Stability of solutions for some classes of nonlinear damped wave equations

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Abstract
We consider two classes of semilinear wave equations with nonnegative damping which may be of type "on-off" or integrally positive. In both cases we give a sufficient condition for the asymptotic stability of the solutions. In the case of integrally positive damping we show that such a condition is also necessary.

Keywords: damped nonlinear wave equations, integrally positive damping, on-off damping.
2000AMS Subject Classification: 35L70, 93D20, 25B35.

1 Introduction
We are concerned with some classes of nonlinear abstract damped wave equations, whose prototype is the usual wave equation in a bounded domain $\Omega \subset \mathbb{R}^N, N \geq 1$,

\begin{align}
\begin{cases}
  u_{tt} = \Delta u - h(t)u_t + f(u) & \text{in } (0, +\infty) \times \Omega, \\
  u(t, x) = 0 & \text{in } (0, +\infty) \times \partial \Omega, \\
  u(0, x) = u_0(x), \; u_t(0, x) = u_1(x) & x \in \Omega,
\end{cases}
\end{align}

(1.1)

*Research supported by the M.I.U.R. project Metodi Variazionali ed Equazioni Differenziali Nonlineari.
though we can handle equations in a more general Banach setting like $u'' + B(t)u' + Au = f(u)$, with $B$ and $A$ suitably given (see Section 2 for the precise setting).

This problem has been already investigated by many authors in the case of ordinary differential equations or systems of ordinary differential equations (i.e. when $u$ depends only on $t$) when $f$ is linear (for example see [5], [8]) and also when $f$ is nonlinear ([16], [18], [19]).

In the case of hyperbolic partial differential equations like (1.1), the problem has been studied when $f$ is linear ([4], [12], . . . ), when $f$ is nonlinear but with linear growth ([22], . . . ), and when $f$ is nonlinear with superlinear growth ([14] in the case of constant damping, [17] for more general cases).

Concerning the damping $h$, different assumptions are alternatively made: on-off ([7]), increasing ([1]), bounded - in many senses - ([4], [20]), integrally positive ([22]), etc. . . . In particular, on-off dampers are suitable to describe a wide variety of communication network models (circuits which can be switched on or off), as well as systems where a control depending on time is necessary.

Very interesting results in the special case $f \equiv 0$ and damping of type on-off can be found in [7], also when the term $a(t)u_t$ in (1.1) is replaced by $a(t)g(u_t)$, where $g$ is a nonlinear function with linear growth (see also [13]). For the case $f \equiv 0$ we also mention a logarithmic decay estimate proved in [3] when the term $a(t)u_t$ in (1.1) is replaced by $(1 + t)g(u_t)$, with $a$ bounded and strictly positive on a subdomain of $\Omega$ and $g$ possibly having superlinear growth at infinity.

In [22] the author shows that, if $f$ has linear growth and $h$ is integrally positive (see Definition 1 below), then any solution $u(u_0, u_1)$ of (2.5) converges to 0 in the norm $\| \nabla u \|_{L^2} + \| u_t \|_{L^2}$ if and only if

$$
\int_0^\infty e^{-H(t)} \int_0^t e^{H(s)} \, ds \, dt = +\infty,
$$

(1.2)

where $H(t) = \int_0^t h(s) \, ds$. Actually the proof contains some gaps, in particular in proving that the $L^2$-norm of $u_t$ converges to 0 as $t \to \infty$. However we re-cover that result in Section 3.

In this paper we show that when $h$ is integrally positive, condition (1.2) is sufficient to prove global stability for problem (1.1), also when $f$ is superlinear and satisfies a sign condition. More precisely, in Theorem 3.1 we prove that condition (1.2) is sufficient under the assumption

$$
u f(u) \leq 0 \quad \text{for every } u \in \mathbb{R},$$

which was already assumed, for example, in [17] and in [20]. Note that such a condition is trivially verified when $f(u) = -|u|^{p-1}u$, $p \geq 1$, which is our prototype, and which was studied, for example, in [14], where a global existence result is proved for $h =$constant. Under a natural non supercritical growth condition on $f$, we also show that (1.2) is also a necessary condition for global stability to hold (see Theorem 3.2). We remark that the requirement $uf(u) \leq 0$ is a bit stronger that $sf(s) < \lambda_1 s^2$, $s \neq 0$, which is essentially assumed in [22],
and also in [9] when the damping is concentrated in a subset of $\omega$ and Neumann–
type conditions are assumed on the boundary – as usual, here $\lambda_1$ denotes the
first eigenvalue of $-\Delta$ on $H^1_0(\Omega)$.

For damping of type on–off, in [7] the following case is considered: let
$(a_n, b_n)_n$ be a sequence of open disjoint intervals of $(0, \infty)$ such that $a_n \to \infty$ and
suppose there exists $M_n \geq m_n > 0$ such that

$$m_n \leq h(t) \leq M_n \quad \forall \ t \in (a_n, b_n). \quad (1.3)$$

If $f \equiv 0$, then any solution $u(u_0, u_1)$ of (2.5) converges to 0 in the norm $\|\nabla u\|_{L^2} + \|u_t\|_{L^2}$ if

$$\sum_{n=1}^{\infty} m_n (b_n - a_n) \min \left( (b_n - a_n)^2, \frac{1}{1 + m_n M_n} \right) = \infty. \quad (1.4)$$

In this result the fact that $f \equiv 0$ is essential in the proof of stability. In the
nonlinear case under consideration, with the assumption

$$sf(s) - \int_0^s f(\sigma) d\sigma \leq 0 \quad \forall \ s \in \mathbb{R},$$

we show that (1.4), which was essentially already introduced in [16] for sys-
tems of ordinary differential equations, is again sufficient for the stability (see
Theorem 4.2). As for the case $f \equiv 0$ in [7], we still don’t know if (1.4) is also
necessary for stability to hold.

However, in the case of integrally positive damping, we give a complete
characterization of stability for signed non supercritical nonlinearities.

Acknowledgments

This work was started while the authors were visiting Prof. László Hatvani at
the Bolyay Institute of the University of Szeged. The first author acknowledges
financial support from GNAMPA. The second author acknowledges financial
support from CNR, in the framework of the bilateral project CNR–MTA 132.07.
The authors wish to thank Prof. Hatvani for some interesting discussions on
this subject and for bringing [22] to their knowledge.

2 The abstract setting

We will use an abstract setting which is a bit less general than the one in [7],
but more natural for our purposes. Let us consider a second order evolution
problem of the form

$$\begin{cases} 
    u'' + B(t)u' + Au = f(u) & t > 0 \\
    u(0) = u_0 \in V, \ u'(0) = u_1 \in H.
\end{cases} \quad (2.5)$$

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Here $H$ denotes a real Hilbert space with scalar product $(\cdot, \cdot)_H$ and norm $\|\cdot\|_H$, $A : D(A) \subset H \to H$ is a linear self-adjoint coercive operator on $H$ with dense domain, and $V = D(A^{1/2})$ with norm $\|v\|_V = \|A^{1/2}v\|_H$ is such that
$$V \hookrightarrow H \equiv H' \hookrightarrow V'$$ with dense embeddings, and
$$\exists \lambda_1 > 0 : \|v\|^2_H \leq \frac{1}{\lambda_1} \|v\|^2_V = \frac{1}{\lambda_1} \|A^{1/2}v\|^2_H \quad \text{for any } v \in V. \quad (2.6)$$

Concerning the time-dependent operator $B$, in Section 3 we assume it is actually a nonnegative function which can be 0 in a set of measure 0 (see Definition 1 below), while for the results of Section 4 we let it be a more particular "positive" nonlinear operator (see below for the precise assumptions), for which we assume that $B \in L^\infty(0, \infty; \text{Lip}(H, H'))$.

