Maximum Principles for Inhomogeneous Elliptic Inequalities on Complete Riemannian Manifolds

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Abstract
We prove some maximum principle results for weak solutions of elliptic inequalities, possibly inhomogeneous, on complete Riemannian manifolds.

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1 Introduction

In this paper we are concerned with weak solutions of differential inequalities on a complete Riemannian manifold $\mathcal{M}$ of dimension $n$. More precisely, our aim is to prove maximum principles for inequalities governed by operators which may be inhomogeneous.

Let us start with three examples of general interest: the first concerns the mean curvature operator, say with global growth $p = 1$, while the latter two involve the $p$–Laplace–Beltrami operator, that is $\Delta_p u := \text{div}(\nabla u |\nabla u|^{p-2})$, $p > 1$, where $\nabla u$ is the Riemannian gradient of $u$ on $\mathcal{M}$. Take the mean curvature equation

$$\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = n\mathcal{H}(x)$$

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In a domain $\Omega \subset M$. Putting $A(\xi) = \xi / \sqrt{1 + |\xi|^2}$, after calculation, we find

$$\langle A(\xi), \xi \rangle = |\xi| + \left( -|\xi| + \frac{|\xi|^2}{\sqrt{1 + |\xi|^2}} \right) \geq |\xi| - \sqrt{5/2 - 11/2}.$$ 

This means that it is not possible to control $\langle A(\xi), \xi \rangle$ from below by a positive quantity, as it is usual and useful to do in obtaining some kind of maximum principle. We also recall that such an operator is natural while studying general relativity (for example see [18]).

For the second prototype of inhomogeneous elliptic operator, we consider the intriguing inequality

$$\Delta_p u - b_1 \Delta_{p-1} u + b_2 |u|^{p-2} u + b \geq 0, \quad p > 2,$$

as well as the homogeneous soliton inequality (see, for example, [3])

$$\Delta_p u + b_1 \Delta_{p-1} u + b_2 |u|^{p-2} u + b \geq 0, \quad p > 2,$$

where in both cases $b_1 > 0, b_2, b \in \mathbb{R}$. Indeed, in (1.1) we have $A(\xi) = |\xi|^{p-2} \xi - b_1 |\xi|^{p-3} \xi$, so that

$$\langle A(\xi), \xi \rangle \geq \frac{1}{p} (|\xi|^p - b_1^p).$$

For these problems the theory of the paper applies and general maximum principles still hold true for their weak solutions.

Our first results are, actually, semi–maximum principles, in the sense that we prove inequalities of the form

$$u \leq C \|u\|_p,$$

$p \geq 1$. Indeed, we underline the fact that the mean curvature operator has linear growth, so that $L^1$–norms are natural and expected. Secondly, from these semi–maximum principles we derive classical maximum principles of the form

$$u \leq C,$$

with $C$ independent of $u$, but depending only on the data of the problem, without growth conditions on the coefficients, as required in [4].

We are interested in maximum principles and for existence of weak solutions on compact Riemannian manifolds we refer to the recent survey [6] and the references therein, while for the noncompact case we recall [19].

The first results about maximum principles in the Euclidean setting go back to the sixties with Serrin ([21], based on earlier work for homogeneous linear equations by Stampacchia [22], Maz’ya [11], Moser [12]), Gilbarg and Trudinger [7]. More recently, general results were proved in [15], [16], [20], and in [13] and [14] for the Riemannian setting. Finally, for inhomogeneous inequalities, in [17] very general results are given in the Euclidean case, and we take those results as inspiring motivations for inequalities on manifolds. Actually, we extend to the Riemannian
setting the semi–maximum principles and the maximum principles proved in [17]
for inhomogeneous inequalities in the Euclidean case. For further comments on the
results, we refer to Section 3, where we discuss some applications after stating the
main theorems.

2 Preliminaries

In this section we introduce the main notation. From now on \( M \) denotes a smooth
complete Riemannian \( n \)-manifold, with metric tensor \( g \in C^\infty(M, T^*M \otimes T^*M) \).

**Definition 2.1** The fibered product bundle of two bundles \( (E, \pi_1, M) \) and \( (F, \pi_2, M) \)
is the manifold

\[
E \times_M F = \{ (e, f) \in E \times F : \pi_1(e) = \pi_2(f) \},
\]

with the induced vector bundle structure.

In the sequel, \( \Omega \) will be any regular domain of \( M \) and, for brevity, we write
\( T\Omega \times_\Omega \mathbb{R} \) in place of \( T\Omega \times_\Omega (\Omega \times \mathbb{R}) \). Of course \( T\Omega \times_\Omega (\Omega \times \mathbb{R}) \cong T\Omega \times \mathbb{R} \), and in turn the notation is not
ambiguous. In analogy with the Euclidean case, elements of \( T\Omega \times_\Omega \mathbb{R} \) are denoted
by \( (x, z, \xi) \), where \( (x, \xi) \in T\Omega \) and \( (x, z) \in \Omega \times \mathbb{R} \).

