f_n : \mathbb{N}_0 \to \mathbb{C}$ are suitable arithmetical functions, the theory of the Jacobian elliptic functions $\mathrm{sn}(z,k)$, $\mathrm{cn}(z,k)$, $\mathrm{dn}(z,k)$, some classes of $q-$ functions, like

$$\zeta_q(s) := \sum_{n\geq 1} \frac{1}{n^{1-s}} \frac{q^n}{1-q^n}, \; |q|<1, \; s \in \mathbb{C},$$

the p, q trigonometric functions, like $\sin_{pq}(x)$, defined as the inverse function of

$$\arcsin_{pq}(x) := \int_0^x \frac{ds}{(1-s^q)^{1/p}}, \; p, q > 1$$

and the p, q Jacobian elliptic functions, like $\mathrm{sn}_{pq}(z,k)$.

Thank you!