Finally, on the nonlinearity $f$ we assume alternatively
$$sf(s) \leq 0 \quad \forall s \in \mathbb{R}, \quad (2.7)$$
which implies
$$F(s) := \int_0^s f(\sigma) \, d\sigma \leq 0 \quad \forall s \in \mathbb{R},$$
or
$$sf(s) - F(s) \leq 0 \quad \forall s \in \mathbb{R}. \quad (2.8)$$

**Remark 2.1.** In both cases, $f(u) = -|u|^{p-1}u, \ p \geq 1$, is the prototype function.

We also remark that the sign assumptions on $f$ look quite reasonable and hard to relax. Indeed, it is well known that solutions of $u_{tt} + a(x, t)u_t - \Delta u = |u|^{p-1}u$ in $\Omega$, $a(x, t) \geq 0$ and $p > 1$, might blow up in finite time (see, for example, [10] or [11]).

By solution of (2.5), we mean a function $u$ such that for any $T > 0$ there holds $u \in L^2(0, T; V) \cap H^1(0, T; H) \cap H^2(0, T; V')$ with $(Bu', u')_H \in L^2(0, T)$ and
$$Au \in L^2(0, T; V'), \quad Bu' \in L^2(0, T; V'), \quad f(u) \in L^2(0, T; H),$$
with $u(0) = u_0, \ u'(0) = u_1$, and such that
$$u'' + Bu' + Au = f(u) \quad \text{in } L^2(0, T; V').$$

**Remark 2.2.** In the case of problem (1.1) the condition $f(u) \in L^2(0, T; H)$ is automatically satisfied when $H = L^2(\Omega), \ V = H^1_0(\Omega)$ and $f(u) = -|u|^{p-1}u, \ p \geq 1$ if $N = 1, 2$ or $1 \leq p \leq N/(N-2)$ if $N \geq 3$.

For any solution $u$ of problem (2.5) we denote by $E_u$, or simply by $E$ if there is no need to specify $u$, the energy associated to such a solution:
$$E(t) = \frac{1}{2} \|u'(t)\|_H^2 + \frac{1}{2} \|u(t)\|_V^2 \ dx - \mathcal{F}(u(t)), \quad (2.9)$$
where $\mathcal{F}(u)$ is the real–valued functional such that $\mathcal{F}(0) = 0$ and $\mathcal{F}'(u)(\phi) = \langle f(u), \phi \rangle_{V', V}$. Of course in the case of problem (1.1) we have $H = L^2(\Omega)$, $V = H^1_0(\Omega)$ and

$$
\mathcal{F}(u) = \int_{\Omega} F(u) \, dx.
$$

The following result, proved in [7] when $f \equiv 0$, still holds true in the non-linear case thanks to the assumption $f(u) \in L^2(0, T; H)$. The proof is an adaptation to the one given therein and is thus omitted.

**Lemma 2.1.** For any solution $u$ of (2.5) we have

- $u \in C([0, T]; V) \cap C^1([0, T]; H)$;
- the associated energy $E_u$ is locally absolutely continuous on $[0, \infty)$ and
  
  $$
  E'_u(t) = -\langle B(t)u'(t), u'(t) \rangle_H \quad \text{a.e. in } [0, \infty). 
  $$

In our setting we will also need the following obvious corollary.

**Lemma 2.2.** If in addition $\langle B(t)w, w' \rangle_H \geq 0$ for a.e. $t \geq 0$ and for every $w \in H$, then for any solution $u$ of (2.5) we have that $E_u$ is non increasing.

**Remark 2.3.** In the case of problem (1.1) equality (2.10) reads

$$
E'(t) = -\int_{\Omega} h(t)u^2_t \, dx,
$$

which is non positive if $h \geq 0$ a.e. in $[0, \infty)$.

## 3 The integrally positive case

Let us start with the following definitions.

**Definition 1.** A function $h : [0, +\infty) \rightarrow [0, +\infty)$ is said *integrally positive* if for every $a > 0$ there exists $\delta > 0$ such that

$$
\int_{t}^{t+a} h(s) \, ds \geq \delta \quad \forall t > 0.
$$

**Remark 3.1.** We underline the fact that according to this definition, the function $h$ may vanish somewhere, but not on any interval.

**Definition 2.** Solutions of (2.5) are said *uniformly bounded* in $D(A) \times D(A^{1/2})$ if for any $B_1 > 0$ there exists $B_2 > 0$ such that

if $(u_0, u_1) \in D(A) \times D(A^{1/2})$ and $\|Au_0\|_H + \|A^{1/2}u_1\|_H \leq B_1$,

and if $u(t, u_0, u_1)$ denotes the solution of (2.5) with initial condition $(u_0, u_1)$, then $\forall t > 0$
• $f(u(t, u_0, u_1)) \in H$ and
• $\|Au(t, u_0, u_1)\|_H + \|A^{1/2}u(t, u_0, u_1)\|_H + \|f(u(t, u_0, u_1))\|_H \leq B_2.$

Let us remark that such a definition is a natural modification of the one introduced in [22], due to the presence of the requirement on $\|f(u)\|_H$.

For the following result we concentrate on (1.1), where $A = -\Delta$, $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$, $D(A^{1/2}) = H^1_0(\Omega)$ and $H = L^2(\Omega)$, and we recall that $\|\Delta u\|_2$ is a norm on $H^2(\Omega) \cap H^1_0(\Omega)$ which is equivalent to the usual one (for example, see [15]), where we have set $\| \cdot \|_2 = \| \cdot \|_{L^2(\Omega)}$.

Having in mind the prototype $f(s) = -|s|^{p-1}s$, $p \geq 1$, we also make the following natural assumption:

if $u$ solves (2.5), $\forall T > 0 \exists C = C(u, T) > 0$ and $q = q(u, T) > 0$ such that

$$\left| \int_{\Omega} F(u(t)) \, dx \right| \leq C\|A^{1/2}u(t)\|_2^2 \quad \forall t \geq T.$$  

Of course, in the model case $f(s) = -|s|^{p-1}s$, $p \geq 1$ and $p \leq N/(N-2)$ if $N \geq 3$ described in Remark 2.2, we have also $p+1 \leq (2N-2)/(N-2)$ if $N \geq 3$; thus the Sobolev inequality can be applied, and we can always take $q = p+1$ and $C$ depending only on the measure of $\Omega$ and the Sobolev constant.

Now we can state our first fundamental result.

**Theorem 3.1.** Assume (1.2), (2.7) and (3.11). If $h$ is integrally positive and solutions of (1.1) are uniformly bounded in $H^1(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$, then every solution $u$ of (1.1) satisfies

$$\|A^{1/2}u(t)\|_2 + \|u'(t)\|_2 \to 0 \quad \text{as} \quad t \to +\infty.$$  

**Proof.** Take $(u_0, u_1) \in D(A) \times D(A^{1/2})$ and denote by $u$ the associated solution of (1.1). By Remark (2.3) there exists $E_\infty \geq 0$ such that

$$\lim_{t \to +\infty} E(t) = \lim_{t \to +\infty} \left( \frac{1}{2} \|u(t)\|_2^2 + \frac{1}{2} \|A^{1/2}u(t)\|_2^2 - \int_{\Omega} F(u) \, dx \right) = E_\infty.$$  

We want to show that $E_\infty = 0$, so let us assume by contradiction that $E_\infty > 0$. By (2.7) $F(s) = \int_0^s f(\tau) \, d\tau \leq 0$ for any $s \in \mathbb{R}$, so that by (3.13) there exists $L \in [0, 2E_\infty]$ such that

$$\lim_{t \to +\infty} \|u'(t)\|_2^2 = L.$$  

Assume by contradiction that $L > 0$. We must distinguish several cases. 