Integrals are taken with respect to the natural Riemannian measure. For ex-
ample, if \((U, \Phi)\) is a coordinate chart and \( u \) is a continuous function compactly
supported in \( U \), we define

\[
\int_U u \, d\mathcal{M} = \int_{\Phi(U)} (\sqrt{G} u) \circ \Phi^{-1} \, dx,
\]

where \( dx \) stands for the Lebesgue measure on \( \mathbb{R}^n \) and \( G \) is the absolute value of
the determinant of the metric tensor in the coordinate chart \((U, \Phi)\). With the help
of smooth partitions of unity, the construction above defines a canonical positive
Radon measure on \( M \), which is the natural Lebesgue measure on \( M \), denoted simply
by \( | \cdot | \). In particular, we can deal with measurable vector fields, i.e. measurable
sections of the tangent bundle. For a section \( V \) defined on \( \Omega \) and \( p \geq 1 \), we introduce
the usual Lebesgue \( p \)-norm as

\[
\| V \|_{p, \Omega} = \left( \int_\Omega |V|^p \, d\mathcal{M} \right)^{1/p},
\]

where \( |V|(x) = \sqrt{g(V(x), V(x))} \). In fact, we use the notation \(| \cdot |\) to denote, according
to the cases if no ambiguity occurs, the real modulus, the Riemannian norm of
tangent vectors and the measure of measurable subsets of \( M \).

Let \( H^{1,p}(\Omega) \) be the closure of \( C^\infty(\Omega) \) in the Sobolev norm \( \| u \| = \| u \|_{p, \Omega} + \| \nabla u \|_{p, \Omega} \), where \( \| u \|_{p, \Omega} = \| u \|_{L^p(\Omega)} \) and let

\[
H^{1,p}_{loc}(\Omega) = \{ u : \Omega \rightarrow \mathbb{R} : u|_{\Omega'} \in H^{1,p}(\Omega') \text{ for all open sets } \Omega' \subset \subset \Omega \}.
\]
Finally, as usual, denote by $H^{1,p}_0(\Omega)$ the closure of $C_c^\infty(\Omega)$ with respect to the Sobolev norm $\| \cdot \|$. We recall that, if $u \in L^1_{\text{loc}}(\Omega)$, a locally integrable vector field $H \in L^1_{\text{loc}}(\Omega, T\Omega)$ is a weak gradient for $u$ if
\[
\int_{\Omega} \langle H, V \rangle \, d\mathcal{H} = - \int_{\Omega} u \, \text{div} \, V \, d\mathcal{H}
\]
for every vector field $V \in C_c^\infty(\Omega, T\Omega)$ (see [10]). Since $H$ is unique, we set $H = \nabla u$. Of course, if $u$ is a smooth function its usual Riemannian gradient is also the weak gradient. We shall also use the following fact:

**Lemma 2.1** ([8], Proposition 2.4) If $u : \Omega \to \mathbb{R}$ is a Lipschitz function, then $u \in H^{1,p}_{\text{loc}}(\Omega)$ for every $p \geq 1$.

### 3 Maximum principles for inhomogeneous inequalities

In this section we extend to a Riemannian setting the results concerning Maximum Principles for solutions of inhomogeneous elliptic inequalities proved in [17, Chapter 6] for the Euclidean case. Therefore, we cover their results, but at the same time we establish more precise a priori estimates, in which we make explicit the dependence on all the coefficients appearing in the inequality.

Our results apply to a large class of differential inequalities, including, in particular, those involving the $p$-Laplace–Beltrami operator defined for a smooth function $u$ as $\Delta_p u := \text{div} \left( |\nabla u|^{p-2} \nabla u \right)$, $p > 1$, or the mean curvature operator given by $\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right)$; but they also apply to more general and sophisticated differential operators on Riemannian manifolds, which are elliptic according to the new definition of ellipticity proposed in [1], to which we refer.

From now on, we assume $\Omega$ to be a bounded and smooth domain of $\mathcal{M}$, so that $\overline{\Omega}$ is a smooth manifold with boundary. However, we can also treat the case of $\Omega$ unbounded but with finite measure; in this case the boundary condition “$u \leq M$ on $\partial \Omega$” is replaced by
\[
\limsup_{|x| \to \infty} u(x) \leq M,
\]
while by $u \leq M$ on $\partial \Omega$ we mean that for every $\delta > 0$ there exists a neighborhood of $\partial \Omega$ in which $u \leq M + \delta$. In fact, $u$ will be assumed only of class $H^{1,p}_{\text{loc}}(\Omega)$, so that it may have no trace on $\partial \Omega$.

