Uniqueness of ground states
for quasilinear elliptic equations in the exponential case

Patrizia Pucci & James Serrin

We consider ground states of the quasilinear equation

\[(1.1) \quad \text{div}(A(|Du|)Du) + f(u) = 0 \quad \text{in} \ R^n, \quad n \geq 2,\]

that is, \(C^1\) solutions of (1.1) such that

\[(1.2) \quad u \geq 0, \quad u \not\equiv 0; \quad u(x) \to 0 \quad \text{as} \ |x| \to \infty.\]

Two particular model operators \(A\) motivate this work, first

\[(1.3) \quad A(t) \equiv 1, \quad n = 2,\]

the Laplace case, and second

\[(1.4) \quad A(t) = t^{m-2}, \quad t > 0, \quad m \geq n,\]

the case of the degenerate Laplacian operator. When \(n = m\) it is well known that radially symmetric ground states can exist in the case (1.4), even for functions \(f(u)\) with exponential growth as \(u \to \infty\), that is

\[f(u) = O(e^{\beta u^{n/(n-1)}}) \quad \text{as} \ u \to \infty,\]

where \(\beta\) is an appropriate positive constant, see [1, 2, 4, 6, 7].

The question of uniqueness of ground states for (1.1) when \(f(u)\) has such exponential behavior has not previously been treated; the purpose of this paper is to make a beginning on this problem.

Appropriate assumptions on the operator \(A\) will be given in the next section. On the other hand, for the nonlinearity \(f\) we consider the specific family of functions

\[(1.5) \quad f(u) = -\alpha u + (e^u - 1 - u), \quad \alpha > 0,\]

or, more generally,

\[(1.6) \quad f(u) = u^{p-1}\{-\alpha u^p + (e^{u^p} - 1 - u^p)\}, \quad \alpha > 0, \quad \frac{1}{2} < p \leq 1,\]

\[This work is partially supported by CNR, Progetto Strategico Modelli e Metodi per la Matematica e l’Ingegneria and by the Italian Ministero della Università e della Ricerca Scientifica e Tecnologica under the auspices of Gruppo Nazionale di Analisi Funzionale e sue Applicazioni of the CNR.\]
of which (1.5) is the special case $p = 1$. For the family (1.6) it is not hard to show (see Section 2) that $f$ is in $C[0, \infty) \cap C^1(0, \infty)$ and that there exists $a > 0$ such that $f(a) = 0$ and

$$f(u) < 0 \text{ for } 0 < u < a, \quad f(u) > 0 \text{ for } a < u < \infty.$$ 

In consequence we are able to apply the principal result of [9] to obtain uniqueness of the corresponding radial ground states of (1.1). Our main conclusions will be given in the next section.

In recent work, e.g. [8, 10], the question whether all ground states of (1.1) are radially symmetric has been considered. In particular, when the operator $A$ is regular on the entire interval $[0, \infty)$, as for example for the Laplacian case (1.3), it is known for wide classes of functions $f$, including the family (1.6), that ground states are radially symmetric with respect to an appropriate origin $O \in \mathbb{R}^n$, see [10]. In such cases the a priori assumption of radial symmetry is then automatically satisfied, and we find for $n = 2$ that ground states of (1.1), (1.6) are unique up to translations, see Theorem 2.

§2. The main result.

We consider the quasilinear equation (1.1), in which the operator $A$ is of class $C^1(0, \infty)$. Let

$$\Omega(t) = tA(t), \quad G(t) = \int_0^t \Omega(s)ds, \quad t > 0,$$

and suppose that the following specific conditions are satisfied:

1. $\Omega'(t) > 0$ for $t > 0$, $\Omega(t) \to 0$ as $t \to 0$,
2. $t\Omega(t)/G(t)$ is (non–strictly) increasing in $(0, \infty)$;

see [9, Section 1].

We define the critical constant

$$m = \lim_{t \to 0} \frac{t\Omega(t)}{G(t)}.$$ 

It is clear that $m \geq 1$, since $G(t) < t\Omega(t)$ for $t > 0$ by (1).

The operator $A$ in (1.4) obviously satisfies conditions (1), (2), where the exponent $m$ there coincides with the constant $m$ given in (2.1).