**First case:** $\|u'(t)\|_2^2 \equiv L \forall t > 0$. Then, by (2.10) and Remark 2.3 we get

$$0 < E_\infty = E(0) + \int_0^\infty E'(\tau) \, d\tau = E(0) - \int_0^\infty h(\tau) \|u'(\tau)\|_2^2 \, d\tau$$

$$= E(0) - L \int_0^\infty h(\tau) \, d\tau.$$  

(3.15)
Since $h$ is integrally positive, there exists $\delta > 0$ such that
\[
\int_{n}^{n+1} h(\tau) \, d\tau \geq \delta \quad \forall n \in \mathbb{N}.
\] (3.16)

Therefore, (3.15) and (3.16) imply
\[
0 < E(0) - L \sum_{n=1}^{\infty} \delta = -\infty,
\]
and a contradiction arises.

**Second case:** $\|u'(t)\|_{2}^2 \neq L$. Set $\liminf_{t \to +\infty} \|u'(t)\|_{2}^2 = \ell \in [0, L]$, and assume first that $\ell < L$. Then, since $u \in C^1([0, T]; H)$ by Lemma 2.1, there exist two sequences $(s_n)_{n}$ and $(t_n)_{n}$ such that

1. $0 < s_n < t_n < s_{n+1} \forall n \in \mathbb{N}$;
2. $s_n \to +\infty$ as $n \to \infty$;
3. $\frac{L + \ell}{2} = \|u'(s_n)\|_{2}^2 < \|u'(t_n)\|_{2}^2 = \frac{3L + \ell}{4} \quad \forall n \in \mathbb{N}$;
4. $\frac{L + \ell}{2} \leq \|u'(t)\|_{2}^2 \leq \frac{3L + \ell}{4} \forall t \in (s_n, t_n)$.

Since solutions of (1.1) are uniformly bounded, there exists $M > 0$, depending on $\|Au\|_{2}$ and $\|A^{1/2}u_1\|_{2}$, such that
\[
\frac{d}{dt}\|u'(t)\|_{2}^2 = 2\langle u'(t), u''(t) \rangle
\]
\[
= -2\langle u'(t), Au(t) \rangle - 2h(t)\|u'(t)\|_{2}^2 + 2\langle u'(t), f(u) \rangle
\]
\[
\leq 2\|u'(t)\|_{2}\|Au(t)\|_{2} + 2\|u'(t)\|_{2}\|f(u)\|_{2} \leq M.
\]

Therefore
\[
\frac{L - \ell}{4} = \frac{3L + \ell}{4} - \frac{L + \ell}{2} = \int_{s_n}^{t_n} \frac{d}{dt}\|u'(t)\|_{2}^2 \, dt \leq M(t_n - s_n),
\]
so that
\[
t_n - s_n \geq \frac{L - \ell}{4M} \quad \forall n \in \mathbb{N}.
\] (3.17)

In this way, by (2.10) and (3.17) we get
\[
0 < E_{\infty} = E(0) + \int_{0}^{\infty} E'(\tau) \, d\tau \leq E(0) - \int_{\cup_n(s_n, t_n)} h(\tau)\|u'(\tau)\|_{2}^2 \, d\tau
\]
\[
\leq E(0) - \frac{L + \ell}{2} \int_{\cup_n(s_n, s_n + \frac{L - \ell}{4M})} h(\tau) \, d\tau.
\] (3.18)
Since $h$ is integrally positive, there exists $\delta > 0$ such that
\[ \int_{\tau_n}^{\tau_{n+1}} h(\tau) \, d\tau \geq \delta \quad \forall n \in \mathbb{N}, \]
so that (3.18) gives again $0 < -\infty$.

Now assume that there exists $\lim_{t \to +\infty} \|u'(t)\|_2^2 = L$. Then there exists $M > 0$ such that
\[ \|u'(t)\|_2^2 \geq \frac{L}{2} \quad \forall t \geq M. \tag{3.19} \]
In this way, by (2.10), Remark 2.3 and (3.19) we get
\[ 0 < E_{\infty} = E(0) + \int_{0}^{\infty} E'(\tau) \, d\tau \leq C_M - \int_{M}^{\infty} h(\tau)\|u'(\tau)\|_2^2 \, d\tau \]
\[ \leq C_M - \frac{L}{2} \int_{M}^{\infty} h(\tau) \, d\tau, \tag{3.20} \]
where $C_M$ is a constant depending on $E(0)$ and $M$. Since $h$ is integrally positive, there exists $\delta > 0$ such that
\[ \int_{\tau_n}^{n+1} h(\tau) \, d\tau \geq \delta \quad \forall n \in \mathbb{N}, \]
so that (3.20) gives $0 < -\infty$, again a contradiction.

Thus we can conclude that $L = 0$, i.e.
\[ \lim_{t \to +\infty} \|u'(t)\|_2 = 0. \tag{3.21} \]

As a consequence, (3.13) implies
\[ \lim_{t \to \infty} \left( \frac{1}{2} \|A^{1/2}u(t)\|_2^2 - \int_{\Omega} F(u(t)) \, dx \right) = E_{\infty} > 0. \]

Then there exists $T > 0$ such that for any $t \geq T$ one has
\[ \frac{1}{2} \|A^{1/2}u(t)\|_2^2 - \int_{\Omega} F(u(t)) \, dx \geq \frac{E_{\infty}}{2}. \]
By (3.11), there exists $\gamma = \gamma(u) > 0$ such that
\[ \|A^{1/2}u(t)\|_2 \geq \gamma \quad \forall t > T. \tag{3.22} \]

Now set
\[ v(t) = \langle u(t), u'(t) \rangle = \frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2, \tag{3.23} \]
so that
\[ v'(t) = \|u'(t)\|^2 + \langle u(t), -h(t)u'(t) - Au + f(u) \rangle_{L^2} \]
\[ = \|u'(t)\|^2 - h(t)v(t) - \|A^{1/2}u\|_2^2 + \langle u, f(u) \rangle_{L^2} \]
\[ \leq \|u'(t)\|^2 - h(t)v(t) - \|A^{1/2}u\|_2^2. \]
by (2.7). Finally, (3.21) and (3.22) imply that there exist
\(T_0 > T\) and \(\delta > 0\) such that
\[v'(t) \leq -\delta - h(t)v(t) \quad \forall t \geq T_0,\]
that is
\[\frac{d}{dt}(v(t)e^{H(t)}) \leq -\delta e^{H(t)} \quad \forall t \geq T_0,\]
where \(H(t) = \int_0^t h(\tau) \, d\tau\). Integrating between \(T_0\) and \(t\) gives
\[v(t) \leq v(T_0)e^{-H(t)+H(T_0)} - \delta e^{-H(t)} \int_{T_0}^t e^{H(\tau)} \, d\tau.\]
Integrating again between \(T_0\) and \(t\), by (3.23) we get
\[\frac{1}{2}\|u(t)\|_2^2 \leq \frac{1}{2}\|u(T_0)\|_2^2 + v(T_0)e^{H(T_0)} \int_{T_0}^t e^{-H(s)} \, ds - \delta \int_{T_0}^t e^{-H(s)} \int_{T_0}^s e^{H(\tau)} \, d\tau \, ds.\]
Letting \(t \rightarrow \infty\), by (1.2) the right hand side of the previous inequality goes to \(-\infty\), and a contradiction arises. In fact, since \(h\) is integrally positive, we get
\[H(t) = \int_0^t h(\tau) \, d\tau > \delta[t] \geq \delta(t-1),\]
where \([t]\) denotes the integer part of \(t\), and then
\[\int_{T_0}^t e^{-H(\tau)} \, d\tau \leq \int_{T_0}^t e^{-\delta(\tau-1)} \, d\tau \leq e^{-\delta(T_0-1)/\delta}.\]
Thus \(E_\infty = 0\), and since \(F(u) \leq 0\) for any \(u\), (3.13) implies that \(\|u'(t)\|_2^2 + \|A^{1/2}u(t)\|_2^2 \rightarrow 0\) as \(t \rightarrow \infty\) and (3.12) clearly follows.