We consider inequalities of the form
\[
\text{div} \, A(x, u, \nabla u) + B(x, u, \nabla u) \geq 0 \quad \text{in} \ \Omega,
\]
where divergence and gradient are taken with respect to the Riemannian structure.
We assume that $A : T\Omega \times \Omega \rightarrow R$, where $T\Omega \times \Omega$ stands for $T\Omega \times \Omega$ as already mentioned, and $A(x, z, \xi) \in T_x V$ for all $x \in \Omega$, $z \in \Omega$ and $\xi \in T_x V$, while $B$ is a real function defined in $T\Omega \times \Omega$. We also suppose that there exist $p \geq 1$, $a_1 > 0$, $a_2$, $b_1$, $b_2$, $b \geq 0$ such that for all $(x, z, \xi) \in T\Omega \times \Omega$ there hold
\begin{equation}
\langle A(x, z, \xi), \xi \rangle \geq a_1|\xi|^p - a_2|z|^p - a^p, \tag{3.3}
\end{equation}
\begin{equation}
B(x, z, \xi) \leq b_1|\xi|^{p-1} + b_2|z|^{p-1} + b^{p-1}
\end{equation}
if $p > 1$, and
\begin{equation}
\langle A(x, z, \xi), \xi \rangle \geq a_1|\xi| - a_2|z| - a, \quad B(x, z, \xi) \leq b
\end{equation}
if $p = 1$. Of course, by a rescaling argument it is enough to consider only the case $a_1 = 1$, so without loss of generality we assume $a_1 = 1$ throughout the rest of the paper.

**Definition 3.1** A (weak) solution of (3.2) is a function $u \in H^1_{loc}(\Omega)$ such that
\begin{equation}
A(\cdot, u, \nabla u) \in L^1_{loc}(\Omega; T\Omega), \quad B(\cdot, u, \nabla u) \in L^p_{loc}(\Omega),
\end{equation}
where $p' = p/(p-1)$ if $p > 1$ and $p' = \infty$ if $p = 1$, and such that
\begin{equation}
\int \langle A(x, u, \nabla u), \nabla \phi \rangle d.m. \leq \int B(x, u, \nabla u) \phi d.m.
\end{equation}
for all nonnegative $\phi \in H^1_{loc}(\Omega)$ such that $\phi = 0$ a.e in some neighborhood of $\partial \Omega$. Furthermore, we say that $u$ is a $p$–regular solution if $u$ is a solution of (3.2) and in addition
\begin{equation}
A(\cdot, u, \nabla u) \in L^p_{loc}(\Omega; T\Omega).
\end{equation}

**Remark 3.1** Condition (3.6) reads $A(\cdot, u, \nabla u) \in L^\infty_{loc}(\Omega; T\Omega)$ when $p = 1$. Of course, we can substitute our results for $p$–regular solutions, $p \geq 1$, assuming for example that $u \in H^1_{loc}(\Omega)$ and, for some nonnegative constants $\gamma_i$, $i = 1, 2, 3$,
\begin{equation}
|A(x, s, \xi)| \leq \gamma_1 + \gamma_2|s|^{p-1} + \gamma_3|\xi|^{p-1}
\end{equation}
for all $(x, z, \xi) \in T\Omega \times \Omega$.

We now list our main results, where we denote by $C$ generic constants which may depend only on $p$, $n$ and $|\Omega|$.

**Theorem 3.1 (Semi–Maximum principle)** Let $u$ be a $p$–regular solution of (3.2) with $A$ and $B$ satisfying (3.3) or (3.4). Assume also that $u \leq M$ on $\partial \Omega$ for some constant $M \geq 0$. Then $u^+ \in L^\infty(\Omega)$ and there exists a universal constant $C = C(n, p, |\Omega|) > 0$ such that
\begin{equation}
u \leq M + C(\|u^+\|_p + a + b + K) \quad a.e. \quad in \ \Omega,
\end{equation}
where $K = K(a_2, b_1, b_2)$ is given by
\begin{equation}
K = \begin{cases}
[a_1 + (a_2 + b_2)^{1/p}]^{n/p} + (a_2^{1/p} + b_2^{1/(p-1)})M, & \text{if } p > 1, \\
[a_2^{n/p} + a_2M, & \text{if } p = 1.
\end{cases}
\end{equation}
The same result can be given if we let the coefficients in (3.3) and (3.4) depend on the $x$–variable with some regularity. More precisely, denoting simply by $\|f\|$ the norm in $L^q(\Omega)$ of $f$ when $q$ is assigned, we have the following generalization.