**Theorem 1.** Suppose that the operator $A$ obeys conditions (1), (2), and let the critical constant $m$ satisfy

$$m \geq n.$$ 

Assume that $f$ in (1.1) is of the form (1.6). Then radially symmetric ground states of (1.1) are unique up to translations.
Proof. By [9, Theorem 1 and Lemma 2.1] it is enough to prove for the family (1.6) that

\( \frac{d}{du} \left[ \frac{F(u)}{f(u)} \right] \geq 0 \quad \text{for } u > 0, \quad u \neq a, \)

where

\[ F(u) = \int_0^u f(s) ds = \frac{1}{p} \left\{ e^{u^p} - 1 - u^p - \frac{(\alpha + 1)}{2} u^{2p} \right\} \]

and \( a \) is the unique positive zero of \( f \).

In fact, to see that \( f \) has exactly one positive zero \( a \), we note that, in the new variable \( v = u^p \),

\[ f = f(u(v)) = v^{(p-1)/p}[e^v - 1 - (\alpha + 1)v]. \]

It is thus enough to show that \( e^v - 1 - (\alpha + 1)v \) has only one zero in \((0, \infty)\), which follows by trivial calculus. Indeed one finds \( a > \log(\alpha + 1)^{1/p} \).

Since

\[ \frac{F'}{F} = \frac{f^2 - f'F}{f^2}, \quad \frac{d}{du}, \]

to show (2.2) it is enough to verify that

\( f^2 - f'F \geq 0 \quad \text{for } u > 0. \)

Again in the \( v \) variable, we find

\[ f^2 - f'F = \frac{1}{p} v^{(p-2)/p}[(1 - p)(e^v - 1)^2 + v(e^v - 1)(A + C v^2) + v^2(D + E v)], \]

where

\[ A = (2p - 1)(\alpha + 1) - 1, \quad B = -\frac{1}{2}(3p + 1)\alpha - \frac{1}{2}(p + 1), \]
\[ C = \frac{1}{2}p(\alpha + 1), \quad D = (1 - 2p)\alpha + 1 - p, \quad E = \frac{1}{2}(\alpha + 1)(\alpha + p + 1). \]

Let \( \varphi \) be the function in brackets in the previous formula, namely

\[ \varphi(v, \alpha; p) = (1 - p)(e^v - 1)^2 + v(e^v - 1)(A + C v^2) + v^2(D + E v). \]

The required condition (2.3) then follows if we show that

\( \varphi(v, \alpha; p) > 0 \quad \text{for } v, \alpha \in (0, \infty) \text{ and } p \in [\frac{1}{2}, 1]. \)

Note moreover that \( \varphi(v, \alpha; p) \) is linear in \( p \), so that it is enough to prove (2.4) at the endpoints \( p = 1/2 \) and \( p = 1 \).
Case 1. $p = 1/2$. Here

$$\varphi(v, \alpha; 1/2) = \frac{1}{2}(e^v - 1)^2 + v(e^v - 1)\{(A + Bv + Cv^2) + v^2(D + Ev^2)\},$$

(2.5)

$$A = -1, \quad B = -\frac{1}{4}(5\alpha + 3), \quad C = \frac{1}{4}(\alpha + 1),$$

$$D = \frac{1}{2}, \quad E = \frac{1}{4}(2\alpha^2 + 5\alpha + 3).$$

We first show

$$\varphi(v, \alpha; 1/2) > 0 \quad \text{for} \quad v \geq 5; \quad \varphi(0, \alpha; 1/2) = 0 \quad \text{for} \quad \alpha > 0.$$  

Indeed

$$A + Bv + Cv^2 = \begin{cases} -1 & \text{when} \quad v = 0 \\ 3/2 & \text{when} \quad v = 5 \end{cases}.$$  

Therefore, since $A + Bv + Cv^2$ is quadratic in $v$, with $C > 0$, we get $A + Bv + Cv^2 \geq \frac{3}{2}$ for $v \geq 5$, $\alpha > 0$. Now (2.6) follows from (2.5).