**Remark 3.2.** 1. In proving the previous result actually what is really needed is that
\[\text{if } (u_0, u_1) \in D(A) \times H \text{ and } \|Au_0\|_H + \|u_1\|_H \leq B_1,\]
then \(\forall t > 0 \, f(u(t, u_0, u_1)) \in H\) and
\[\|Au(t, u_0, u_1)\|_H + \|u'(t, u_0, u_1)\|_H + \|f(u(t, u_0, u_1))\|_H \leq B_2.\]
However, we preferred to maintain the definition proposed in [22], since it is natural to deal with solutions whose time derivative is still in \(H_0^1(\Omega)\), as it happens when it is possible to apply a regularity result.

2. Moreover, the proof above extends immediately to the abstract case, and this is the reason why we maintained the abstract formulation, writing for example \(\|A^{1/2}u\|_2\) in place of \(\|Du\|_2\).

**Remark 3.3.** In proving the analogue of Theorem 3.1 in [22] for a sublinear \(f\), the author didn’t take into account the different possibilities about the limit \(L\) defined in (3.14). However, adapting our proof to any function \(f\) with sublinear growth and such that \(sf(s) < \lambda_1 s^2\), \(s \neq 0\), like in [22], we can recover the stability result quoted therein.
As in [22], we prove that condition (1.2) is also necessary for asymptotical stability to hold, even without the assumption that $h$ is integrally positive and without the sign assumption on $f$, though we need $f$ to be non supercritical, in the usual sense. Moreover, we can even require a weaker a priori bound condition.

**Definition 3.** Solutions of (2.5) are said weakly uniformly bounded in $D(A^{1/2}) \times H$ if for any $B_1 > 0$ there exists $B_2 > 0$ such that

$$
\text{if } (u_0, u_1) \in D(A^{1/2}) \times H \text{ and } \|A^{1/2}u_0\|_H + \|u_1\|_H \leq B_1,
$$

then $\forall \ t > 0$

- $f(u(t, u_0, u_1)) \in H$
- $\|A^{1/2}u(t, u_0, u_1)\|_H + \|u'(t, u_0, u_1)\|_H + \|f(u(t, u_0, u_1))\|_H \leq B_2,$

where $u(t, u_0, u_1)$ denotes the solution of (2.5) with initial condition $(u_0, u_1)$.

**Theorem 3.2.** Suppose that (2.7) holds and that solutions of (1.1) are weakly uniformly bounded in $D(A^{1/2}) \times L^2(\Omega)$. In addition, assume that

$$
\exists a, b \geq 0, p \in \begin{cases} [1, \infty) & \text{if } N = 1, 2, \\ [1, \frac{N}{N - 2}] & \text{if } N \geq 3, \end{cases} \text{ such that } |f(s)| \leq a + b|s|^p \forall s \in \mathbb{R}. \tag{3.24}
$$

If every solution of (1.1) satisfies (3.12), then (1.2) holds.

**Proof.** Since all solutions $u$ of (1.1) are weakly uniformly bounded in $D(A^{1/2}) \times L^2(\Omega)$, for any $D > 0$ there exists $M > 0$ such that, for any $u_0, u_1 \in D(A^{1/2}) \times L^2(\Omega)$ with $\|A^{1/2}u_0\|_2^2 + \|u_1\|_2^2 \leq D$, there holds in particular

$$
\|A^{1/2}u(t, u_0, u_1)\|_2 \leq M \forall t > 0. \tag{3.25}
$$

Moreover, by the Hölder and Sobolev inequalities, there exist $S_1, S_{p+1} > 0$ such that for any $u \in H_0^1(\Omega)$ there holds

$$
\int_{\Omega} |u| \, dx \leq S_1 \|A^{1/2}u\|_2 \quad \text{and} \quad \int_{\Omega} |u|^{p+1} \, dx \leq S_{p+1} \|A^{1/2}u\|_2^{p+1}. \tag{3.26}
$$

In fact, note that if $N \geq 3$, then $p + 1 \leq (2N - 2)/(N - 2)$ and Sobolev’s Theorem can be applied.

Assume by contradiction that

$$
\int_0^\infty e^{-H(t)} \int_0^t e^{H(s)} \, ds \, dt < \infty.
$$

Then, for any $\gamma > 0$ there exists $t_0$ such that

$$
\int_{t_0}^\infty e^{-H(t)} \int_{t_0}^t e^{H(s)} \, ds \, dt < \frac{D}{8\gamma}. \tag{3.27}
$$
Now take $\phi \in H_0^1(\Omega)$ such that $\|A^{1/2}\phi\|_2^2 = D/2$, $\|\phi\|_2^2 = D/2$, and consider the solution $u$ of (1.1) such that $u(t_0) = \phi$ and $u_t(t_0) = 0$, so that $(\phi,0)$ guarantees $\|A^{1/2}u(t_0)\|_2^2 + \|u_t(t_0)\|_2^2 \leq D$. Finally, for $t \geq t_0$, set $w(t) = \langle u(t), u'(t) \rangle_{L^2} = \frac{1}{2}(\|u(t)\|_2^2)'$. Differentiating, we get
\[ w'(t) = \|u'(t)\|_2^2 + \langle u(t), u''(t) \rangle_{L^2} \]
\[ = \|u'(t)\|_2^2 + (u(t), -Au) - h(t)w(t) + \langle u(t), f(u) \rangle_{L^2} \]
\[ = \|u'(t)\|_2^2 - \|A^{1/2}u(t)\|_2^2 - h(t)w(t) + \langle u(t), f(u) \rangle_{L^2} \]
\[ \geq -\|A^{1/2}u(t)\|_2^2 - h(t)w(t) - \int_0^1 (a|u(t)| + b|u(t)|^{p+1}) \, dt. \]  
(3.28)

By (3.26) we get
\[ w'(t) \geq -\|A^{1/2}u(t)\|_2^2 - h(t)w(t) - aS_1\|A^{1/2}u\|_2 - bS_{p+1}\|A^{1/2}u\|_2^{p+1}. \]  
(3.29)

By the Young inequality we can find $\eta > 0$ such that (3.29) gives
\[ w'(t) \geq -\eta(1 + \|A^{1/2}u(t)\|_2^{p+1}) - h(t)w(t) - bS_{p+1}\|A^{1/2}u\|_2^{p+1}, \]
and by (3.25),
\[ w'(t) \geq -(\eta + \eta M^{p+1} + bS_{p+1}M^{p+1}) - h(t)w(t). \]  
(3.30)

Setting $\gamma = \eta + \eta M^{p+1} + bS_{p+1}M^{p+1} > 0$ (which is independent of $t_0$), integrating (3.30) twice between $t_0$ and $t$ gives
\[ \frac{1}{2}\|u(t)\|_2^2 \geq \frac{1}{2}\|u(t_0)\|_2^2 - \gamma \int_{t_0}^t e^{-H(s)} \int_{s_0}^s e^{H(\tau)} \, d\tau \, ds. \]

Finally, by (3.27), we get
\[ \frac{1}{2}\|u(t)\|_2^2 \geq \frac{D}{8}. \]

By Poincaré’s inequality we obtain
\[ \|A^{1/2}u\|_2^2 \geq \frac{\lambda_1 D}{4}, \]
so that (3.12) cannot hold. \qed

**Remark 3.4.** 1. Of course a uniform bound implies a weak uniform bound by Poincaré’s inequality. However, we preferred to present Theorem 3.2 under the more general assumption of a weak uniform bound on the set of solutions.