**Theorem 3.2** Let $a, a_2, b, b_1, b_2$ belong to some Lebesgue spaces, in particular 
\[ a, b_1 \in L^{\alpha}(\Omega), \quad b \in L^{\alpha(p-1)}(\Omega), \quad a_2, b_2 \in L^\alpha(\Omega), \]
\[ \alpha = \max\{n/p, 1\}, \quad \varepsilon \in (0, 1], \]
and let $u$ be a $p$–regular solution of (3.2) with $A$ and $B$ satisfying (3.3) or (3.4). If $u \leq M$ on $\partial \Omega$ for some constant $M \geq 0$, then $u^+ \in L^\infty(\Omega)$ and there exists a universal constant $C = C(n, p, |\Omega|, \varepsilon) > 0$ such that 
\[ u \leq M + C \left\{ (1 + \|b_1\| + \|a_2 + b_2\|^{1/p})^\nu (\|u^+\|_p + \mathcal{K}), \quad \text{if } p > 1, \right. 
\[ \left. (1 + \|a_2 + b\|)^\nu (\|u^+\|_1 + \|a_2\| + M\|a_2\|), \quad \text{if } p = 1, \right. \]  
where $\nu = n/\varepsilon p$ if $n > p$ and $\nu = 4/\varepsilon$ if $p \geq n$, and for brevity 
\[ \mathcal{K} = \|a\| + \|b\| + (\|a_2\|^{1/p} + \|b_2\|^{1/(p-1)})M. \]

The limit case $\varepsilon = 1$ is also allowed, provided that $\alpha$ is replaced by $\infty$.

The embedding of $H^1_0(\Omega)$ in $L^{n/(n-1)}(\Omega)$ is used below, with an obvious meaning when $n = 1$, and $S = S(1, n)$ denotes the corresponding Sobolev constant. With the aid of the previous estimates (3.8), one can prove the following general results.

**Theorem 3.3** (Maximum principle) Let $u$ be a $p$–regular solution of (3.2) in $\Omega$, where $A$ and $B$ satisfy (3.3), with $b_1 = b_2 = 0$, or (3.4). Suppose $u \leq M$ on $\partial \Omega$ for some constant $M \geq 0$ and, if $p = 1$, we also assume that 
\[ a_2 + b \leq |\Omega|^{-1/n}(1-\delta)/S, \]  
where $\delta \in (0, 1)$. Then $u^+ \in L^\infty(\Omega)$ and there exists $C = C(n, p, |\Omega|) > 0$ such that 
\[ u \leq M + C(a + b + Ma_2^{1/p})e^{a_2^{(p-1)/p}} \quad \text{a.e. in } \Omega, \]  
if $p > 1$, while 
\[ u \leq M + \frac{Ca}{\delta}(1 + a_2 + b) \quad \text{a.e. in } \Omega \]  
if $p = 1$.

**Theorem 3.4** (Maximum principle: case $p > 1$) Let $u$ be a $p$–regular solution of (3.2) in $\Omega$, where $A$ and $B$ satisfy (3.3), with $a_2 = b_2 = 0$. If $u \leq M$ on $\partial \Omega$ for some constant $M \geq 0$, then $u^+ \in L^\infty(\Omega)$ and 
\[ u \leq M + (a + b)e^{C(1 + b_1)^{p-1}} \quad \text{a.e. in } \Omega. \]
Again, the result can be generalized if the coefficients in (3.3) or (3.4) belong to some suitable Lebesgue spaces.

**Theorem 3.5** Theorems 3.3 and 3.4 continue to be valid if the coefficients \(a, b, a_2\) and \(b_1\) are functions in the following Lebesgue spaces:

\[
\beta = \begin{cases} 
\frac{n}{p}(1 - \varepsilon), & \text{if } 1 < p \leq n, \\
1, & \text{if } p > n,
\end{cases}
\quad \varepsilon \in (0, 1],
\]

and \(a, b, a_2 \in L^\sigma(\Omega)\) for some \(\sigma > n\) if \(p = 1\). In the latter case also assume that

\[
\|a_2 + b\|_n \leq \frac{1 - \delta}{S}.
\]

Under these hypotheses, (3.10) becomes

\[
u \leq M + C\|a\| + \|b\| + M\|a_2\|^{1/p}\|a_2\|^{(n+1)/p} \quad \text{a.e. in } \Omega,
\]
while (3.11) is replaced by

\[
u \leq M + \frac{C\|a\|_1}{\delta} (1 + \|a_2 + b\|_\sigma) \quad \text{a.e. in } \Omega,
\]
and (3.12) changes into

\[
u \leq M + (\|a\| + \|b\|)e^{C(1 + \|b_1\|)^{\nu+1}} \quad \text{a.e. in } \Omega,
\]
with \(\nu\) in (3.15) and (3.17) defined as in Theorem 3.2.