Next, observe that

$$\frac{1}{2}(e^v - 1)^2 + v(e^v - 1)A + Dv^2 = \frac{1}{2}(e^v - 1 - v)^2 > 0.$$  

Therefore, since $B$ is linear in $\alpha$ and $E$ quadratic, and also $e^v - 1 \leq \frac{1}{5}(e^5 - 1)v$ for $v \in [0, 5]$, it is clear from (2.5) that there is $\alpha_1 > 0$ such that

$$\varphi(v, \alpha; 1/2) > 0 \quad \text{if} \quad 0 < v \leq 5 \quad \text{and} \quad \alpha \geq \alpha_1.$$  

Also, by a straightforward calculation we obtain

$$\varphi(v, 0; 1/2) = \frac{1}{8}v^5 + \left(\frac{e^v - 1 - v - v^2/2}{2}\right) \cdot \left\{ -\frac{1}{4}v^2 + \frac{1}{3}v^3 + \frac{1}{2}(e^v - 1 - v - v^2/2 - v^3/6) \right\}.$$

Therefore

(2.8) \quad $\varphi(v, 0; 1/2) > 0$ \quad \text{for all} \quad v > 0.$$

From (2.6)–(2.8) it follows that if $\varphi(v, \alpha; 1/2)$ takes negative values in $(0, \infty) \times (0, \infty)$, then it attains a negative minimum at some point $(v_0, \alpha_0)$, with $0 < v_0 < 5$ and $0 < \alpha_0 < \alpha_1$. Moreover, by direct calculation we find

$$\frac{\partial \varphi}{\partial v}(v, \alpha; 1/2) = (e^v - 1)^2 + v(e^v - 1)\{(A + 2B) + (B + 3C)v + Cv^2\} + (B + 3E)v^2 + Cv^3.$$  

(2.9)

$$\frac{\partial \varphi}{\partial \alpha}(v, \alpha; 1/2) = \frac{1}{4}v^2\{(e^v - 1)(v - 5) + (4\alpha + 5)v\}.$$  

(2.10)
We claim that at the minimum point \((v_0, \alpha_0)\) there holds

\[
\varphi(v_0, \alpha_0; 1/2) = \frac{v_0^2}{4(5-v_0)^2} \left[ 32\alpha_0^2 - 2\alpha_0(25\alpha_0 + 22)v_0 
+ 4(5\alpha_0^2 + 8\alpha_0 + 3)v_0^2 - 2(\alpha_0 + 1)^2v_0^3 \right].
\]

Indeed, since \((v_0, \alpha_0)\) is a critical point, by (2.10) and the fact that \(0 < v_0 < 5\) there follows

\[
e^{v_0} - 1 = \frac{4\alpha_0 + 5}{5-v_0} v_0.
\]

The claim (2.11) now arises by eliminating \(e^{v_0} - 1\) from the main formula (2.5).

By (2.9), we also have at \((v_0, \alpha_0)\)

\[
(e^{v_0} - 1)^2 = -v_0(e^{v_0} - 1)\{(A + 2B) + (B + 3C)v_0 + Cv_0^3\} - (B + 3E)v_0^2 - Cv_0^3.
\]

Eliminating \((e^{v_0} - 1)^2\) from the main formula (2.5) and then using (2.12) once more, we get

\[
\varphi(v_0, \alpha_0; 1/2) = \frac{v_0^2}{4} \left[ (70\alpha_0^2 + 108\alpha_0 + 75) - (28\alpha_0^2 + 44\alpha_0 + 23)v_0 
+ (2\alpha_0^2 + 3\alpha_0 + 1)v_0^2 \right].
\]

Define

\[
\psi(v, \alpha) = 70\alpha_0^2 + 108\alpha_0 + 75 - (28\alpha_0^2 + 44\alpha_0 + 23) v + (2\alpha_0^2 + 3\alpha_0 + 1) v^2.
\]

Clearly

\[
\psi(v, 0) = 75 - 23v + v^2 > 0 \quad \text{for } v \in [0, 3.9],
\]

\[
\frac{\partial \psi}{\partial \alpha}(v, 0) = 108 - 44v + 3v^2 > 0 \quad \text{for } v \in [0, 3.1],
\]

\[
\frac{\partial^2 \psi}{\partial \alpha^2}(v, \alpha) = 4[35 - 14v + v^2] > 0 \quad \text{for } v \in [0, 3.2] \text{ and all } \alpha > 0.
\]

Consequently, by integration with respect to \(\alpha\) from 0 to any \(\alpha > 0\), we get \(\psi(v, \alpha) > 0\) on \([0, 3] \times [0, \infty)\). But \(\psi(v_0, \alpha_0) = 2v_0^{-2}\varphi(v_0, \alpha_0; 1/2) < 0\), so that

\[
3 < v_0 < 5.
\]