2. Again, in the proof of Theorem 3.2 we maintained the abstract form $A$ for $-\Delta$, since an analogous version for the abstract problem (2.5) can be provided at once, where $\lambda_1$ is now given by (2.6).
Summing up, in view of Theorems 3.2 and 3.1, and recalling that (3.24) implies (3.11) when \( a = 0 \), we can conclude with the following result.

**Theorem 3.3.** Let \( \Omega \) be a bounded domain of \( \mathbb{R}^N, N \geq 1 \). Assume (2.7), (3.11) and (3.24); moreover, suppose that \( h \) is integrally positive and that solutions of (1.1) are uniformly bounded in \( H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega) \). Then every solution of (1.1) verifies

\[
\| Du(t) \|_2 + \| u'(t) \|_2 \to 0 \quad \text{as} \quad t \to +\infty
\]

if and only if (1.2) holds.

4 The on–off case

As in Section 2, we set \( V = D(A^{1/2}) \subset H \) and by (2.6) there exists \( \lambda_1 > 0 \) such that

\[
\| v \|_H^2 \leq \frac{1}{\lambda_1} \| v \|_V^2 = \frac{1}{\lambda_1} \| A^{1/2}v \|_H^2 \quad \text{for any} \quad v \in V.
\]

We remark that in the standard case of problem (1.1) we have \( H = L^2(\Omega), V = H^1_0(\Omega) \), \( \lambda_1 \) is the first eigenvalue of \( -\Delta \) on \( H^1_0(\Omega) \), the inequality above being the usual Poincare’s inequality.

The inequality expressed in (2.8) can be replaced in the abstract case by the condition that

\[
\langle f(u), u \rangle_{V', V} - \mathcal{F}(u) \leq 0 \quad \forall \ u \in V,
\]

though it would suffice to hold only for those \( u \)'s which solve (2.5). Of course this condition, in effect, reduces to (2.8) in the model case of problem (1.1).

**Theorem 4.1.** Fix \( T > 0 \) and assume there exist \( M, m > 0 \) such that

\[
m \| v \|_H^2 \leq \langle B(t)v, v \rangle_H \quad \forall \ t \in [0, T], \forall \ v \in H
\]

\[
\| B(t)v \|_H^2 \leq M \langle B(t)v, v \rangle_H \quad \forall \ t \in [0, T], \forall \ v \in H.
\]

Moreover, suppose that (2.6) and (4.31) hold. Then for every \( (u_0, u_1) \in V \times H \) the solution \( u \) of problem (2.5) satisfies

\[
E_u(T) \leq \frac{1}{1 + \frac{T^3}{30 \lambda_1 m} + \frac{MT^2}{32m} + \frac{MT^2}{16 \lambda_1}} E_u(0).
\]

**Proof.** As in [6] and [7], for any \( t \in [0, T] \) we set \( \theta(t) = t^2(T - t)^2 \), so that \( \theta'(t) = 2t(T - t)(T - 2t) \) and

\[
|\theta'(t)| \leq 2T\theta^{1/2}(t) \quad \text{for all} \quad t \in [0, T],
\]

\[
\max_{t \in [0, T]} \theta(t) = \frac{T^4}{16}
\]
\[
\int_0^T \theta(t) \, dt = \frac{T^5}{30}.
\] (4.37)

Therefore (2.10) gives

\[
E(0) - E(T) = \int_0^T \langle Bu', u' \rangle_H \, dt \geq 0.
\] (4.38)

Multiplying equation (2.5) by \(\theta u\) gives

\[
\int_0^T \theta \left\{ \langle u'' + Bu', u \rangle_{V', V} + \|A^{1/2}u\|_H^2 - \langle f(u), u \rangle_{V', V} \right\} \, dt = 0.
\]

Therefore, integrating by parts,

\[
\int_0^T \theta \|A^{1/2}u\|_H^2 \, dt = \int_0^T \left\{ \theta \|u''\|_H^2 + \theta' \langle u, u' \rangle_H - \theta \langle Bu', u \rangle_H + \theta \langle f(u), u \rangle_{V', V} \right\} \, dt,
\]

that is

\[
\int_0^T \theta \|A^{1/2}u\|_H^2 \, dt = \int_0^T \left\{ \theta \|u''\|_H^2 + \theta' \langle u, u' \rangle_H - \theta \langle Bu', u \rangle_H + \theta \langle f(u), u \rangle_{V', V} \right\} \, dt.
\]

Thus for any \(\varepsilon, \eta > 0\) we get

\[
\int_0^T \theta \|A^{1/2}u\|_H^2 \, dt \leq \int_0^T \left\{ \varepsilon \|u''\|_H^2 + \frac{1}{4\varepsilon} \theta \|Bu'\|_H^2 + \frac{1}{4\eta} \|u'\|_H^2 + \theta \|u\|_H^2 + \theta \langle f(u), u \rangle_{V', V} \right\} \, dt.
\]

By (4.35), (4.36), (4.33) and (2.6) we get

\[
\int_0^T \theta \|A^{1/2}u\|_H^2 \, dt \leq \int_0^T \left\{ \varepsilon \|A^{1/2}u\|_H^2 + \frac{MT^4}{64\varepsilon} \langle Bu', u' \rangle_H + \frac{4\eta T^2}{\lambda_1} \theta \|A^{1/2}u\|_H^2 + \frac{1}{4\eta} \|u'\|_H^2 + \frac{T^4}{16} \|u'\|^2_H + \theta \langle f(u), u \rangle_{V', V} \right\} \, dt.
\] (4.39)

Let us choose \(\varepsilon\) and \(\eta\) so that

\[
\frac{4\eta T^2}{\lambda_1} = \frac{\varepsilon}{\lambda_1} = \frac{1}{4}.
\]

Then (4.39) reads

\[
\int_0^T \theta \|A^{1/2}u\|_H^2 \, dt \leq \int_0^T \left\{ \frac{1}{4} \theta \|A^{1/2}u\|_H^2 + \frac{MT^4}{16\lambda_1} \langle Bu', u' \rangle_H + \theta \langle f(u), u \rangle_{V', V} + \frac{1}{4} \theta \|A^{1/2}u\|_H^2 + \frac{4T^2}{\lambda_1} \|u'\|_H^2 + \frac{T^4}{16} \|u'\|_H^2 \right\} \, dt,
\]