Let us note that the Lebesgue spaces allowed for the coefficients in Theorems 3.2 and 3.5 are different.

For the applications and examples of Theorem 3.5 in the Euclidean case when \(p \geq 1\), we refer to [17, Sections 6.1 and 6.5]. In the context of \(n\)-dimensional Cartan–Hadamard manifolds\(^1\) Croke establishes in [5], see also [9, Section 8.2], the explicit formula for an upper bound of the best Sobolev constant \(S = S(1, n)\), that is

\[
C(n) := \frac{\omega_n^{-2}}{\omega_n^{-1}} \left( \int_0^{\pi/2} (\cos t)^{n/2}(\sin t)^{n-2}dt \right)^{n-2} \geq S(1, n),
\]

where \(\omega_n\) denotes the \(n\)-dimensional surface measure of the unit sphere \(S^n\) of \(\mathbb{R}^{n+1}\). Moreover, Croke shows that \(C(4) = S(1, 4) = 2^{-7}\pi^{-2}\), and for the main prototype when \(p = 1\) and \(a_2 = 0\), that is

\[
\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) + B(x, u, \nabla u) \geq 0,
\]

\(^1\)A Cartan–Hadamard manifold is, by definition, a smooth complete simply connected Riemannian manifold of nonpositive sectional curvature.
conditions (3.9) and (3.14), when \( n = 4 \), read respectively
\[
b \leq 2^7 \pi^2 (1 - \delta) |\Omega|^{-1/4},
\]
and
\[
\|b\|_4 \leq 2^7 \pi^2 (1 - \delta),
\]
where \( b \) is the bound in (3.4). Thus Theorems 3.3 and 3.5 apply under these restrictions. In particular, any solution \( u \) of (3.18) with \( u \leq M \) on \( \partial \Omega \), \( M \geq 0 \), verifies
\[
u \leq M + C a (1 + \|b\|_4) \quad \text{a.e. in } \Omega, \text{ where } a = \sqrt{\frac{5\sqrt{5} - 11}{2}},
\]
and \( C = C(|\Omega|) > 0 \) is a universal constant.

## 4 Proofs of theorems 3.1 and 3.2

At a first step we prove that \( M \) can be taken 0, so that in the following lemmas it will be a general assumption. Indeed, if \( p > 1 \) and \( \tilde{u} = u - M \), then \( \tilde{u} \leq 0 \) on \( \partial \Omega \). Furthermore \( \tilde{u} \) satisfies (3.2), with \( A \) replaced by \( \tilde{A} \) and \( \bar{A} \) defined as
\[
\tilde{A}(x, s, \xi) = A(x, s - M, \xi).
\]
Moreover, \( \tilde{A} \) satisfies inequalities like those in (3.3) with the coefficients \( a_2, b_2, a, b \) respectively replaced by
\[
\tilde{a}_2 = 2^{p-1} a_2, \quad \tilde{a} = (a^p + 2^{p-1} a_2 M^p)^{1/p},
\]
\[
\tilde{b}_2 = 2^{p-1} b_2, \quad \tilde{b} = 2^p (b + b_2^{1/(p-1)} M),
\]
while \( a_1 \) is again kept equal to 1 and \( b_1 \) is not changed. Indeed, by convexity,
\[
a_2 u^p = a_2 |\tilde{u} + M|^p \leq 2^{p-1} a_2 (|\tilde{u}|^p + M^p).
\]
Thus \( \tilde{a}_2 = 2^{p-1} a_2 \), and \( \tilde{a}^p = a^p + 2^{p-1} a_2 M^p \), as required by (4.19). In the same way \( b_2 \) and \( b \) can be treated, proving (4.19) completely.

Now take \( k = \|\tilde{a}\| + \|\tilde{b}\| \). Then \( \tilde{u} = u - M \) obeys (3.8) or analogous a priori estimates with the constant depending on \( p, n, |\Omega|, \varepsilon, \|b_1\| \) and \( \|\tilde{a}_2 + \tilde{b}_2\| \). The conclusions of Theorems 3.1 and 3.2 are thus easily obtained for \( p > 1 \). The case \( p = 1 \) can be treated similarly and more simply.

The following result, which is a generalization of [1, Lemma 3.2] and whose proof is similar to the one therein, will be used later.