Next define

\[
\omega(v, \alpha) = 32\alpha_0^2 - 2\alpha(25\alpha + 22)v + 4(5\alpha_0^2 + 8\alpha_0 + 3)v^2 - 2(\alpha_0 + 1)^2 v^3.
\]
Then
\[
\omega(v, 0) = 2v^2(6 - v) > 0 \quad \text{for } v \in (0, 6),
\]
\[
\frac{\partial \omega}{\partial \alpha}(v, 0) = -4v(11 - 8v + v^2) > 0 \quad \text{for } v \in [1.8, 6.2],
\]
\[
\frac{\partial^2 \omega}{\partial \alpha^2}(v, \alpha) = 4(1 - v)(v^2 - 9v + 16) > 0 \quad \text{for } v \in [2.5, 6.5] \text{ and all } \alpha > 0.
\]

Hence, by integration with respect to \( \alpha \) from 0 to \( \alpha > 0 \), we get \( \omega(v, \alpha) > 0 \) on \([2.5, 6.5] \times (0, \infty)\). Since \( \omega(v_0, \alpha_0) = 4(5 - v_0^2)\varphi(v_0, \alpha_0; 1/2) < 0 \) by (2.11) and since \( v_0 < 5 \), this implies
\[
0 < v_0 < 2.5,
\]
which contradicts (2.14). Therefore \( \varphi(v, \alpha; 1/2) \) cannot take negative values. This completes the proof of Case 1.

**Case 2.** \( p = 1 \). Here \( v = u \) and
\[
\varphi(v, \alpha; 1) = v(e^v - 1)\{A + Bv + Cv^2\} + v^2\{D + Ev\},
\]
(2.15)
\[
A = \alpha, \quad B = -2\alpha + 1, \quad C = \frac{1}{2}(\alpha + 1),
\]
\[
D = -\alpha, \quad E = \frac{1}{2}(\alpha + 1)(\alpha + 2).
\]

By Taylor’s expansion and use of the formula (2.15) for the coefficients, \( A, \ldots, E \), one finds
\[
\varphi(v, \alpha; 1) = \frac{1}{2}v^3 - \frac{1}{3}v^4 + \sum_{k=2}^{\infty}v^{k+3}\left\{-\frac{\alpha}{(k+2)!} - \frac{2\alpha + 1}{(k+1)!} + \frac{\alpha + 1}{2k!}\right\}
\]
\[
= \frac{1}{2}v^3\left\{\alpha^2 - \frac{2}{3}\alpha + \sum_{k=2}^{\infty}\frac{v^k}{(k+2)!}(k^2 - k - 4)\alpha + k^2 + k - 2\right\}.
\]

For all \( k \geq 3 \) the coefficients \( (k^2 - k - 4)\alpha + k^2 + k - 2 \) are non-negative. Hence, dropping all terms with \( k \geq 4 \), we obtain
\[
\varphi(v, \alpha; 1) \geq \frac{1}{2}v^3\left\{\alpha^2 - \frac{2}{3}\alpha v + \frac{1}{12}(2 - \alpha)v^2 + \frac{1}{60}(\alpha + 5)v^3\right\}.
\]
(2.16)

By (2.16), to prove the assertion it is enough to show that
\[
60\alpha^2 - 40\alpha v + 5(2 - \alpha)v^2 + (\alpha + 5)v^3 > 0 \quad \text{on } (0, \infty) \times (0, \infty).
\]
(2.17)

By Cauchy’s inequality \( 40\alpha v \leq 20\alpha^2 + 20v^2 \). We are thus led to consider the function
\[
\psi(v, \alpha) = 40\alpha^2 - 5(\alpha + 2)v^2 + (\alpha + 5)v^3 \quad v > 0, \quad \alpha > 0.
\]
(2.18)
For fixed $\alpha$ the minimum of $\psi(\cdot, \alpha)$ is attained at

$$v_{\alpha} = \frac{10 \alpha + 2}{3 \alpha + 5}.$$  

Hence

$$\psi(v, \alpha) \geq \psi(v_{\alpha}, \alpha) = \frac{20}{27} \left( \frac{1}{\alpha + 5} \right)^2 \{54\alpha^2(\alpha + 5)^2 - 25(\alpha + 2)^3\}.$$  

The expression in braces is the quartic function

$$Q(\alpha) = 54\alpha^4 + 515\alpha^3 + 1200\alpha^2 - 300\alpha - 200.$$  

Observe that $Q(2/3) > 0$, $Q'(2/3) > 0$ and $Q''(\alpha) > 0$ on $(0, \infty)$. Consequently, $Q(\alpha) > 0$ for all $\alpha \geq 2/3$. It is now evident that $\psi(v, \cdot) > 0$ for $\alpha \geq 2/3$, and so (2.17) holds in the subset $(0, \infty) \times [2/3, \infty)$.