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so that
\[
\frac{1}{2} \int_0^T \theta \|A^{1/2} u\|_H^2 \, dt \leq \int_0^T \left\{ C_T \|u'\|_H^2 + \frac{MT^4}{16\lambda_1} \langle Bu', u' \rangle_H + \theta \langle f(u), u \rangle_{V', V} \right\} \, dt,
\]
where \( C_T = \frac{4T^2}{\lambda_1^2} + \frac{T^4}{16}. \) By (4.38) this means
\[
\frac{1}{2} \int_0^T \theta \|A^{1/2} u\|_H^2 \, dt \leq \int_0^T \left\{ C_T \|u'\|_H^2 + \theta \langle f(u), u \rangle_{V', V} \right\} \, dt + \frac{MT^4}{16\lambda_1} (E(0) - E(T)).
\]
By (2.9)
\[
\frac{1}{2} \int_0^T \theta \|A^{1/2} u\|_H^2 \, dt = \int_0^T \theta E(t) \, dt - \frac{1}{2} \int_0^T \theta \|u'\|_H^2 \, dt + \int_0^T \theta \mathcal{F}(u) \, dt
\]
and by Lemma 2.2
\[
\frac{1}{2} \int_0^T \theta \|A^{1/2} u\|_H^2 \, dt \geq E(T) \int_0^T \theta(t) \, dt - \frac{1}{2} \int_0^T \theta \|u'\|_H^2 \, dt + \int_0^T \theta \mathcal{F}(u) \, dt.
\]
Therefore (4.37), (4.40), (4.36) and (4.41) give
\[
\frac{T^5}{30} E(T) \leq \int_0^T \theta \langle f(u), u \rangle_{V', V} - \mathcal{F}(u) \rangle \, dt + \left( C_T + \frac{T^4}{32} \right) \int_0^T \|u'\|_H^2 \, dt + \frac{MT^4}{16\lambda_1} (E(0) - E(T)),
\]
which implies, together with (4.31),
\[
\left( \frac{T^5}{30} + \frac{MT^4}{16\lambda_1} \right) E(T) \leq \left( C_T + \frac{T^4}{32} \right) \int_0^T \|u'\|_H^2 \, dt + \frac{MT^4}{16\lambda_1} E(0).
\]
By (4.32) we get
\[
\left( \frac{T^5}{30} + \frac{MT^4}{16\lambda_1} \right) E(T) \leq \frac{1}{m} \left( C_T + \frac{T^4}{32} \right) \int_0^T \langle Bu', u' \rangle_H \, dt + \frac{MT^4}{16\lambda_1} E(0).
\]
By (2.10) this implies
\[
\left( \frac{T^5}{30} + \frac{MT^4}{16\lambda_1^4} \right) E(T) \leq \frac{1}{m} \left( C_T + \frac{T^4}{32} \right) \left( E(0) - E(T) \right) + \frac{MT^4}{16\lambda_1^4} E(0),
\]
and so
\[
\left( \frac{T^5}{30} + \frac{C_T}{m} + \frac{T^4}{32m} + \frac{MT^4}{16\lambda_1^4} \right) E(T) \leq \left( \frac{C_T}{m} + \frac{T^4}{32m} + \frac{MT^4}{16\lambda_1^4} \right) E(0)
\]
and (4.34) follows.

**Remark 4.1.** Theorem 4.1 can be generalized to any interval \([a, b]\), obtaining
\[
E_u(b) \leq \frac{1}{1 + \frac{T^3}{30} \frac{4}{\lambda_1 m} + \frac{MT^4}{32m} + \frac{MT^4}{16\lambda_1^4}} E_u(a),
\]
with \(T = b - a\). Indeed, in \([a, b]\) take \(\theta(t) = (t - a)^2(b - t)^2\); then (4.35), (4.36), (4.37) are replaced by \(|\theta'(t)| \leq 2(b - a)\theta^{1/2}_{[a, b]}(t)\), \(\max \theta = \frac{(b - a)^4}{16}\) and
\[
\int_a^b \theta(t) \, dt = \int_0^T x^2(T - x)^2 \, dx = \frac{(b - a)^5}{30},
\]
respectively, so that they give the same estimates, since \(b - a = T\), while (4.38) is replaced by \(E(a) - E(b) = \int_a^b \langle Bu', u' \rangle_H \, dt \geq 0\).

Now multiply equation (2.5) by \(\theta u\) and integrate in \([a, b]\); performing the same estimates as in the proof of Theorem 4.1 we get the desired result.

Theorem 4.1 is the essential tool for the following stability result, which can be proved extending the method of Smith (see [21]) as already done in [7].

**Theorem 4.2.** Let \((a_n, b_n)\) be a sequence of disjoint open intervals in \((0, +\infty)\) with \(a_n \to +\infty\) and assume that (1.3), (1.4), (2.6) and (4.31) hold. Then for every \((u_0, u_1) \in D(A^{1/2}) \times H\) the solution \(u\) of problem (2.5) is such that \(E_u(t) \to 0\) as \(t \to +\infty\).

**Proof.** The proof of Theorem is now similar to the proof of [7, Theorem 3.2], since the main tool in the proof is an inequality of the type of (4.34) proved in Theorem 4.1. We sketch it for completeness.

Apply Theorem 4.1 in the form of Remark 4.1 in the interval \((a_n, b_n)\) instead of the interval \((0, T)\), obtaining
\[
E_u(b_n) \leq \frac{1}{1 + \frac{T_n^3}{30} \frac{4}{\lambda_1 m_n} + \frac{MT_n^4}{32m_n} + \frac{M_n T_n^6}{16\lambda_1^4}} E_u(a_n),
\]
where we have set \(T_n = b_n - a_n\).
Defining
\[ k_n := \frac{m_n T_n^3}{128 + 3 \lambda_1 T_n^2 + 2 m_n T_n^2} \text{ and } c = \frac{16 \lambda_1}{15}, \]
equation (4.42) can be rewritten as
\[ E_u(b_n) \leq \frac{1}{1 + c k_n} E_u(a_n). \]

Since \( E \) is non increasing, by iteration we get that for any \( n \in \mathbb{N} \)
\[ E(a_{n+1}) \leq E(b_n) \leq \frac{1}{1 + c k_n} E_u(a_u) \leq E(a_0) \prod_{j=0}^{n} \frac{1}{1 + c k_j} \leq E(0) \prod_{j=0}^{n} \frac{1}{1 + c k_j}. \]

Since \( E \) is non increasing, Theorem 4.2 will be proved if we show that \( E(a_{n+1}) \to 0 \) as \( n \to \infty \); therefore, let us show that
\[ \prod_{j=0}^{\infty} \frac{1}{1 + c k_j} = 0, \quad \text{or equivalently,} \quad \sum_{j=0}^{\infty} \log \frac{1}{1 + c k_j} = -\infty. \]

This condition obviously holds if \( k_j \not\to 0 \) as \( j \to \infty \), while if \( k_j \to 0 \) it means that
\[ \sum_{j=1}^{\infty} k_j = +\infty. \]

This last condition is equivalent to (1.4): indeed, if \( T_j^2(3 \lambda_1 + 2 m_j M_j) := T_j^2 c_j \geq 128 \), then \( k_j \geq \frac{1}{2 c_j} \), while if \( T_j^2 c_j \leq 128 \), then \( k_j \geq \frac{T_j^2}{2 \times 128} \), concluding that
\[ k_j \geq \frac{1}{2} m_j (b_j - a_j) \min \left( \frac{(b_j - a_j)^2}{128}, \frac{1}{\lambda_1 + 2 m_j M_j} \right). \]

On the other hand,
\[ k_j \leq m_j (b_j - a_j) \min \left( \frac{(b_j - a_j)^2}{128}, \frac{1}{\lambda_1 + 2 m_j M_j} \right), \]
and the claim follows. \( \square \)

5 Some concrete applications

5.1 The integrally positive case

Consider again problem (1.1), where \( \Omega \) is a bounded domain of \( \mathbb{R}^N \), \( N \geq 1 \). First, let us briefly show that the set of solutions is weakly uniformly bounded in \( H_0^1(\Omega) \times L^2(\Omega) \) under the subcritical growth assumption on \( f \) of Theorem
3.2, even without any sign condition on $f$. Indeed, by Lemma 2.2, if $u$ solves (1.1), we get

$$E(t) \leq E(0) = \frac{1}{2} \|u_1\|^2 + \frac{1}{2} \|u_0\|^2_{H^1_0(\Omega)} - \int_{\Omega} F(u_0) \, dx.$$ 