**Lemma 4.1** Let \( \psi : \mathbb{R} \to \mathbb{R}_0^+ \) be a non decreasing continuous function such that \( \psi(t) = 0 \) for \( t \in (-\infty, \ell] \), \( \ell > 0 \), and \( \psi \) is of class \( C^1 \) in \( [\ell, \infty) \) with \( \psi' \) bounded. Assume that \( u \) is a \( p \)-regular solution of (3.2), \( p \geq 1 \), and \( v \in H^{1,p}_{loc}(\Omega) \) is such that \( v \leq \ell' < \ell \) on \( \partial \Omega \). Then (3.5) is valid for \( \phi = \psi \circ v \), in the sense that
\[
\int_\Omega <A(x, u, \nabla u), \nabla \phi> \, d.M \leq \int_\Omega [B(x, u, \nabla u)]^+ \phi \, d.M, \tag{4.20}
\]
where \( \nabla \phi = \psi'(v) \nabla v \) when \( v \neq \ell \).
We begin with the case of constant coefficients. Set \( p^* = np/(n-p) \) and denote by \( S \) the Sobolev constant for the embedding \( H^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \).

**Lemma 4.2** Let \( u \) be a \( p \)-regular solution of inequality (3.2) such that \( u \leq 0 \) on \( \partial \Omega \), and suppose that \( A \) and \( B \) satisfy (3.3) or (3.4) with \( 1 \leq p < n \). Set \( w = u^+ + k \), where \( k = a + b \geq 0 \). Then \( w \in L^\infty(\Omega) \) and

\[
w \leq C \left[ 1 + b_1 + p^{1/p}(a_2 + b_2 + 2)^{1/p} \right] w_p \quad \text{a.e. in } \Omega,
\]

where \( C = C(n, p, |\Omega|) > 0 \).

**Proof.** Clearly \( w \geq k \) in \( \Omega \), \( w = k \) on \( \partial \Omega \), and of course \( w \in H^{1,p}_{loc}(\Omega) \).

**Step 1.** Let \( \ell, m \) be fixed with \( 0 \leq k < \ell < m \) and take \( q \) and \( r \) in \( \mathbb{R} \), with \( q, r \geq 1 \) such that

\[
q - 1 = p(r - 1).
\]

Set

\[
\psi(t) = \begin{cases} 
0, & \text{if } t \leq \ell, \\
\frac{r^p}{q} t^q - \ell^q, & \text{if } \ell < t < m, \\
qm^{p-1}t - (q-1)m^q - \ell^q, & \text{if } t \geq m;
\end{cases}
\]

\[
\zeta(t) = \begin{cases} 
\ell^r, & \text{if } t \leq \ell, \\
r^{r-1}t - (r-1)m^r, & \text{if } t \geq m.
\end{cases}
\]

In this way, \( \psi \) and \( \zeta \) are convex, piecewise smooth except at the corner \( t = \ell \), and linear when \( t \geq m \).

By Lemma 4.1, with \( v = w \) and \( \ell' = k \), we can take \( \varphi = \psi(w) \) as test function for (3.2), and by (4.20) we get

\[
\int_\Omega \langle A(x, u, \nabla u), \nabla \varphi \rangle \, d\mathcal{H} \leq \int_\Omega [B(x, u, \nabla u)]^\top \varphi \, d\mathcal{H}.
\]

When \( w \leq \ell \) we have \( \varphi = 0 \), so that the integrals are evaluated only over the set \( \Lambda = \{ x \in \Omega : \ell < w(x) \} \). But in \( \Lambda \) necessarily \( u(x) > 0 \), so that \( u^+ = u \), in other words

\[
u = w - k \quad \text{and} \quad \nabla u = \nabla w \quad \text{in } \Lambda.
\]

Moreover, \( \nabla \varphi = \psi'(w)\nabla w \); hence by (3.3) and (4.25), since \( w > u > 0 \) in \( \Lambda \),

\[
\langle A(x, u, \nabla u), \nabla \varphi \rangle \geq \psi'(w)(|\nabla w|^p - a_2 w^p - a^p),
\]

\[
[B(x, u, \nabla u)]^\top \varphi \leq \psi(w)(b_1 |\nabla w|^{p-1} + b_2 w^{p-1} + b^{p-1})
\]

if \( p > 1 \), and if \( p = 1 \)

\[
\langle A(x, u, \nabla u), \nabla \varphi \rangle \geq \psi'(w)(|\nabla w| - a_2 w - a),
\]

\[
[B(x, u, \nabla u)]^\top \varphi \leq b \psi(w).
\]
To evaluate the right hand sides of (4.26) and (4.27) we require some preliminary estimates. First, by (4.21) we have

$$
\psi'(t) = [\zeta'(t)]^p,
$$

(4.28)

and, since $\zeta$ is non decreasing,

$$
\psi(t) = \int_0^t [\zeta'(s)]^p ds \leq [\zeta'(t)]^p - 1 \int_0^t \zeta'(s) ds \leq \zeta(t)[\zeta'(t)]^p - 1,
$$

(4.29)

which holds for any $p \geq 1$.