On the other hand,

$$60\alpha^2 - 40\alpha v + 5(2 - \alpha)v^2 + (\alpha + 5)v^3 \geq 5\{12\alpha^2 - 8\alpha v + (2 - \alpha)v^2\}.$$  

The quadratic function on the right side has minimum value

$$\frac{20\alpha^2}{2 - \alpha}(2 - 3\alpha) > 0$$  

for $\alpha < 2/3$. Hence (2.17) holds on the entire set $(0, \infty) \times (0, \infty)$, as required.

This completes the proof of Case 2, and in turn of the theorem.

It is surprising that the proof of Case 1 is so tricky. On the other hand, the function $\varphi(\cdot, \cdot; 1/2)$ in (2.5) is very close to zero when $(v, \alpha)$ is in $[0, 2]$. Indeed

$$\varphi(0.01, 0.01; 1/2) = 21 \times 10^{-12}, \quad \varphi(0.1, 0.1; 1/2) = 22 \times 10^{-7},$$  

$$\varphi(1, 1/2; 1/2) = 0.039, \quad \varphi(1/2, 1; 1/2) = 0.040, \quad \varphi(1, 1; 1/2) = 0.181.$$  

Instead of (1.6), another natural choice for the nonlinearity $f(u)$ in (1.1) would be

$$f(u) = -\alpha u + (e^{u^p} - 1 - u^p), \quad \alpha > 0, \quad \frac{1}{2} < p \leq 1.$$  

For this family, however, the verification of (2.2) does not seem to be obvious.

Since in Theorem 1 we treat radially symmetric ground states $u = u(r)$ of (1.1), it is clear that the dimension $n$ may be taken as any real number greater than 1. In fact, it is precisely this point of view which was used in Theorem 1 of [9]. With this interpretation the condition $m \geq n$ in Theorem 1 then allows values $m < 2$ when $n < 2$.  

When the operator $A$ is smooth on the entire interval $[0, \infty)$, the result of Theorem 1 can be improved. In particular we have the following

**Theorem 2.** Let $n = 2$. Suppose that $A$ is of class $C^{1,1}[0, \infty)$, that $\Omega'(t) > 0$ for $t \geq 0$, and that condition (2) is satisfied. Also let $f$ in (1.1) have the form (1.6). Then ground states of (1.1) having connected support are radially symmetric, and unique up to translations.

When $p = 1$ in (1.6), the condition of connected support can be omitted.

**Proof.** First observe that $A$ satisfies conditions (1), (2). Moreover $\Omega(t) = t\Omega'(0) + o(t)$ as $t \to 0$, with $\Omega'(0) > 0$. Hence from the definition (2.1) and use of condition (2) we get $m = 2$.

The family (1.6) has the property that $f'(u) < 0$ for $0 < u < \delta$ and some $\delta > 0$; indeed, it is not hard to see that (using the variable $v = u^p$)

$$f'(u) \leq v^2(p-1/p)e^v - 1 - \frac{2p-1}{p} \alpha < 0$$

for $v \in (0, \eta)$, $\eta = \log(1 + \frac{2p-1}{p} \alpha) > 0$ since $p > 1/2$. Thus we can take $\delta = \eta^{1/p}$. Hence $f$ is nonincreasing for $u$ near zero (note also that $f'(0) = -\infty$ when $1/2 < p < 1$), and by Theorem 1 of [10] any corresponding ground state $u$ of (1.1) having connected support is radially symmetric about some origin $O = O(u) \in \mathbb{R}^n$. The first part of Theorem 2 now follows from Theorem 1, since for $n = 2$ the required condition $m \geq n$ is fulfilled.

To obtain the second part of the theorem, note that when $p = 1$ the function $f$ in (1.6), see also (1.5), is uniformly Lipschitz continuous on $[0, \delta]$ for any $\delta > 0$. Hence by Corollary 1.1 of [10] any ground state is radially symmetric and the result follows from Theorem 1 above.

Similar results can be proved for singular operators $A$ such as the $m$–Laplacian. We omit the details since the required conditions are somewhat complicated to state (see in particular [5, 10]).

**References**