But

$$\int_{\Omega} |F(u_0)| \, dx \leq a \int_{\Omega} |u_0| \, dx + \frac{b}{p+1} \int_{\Omega} |u_0|^{p+1} \, dx,$$

and by (3.26) there exists $S > 0$

$$\int_{\Omega} |F(u_0)| \, dx \leq S(\|u_0\|_{H^1_0(\Omega)} + \|u_0\|_{H^1_0(\Omega)}^{p+1}).$$

Therefore, if $\|u_0\|_{H^1_0(\Omega)} + \|u_1\|_2 \leq B_1$, then

$$E(0) \leq \frac{B_1^2}{2} + S(B_1 + B_1^p) = B_2.$$

Hence for every $t \geq 0$

$$\|u(t, u_0, u_1)\|^2_{H^1_0(\Omega)} + \|u'(t, u_0, u_1)\|^2_2 = 2E(t) + 2 \int_{\Omega} F(u) \, dx \leq 2E(0) = 2B_2,$$

that is Definition 3 is verified, as claimed.

Finally, we recall that the request that the set of solutions of (1.1) is uniformly bounded in $H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$ is not so strange. Several examples are considered in [22], and we refer to those cases therein for a sublinear $f$, simple recalling the following

**Example 5.1** ([22], Example 3.1). Assume that $f(s) = \alpha s$ for some constant $\alpha < \lambda_1$, where $\lambda_1$ is the first eigenvalue of $-\Delta$ on $H^1_0(\Omega)$. Then the set of solutions of (1.1) is uniformly bounded in $H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$.

Finally, we show that the set of solutions of (1.1) is uniformly bounded in $H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$ also in more general cases. This result appears in [22] for $f$ having linear growth, under the additional assumptions that $h$ is bounded above and below by strictly positive constants (though there is a mistake in the final step on page 198). We take the latter assumption, but we let $f$ have superlinear growth. However, let us note that the growth condition we give on $f$ immediately implies condition (3.11), which is therefore useless from now on.

**Lemma 5.1.** Suppose that $N = 1$, $f$ is an absolutely continuous function satisfying (2.7) such that

$$\exists b_1 > 0, p \in [1, \infty) : \quad |f'(s)| \leq b_1 |s|^{p-1} \quad \forall s \in \mathbb{R}.$$ 

Assume that there exist two positive constants $\alpha < \beta$ such that $\alpha \leq h(t) \leq \beta$ for any $t \geq 0$. Finally assume that $\forall P > 0$ there exists $Q > 0$ such that $\|Du\|_2 \leq P$ implies $\|Df(u)\|_2 \leq Q\|Du\|_2$. Then the set of solutions of (1.1) is uniformly bounded in $H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$.
Proof. First let us note that the condition on \( f' \) implies that \(|f(s)| \leq b_2 |s|^p \) for any \( s \). Now take \( B_1 > 0 \) and \((u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)\) such that \( \|\Delta u_0\|_2 + \|Du_1\|_2 \leq B_1 \), and observe that the growth condition on \( f \) and the Sobolev inequality immediately ensure that \( f(u) \in L^2(\Omega) \) for any \( t > 0 \).

By Lemma 2.2 and Remark 2.3 we get

\[
\|u'(t)\|_2^2 + \|Du(t)\|_2^2 \leq 2E(0) + 2 \int_\Omega F(u(t)) \, dx \\
\leq 2E(0) + \|u_1\|_2^2 + \|Du_0\|_2^2 - 2 \int_\Omega F(u_0) \, dx.
\]

By the Poincaré and Sobolev inequalities and the growth condition on \( f \), we get

\[
\|u_1\|_2^2 + \|Du_0\|_2^2 - 2 \int_\Omega F(u_0) \, dx \leq C \left( \|Du_1\|_2^2 + \|\Delta u_0\|_2^2 + \|u_1\|_1 + \|u\|_p^{p+1} \right)
\]

Applying again the Poincaré and Sobolev inequalities, since \( \|\Delta u_0\|_2 + \|Du_1\|_2 \leq B_1 \), we get the existence of a constant \( B > 0 \) such that

\[
\|u'(t)\|_2^2 + \|Du(t)\|_2^2 \leq B \quad \forall t > 0.
\] (5.43)

Now set

\[
V(t) = \int_\Omega u(t) u'(t) \, dx.
\]

Proceeding as in [22], we can find

\[
V'(t) = \int_\Omega \left( u'(t)^2 + u''(t)u(t) \right) \, dx,
\]

and using the equation in (1.1)

\[
V'(t) = \int_\Omega u'(t)^2 \, dx - \int_\Omega |Du(t)|^2 \, dx - h(t) \int_\Omega u'(t)u(t) \, dx + \int_\Omega f(u(t))u(t) \, dx
\]

By (2.7) and the Hölder, Young and Poincaré inequalities

\[
V'(t) \leq \left( 1 + \frac{\beta^2}{2\epsilon} \right) \int_\Omega u'(t)^2 \, dx + \left( \frac{\epsilon}{2\lambda_1} - 1 \right) \int_\Omega |Du(t)|^2 \, dx
\]

for any \( \epsilon > 0 \). Therefore

\[
V(t) \leq V(0) + \left( 1 + \frac{\beta^2}{2\epsilon} \right) \int_0^t \|u'(\tau)\|_2^2 \, d\tau + \left( \frac{\epsilon}{2\lambda_1} - 1 \right) \int_0^t \|Du(\tau)\|_2^2 \, d\tau.
\]

Choosing \( \epsilon < 2\lambda_1 \) we get the existence of a positive constant \( C \) such that

\[
\int_0^t \|Du(\tau)\|_2^2 \, d\tau \leq C \left( V(0) + \int_0^t \|u'(\tau)\|_2^2 \, d\tau \right) - V(t) \quad \forall t > 0.
\]

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But

$$|V(t)| \leq \|u'(t)\|_2 \|u(t)\|_2 \leq \frac{1}{\sqrt{\lambda_1}} \|u'(t)\|_2 \|Du(t)\|_2 \leq \frac{B}{\sqrt{\lambda_1}} \quad \forall t > 0$$

by (5.43), so that

$$\int_0^t \|Du(\tau)\|_2^2 d\tau \leq C_1 \left(1 + \int_0^t \|u'(\tau)\|_2^2 d\tau\right) \quad \forall t > 0 \quad (5.44)$$

for some positive constant $C_1$.