Moreover, using (4.23) and the fact that $r \geq 1$, one finds that

$$
t\zeta'(t) \leq r\zeta(t).
$$

(4.30)

Setting $v = \zeta \circ w$, the terms on the right side of (4.26) and (4.27) have the following estimates:

$$
\begin{align*}
\psi'(w)|\nabla w|^p &= |\zeta'(w)|^p|\nabla w|^p = |\nabla v|^p, \\
\psi'(w)w^p &= |w\zeta'(w)|^p \leq r^pv^p, \\
\psi(w)|\nabla w|^{p-1} &\leq v[\zeta'(w)]^{p-1} = v|\nabla v|^{p-1} \quad \text{by (4.28) and (4.30),} \\
\psi(w)w^{p-1} &\leq \zeta(w)|w\zeta'(w)|^{p-1} \leq r^{p-1}v^p \quad \text{by (4.29) and (4.30),}
\end{align*}
$$

which hold true also when $p = 1$.

If $p = 1$ we note that the last inequality above gives

$$
\psi(w) \leq v,
$$

(4.31)

while in general we need the following estimates: by (4.28) and (4.30)

$$
\psi'(w) = [\zeta'(w)]^p \leq r^pv^p \leq \frac{r^pv^p}{w^p},
$$

(4.32)

since $w \geq k$. In addition (4.29) gives similarly

$$
\psi(w) \leq v[\zeta'(w)]^{p-1} \leq \frac{r^{p-1}v^p}{k^{p-1}}.
$$

With the estimates above, using the fact that $r \geq 1$, then (4.24) becomes

$$
\int_\Omega |\nabla v|^p d.M \leq \int_\Omega b_1v|\nabla v|^{p-1} d.M + r^p \int_\Omega (a_2 + b_2 + c_2)v^p d.M,
$$

(4.33)

where (recalling that $w \geq k$)

$$
c_2 = (a/k)^p + (b/k)^{p-1}, \quad \text{or} \quad c_2 = 0 \quad \text{if} \quad k = 0, \quad \text{i.e.} \ a = b = 0.
$$

(4.34)

Moreover, if $p = 1$ then (4.26) gives

$$
\|\nabla v\|_1 \leq \int_\Omega [a_2w + b\psi'(w) + b\psi(w)] d.M.
$$
By (4.22) one has $\psi'(w) \leq 1$ and by (4.31) inequality (4.33) is replaced by
\[
\int_{\Omega} |\nabla v| \, dM \leq \int_{\Omega} [(a_2 + b)v + a] \, dM, \tag{4.35}
\]
which will be used later.

The integrals in (4.33) and (4.35) are well defined, since $v \in H^{1,p}(\Omega)$ as a consequence of (4.23), being $w = k$ on $\partial \Omega$.

**Step 2.** The argument requires the following two sublemmas.

**Lemma 4.3 (Lemma 6.2.2 of [17])** Let $\alpha, \beta > 0$ and $p \geq 1$. If $z^p \leq \alpha z^p - 1 + \beta$, then also
\[
z \leq \alpha + (p \beta)^{1/p}, \quad z^p \leq \alpha^p + p \beta.
\]

**Lemma 4.4** With the notations of Lemma 4.2, we have $\|v\|_{p^{\ast}} < \infty$ and
\[
\|v\|_{p^{\ast}} \leq S\|\nabla v\|_p + |\Omega|^{1/p^{\ast} - 1/p}\|v\|_p. \tag{4.36}
\]

**Proof.** Since $v \in H^{1,p}(\Omega)$ and $v \equiv \ell^{r}$ near $\partial \Omega$, we immediately get that $v \in H^{1,p}(\Omega)$ and $\|v - \ell^{r}\|_{p^{\ast}} \leq S\|\nabla v\|_p$ by Sobolev’s inequality. Moreover, $\|\ell^{r}\|_{p^{\ast}} = |\Omega|^{1/p^{\ast} - 1/p}\|\ell^{r}\|_p$. Therefore
\[
\|v\|_{p^{\ast}} \leq \|v - \ell^{r}\|_{p^{\ast}} + \|\ell^{r}\|_{p^{\ast}} \leq S\|\nabla v\|_p + |\Omega|^{1/p^{\ast} - 1/p}\|\ell^{r}\|_p
\]
and the lemma now follows since $v \geq \ell^{r}$ in $\Omega$. \hfill \square