Using again Lemma 2.2 and Remark 2.3, we have

$$E(t) = E(0) - \int_0^t h(\tau) \|u'(\tau)\|_2^2 d\tau,$$

so that

$$\int_0^t \|u'(\tau)\|_2^2 d\tau \leq \frac{E(0)}{a} \leq C_2 \quad \forall t > 0$$

for some positive constant $C_2$. In this way (5.44) implies the existence of $C_4 > 0$ such that

$$\int_0^t \|Du(\tau)\|_2^2 d\tau \leq C_3 \quad \forall t > 0. \quad (5.45)$$

Finally, introduce

$$W(t) = \frac{1}{2} \int_\Omega |Du'(t)|^2 dx + \frac{1}{2} \int_\Omega |\Delta u(t)|^2 dx.$$

As in [22], we find

$$W(t) = W(0) - \int_0^t h(\tau) \|Du'(\tau)\|_2^2 d\tau + \int_0^t \int_\Omega Du'(\tau) \cdot Df(u(\tau)) dx d\tau,$$

so that by Cauchy’s inequality

$$W(t) \leq W(0) - \int_0^t h(\tau) \|Du'(\tau)\|_2^2 d\tau + \int_0^t \|Du'(\tau)\|_2 \|Df(u(\tau))\|_2 d\tau.$$

By Young’s inequality, for any $\varepsilon > 0$

$$W(t) \leq W(0) - \int_0^t h(\tau) \|Du'(\tau)\|_2^2 d\tau + \varepsilon \int_0^t \|Du'(\tau)\|_2^2 d\tau + \frac{1}{2\varepsilon} \int_0^t \|Df(u(\tau))\|_2^2 d\tau.$$

By assumption on $h$, we finally get

$$W(t) \leq W(0) + (\varepsilon - \alpha) \int_0^t h(\tau) \|Du'(\tau)\|_2^2 d\tau + \frac{1}{2\varepsilon} \int_0^t \|Df(u(\tau))\|_2^2 d\tau.$$
Choosing \( \varepsilon < \alpha \) and recalling that \( \|\Delta u_0\|_2 + \|Du_1\|_2 \leq B_1 \), we get

\[
W(t) \leq C + C \int_0^t \|Df(u(\tau))\|_2 \, d\tau \quad \forall \, t > 0
\]

for some positive constant \( C \).

By assumption, taking \( P := \sqrt{B} \) in (5.43), from (5.46) we get

\[
W(t) \leq C + CQ \int_0^t \|Du(\tau)\|_2^p \, d\tau \quad \forall \, t > 0.
\]

By (5.45) we find \( D > 0 \) such that

\[
W(t) \leq D \quad \text{for any } t, \text{i.e. the set of solutions is uniformly bounded in } H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega), \text{as claimed.}
\]

As a final application, Lemma 5.1 gives:

**Proposition 5.1.** Suppose that \( N = 1 \), \( f \) is an absolutely continuous function satisfying (2.7) such that

\[
\exists b_1 > 0, \, p \in [1, \infty) : \quad |f'(s)| \leq b_1 |s|^{p-1} \quad \forall \, s \in \mathbb{R}.
\]

Finally assume that there exist two positive constants \( \alpha < \beta \) such that \( \alpha \leq h(t) \leq \beta \) for any \( t \geq 0 \). Then the set of solutions of (1.1) is uniformly bounded in \( H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega) \).

**Proof.** The only thing to prove is that for any \( P > 0 \) there exists \( Q > 0 \) such that \( \|Du\|_2 \leq P \) implies \( \|Du\|_2 \leq Q \). But this is just an application of Sobolev and Poincaré’s inequality. Indeed, if \( \|Du\|_2 \leq P \), then

\[
\|Df(u)\|_2 = \|f'(u)Du\|_2 \leq b_1 \|u|^{p-1}Du\|_2 \leq C\|Du\|_2^p \leq CP^{p-1}\|Du\|_2 = Q\|Du\|_2,
\]

as soon as \( Q = CP^{p-1} \). Now apply Lemma 5.1.

Summing up, Theorems 3.1, 3.2 and Proposition 5.1 imply the following final result.

**Proposition 5.2.** Assume \( N = 1 \) and suppose that \( f \) is an absolutely continuous function satisfying (2.7) and such that

\[
\exists b_1 > 0, \, p \in [1, \infty) : \quad |f'(s)| \leq b_1 |s|^{p-1} \quad \forall \, s \in \mathbb{R}.
\]

Moreover, assume that there exist two positive constants \( \alpha < \beta \) such that \( \alpha \leq h(t) \leq \beta \) for any \( t \geq 0 \). Then every solution of (1.1) satisfies

\[
\|u\|_{H^1_0(\Omega)} + \|u'\|_{L^2(\Omega)} \to 0 \quad \text{as } \, t \to +\infty
\]

if and only if (1.2) holds.
5.2 The on–off case

A particular case of the abstract problem considered in the Section 4, is the following nonlinear wave system in a bounded domain \(\Omega\) of \(\mathbb{R}^N\), \(N \geq 1\):

\[
(W) \quad \begin{cases}
    u_{tt} = \Delta u - h(t)g(u_t) + f(u) & \text{in } (0, +\infty) \times \Omega, \\
    u(t, x) = 0 & \text{in } (0, +\infty) \times \partial \Omega, \\
    u(0, x) = u_0(x), \ u_t(0, x) = u_1(x) & x \in \Omega,
\end{cases}
\]

where the following assumptions are made:

\[
(A) \quad \begin{cases}
    g : \mathbb{R} \rightarrow \mathbb{R} \text{ is a } C^1 \text{ function with } g(0) = 0, \\
    \exists B \geq A > 0 \text{ such that } 0 < A \leq g'(v) \leq B \forall v \in \mathbb{R}, \\
    f \text{ satisfies (2.8)},
\end{cases}
\]

while \(u_0 \in H^1_0(\Omega)\) and \(u_1 \in L^2(\Omega)\).

In this case Theorem 4.1 can be easily generalized as follows.

**Theorem 5.1.** Fix \(T > 0\) and assume there exist \(M, m > 0\) such that

\[
0 < m \leq h(t) \leq M \quad \forall t \in [0, T].
\]

Suppose (A) holds. Then for every \((u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)\) the solution \(u\) of problem \((W)\) satisfies

\[
E(t) \leq \frac{1}{1 + \frac{T^2}{3M} \frac{2}{m^2} \frac{1}{M^2} + \frac{m^4}{T^4M}} E(0).
\]

In the same way as Theorem 4.2 is implied by Theorem 4.1, Theorem 5.1 immediately gives the following fundamental application:

**Theorem 5.2.** Let \((a_n, b_n)\) be a sequence of disjoint open intervals in \((0, +\infty)\) with \(a_n \rightarrow +\infty\) and assume that (1.3), (1.4) and (A) hold. Then for every \((u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)\) the solution \(u\) of problem \((W)\) is such that \(E_n(t) \rightarrow 0\) as \(t \rightarrow +\infty\).

Of course the abstract setting we gave in Section 2 lets us deal with higher order problem. Indeed, consider the problem

\[
(H) \quad \begin{cases}
    u_{tt} = \Delta^{2m} u - h(t)g(u_t) + f(u) & \text{in } (0, +\infty) \times \Omega, \\
    C u(t, x) = 0 & \text{in } (0, +\infty) \times \partial \Omega, \\
    u(0, x) = u_0(x) \in D(\Delta^m), \ u_t(0, x) = u_1(x) & x \in \Omega,
\end{cases}
\]

where \(m \in \mathbb{N}, g\) is as before and \(C\) is a boundary operator. For example, if \(m = 1\) and \(C u = (u, \partial u/\partial \nu)\), \(\nu\) being the unit outward normal to \(\partial \Omega\), we have \(D(\Delta^m) = H^1_0(\Omega)\), while, in the case \(C u = (u, \Delta u)\) we have \(D(\Delta^m) = H^1_0(\Omega) \cap H^2(\Omega)\). Other generalization are easy to do. For this case we have the following

**Theorem 5.3.** Let \((a_n, b_n)\) be a sequence of disjoint open intervals in \((0, +\infty)\) with \(a_n \rightarrow +\infty\) and assume that (1.3), (1.4) and (2.8) hold. Then for every \((u_0, u_1) \in D(\Delta^m) \times L^2(\Omega)\) the solution \(u\) of problem \((H)\) is such that \(E_n(t) \rightarrow 0\) as \(t \rightarrow +\infty\).
References