**Step 3.** By Hölder’s inequality
\[
\int_{\Omega} v|Dv|^{p-1} \, dM \leq \|v\|_p\|\nabla v\|_p^{p-1}.
\]
Set
\[
z = \|\nabla v\|_p/\|v\|_{p^{\ast}}, \quad y = \|v\|_p/\|v\|_{p^{\ast}},
\]
which are well defined, since $\|v\|_{p^{\ast}} > 0$. Then from (4.33) there follows, after division by $\|v\|_{p^{\ast}}$,
\[
z^p \leq b_1 y z^{p-1} + cr^p y^p,
\]
where $c = a_2 + b_2 + 2$. To see this we recall that $k = a + b$; hence by (4.34) there holds $c_2 \leq 2$ and $a_2 + b_2 + c_2 \leq c$. This being shown, from Lemma 4.3 we obtain
\[
z \leq [b_1 + (pc)^{1/p}] y \leq dy, \tag{4.37}
\]
where $d = b_1 + (pc)^{1/p}$, since $r \geq 1$.

Inequality (4.36) can be rewritten in the form
\[
1 \leq Sz + Cy. \tag{4.38}
\]
Consequently, by (4.37) we get $1 \leq (C + Sdr)y$. In turn, using the definition of $y$, there results
\[ \|v\|_{p^r} \leq (C + Sdr)\|v\|_p. \] (4.39)

By (4.23), the left hand side of (4.39) can be replaced by the smaller norm $\|w\|_{p^{r}, \Gamma}$, where $\Gamma = \{x \in \Omega : t \leq w(x) < m\}$; while on the right the term $\|v\|_p$ can be replaced by the larger one $\|w\|_{p^r} + (\ell^r - k^r)\|\Omega\|^{1/p}$. Indeed, this is a consequence of the fact that $\|v\|_{p} \leq \|v\|_{p^r} + (\ell^r - k^r)$, as one can prove considering that $\theta = \|v\|_{p^r} + (\ell^r - k^r)$, which is meaningful provided that $\beta \in L^{p^r}(\Omega)$.

Indeed, we claim that $\|w\|_{p^r} \leq (Kr)^{1/\kappa}\|w\|_{p^r}$; $\kappa = p^*/p = n/(n - p)$, $K = 1 + Sd$, (4.40) which is meaningful provided that $w \in L^{p^r}(\Omega)$.

Next, take $r = p^*/p = \kappa$, so that (4.40) and (4.41) give
\[ \|w\|_{p^{\kappa r}} \leq (K\kappa)^{1/\kappa}\|w\|_{p^r} \leq (K\kappa)^{1/\kappa}K\|w\|_p = K^{1+1/\kappa}\kappa^{1/\kappa}\|w\|_p. \] (4.42)

Iterating this process, with $r$ successively equal to $\kappa$, $\kappa^2$, etc., we get
\[ \|w\|_{p^{\kappa^j r}} \leq K^{\Sigma_j} \kappa^{\Sigma^\prime_j}\|w\|_p, \] (4.43)

where
\[ \Sigma_j = \sum_{i=0}^{j-1} \frac{i}{\kappa^i}, \quad \Sigma^\prime_j = \sum_{i=1}^{j-1} \frac{i}{\kappa^i}. \]

The series $\Sigma_j$ converges to $\kappa/(\kappa - 1) = n/p$ as $j \to \infty$. Similarly the series $\Sigma^\prime_j$ converges to $\kappa/(\kappa - 1)^2 = n(n - p)/p^2$. Thus letting $j \to \infty$ in (4.42) gives
\[ \|w\|_{p^\infty} \leq K^{n/p} \left( \frac{n}{n - p} \right)^{n(n - p)/p^2} \|w\|_p \]
\[ = \left[ 1 + \frac{p}{n - p} \right]^{(n-p)/p} K^{n/p} \|w\|_p \]
\[ \leq (Ke)^{n/p}\|w\|_p, \]

since $(1 + 1/t)^t < e$ for all $t > 0$. Here
\[ K = 1 + Sd = 1 + S[b_1 + (pc)^{1/p}] = 1 + S[b_1 + p^{1/p}(a_2 + b_2 + 2)^{1/p}], \]
and the lemma is proved. \(\square\)

In the general case, that is when $p \geq 1$ and the coefficients in (3.3) or (3.4) are $x$-dependent, similar but more sophisticated calculations are needed.
Lemma 4.5  Let the hypotheses of Lemma 4.2 be satisfied for \( p \geq 1 \), and assume that the coefficients in (3.3) or (3.4) are functions in the Lebesgue spaces indicated in (3.7). Let \( k = \|a\| + \|b\| \). Then \( w = u^+ + k \in L^\infty(\Omega) \) and

\[
w \leq C\|w\|_p \left\{ \begin{array}{ll}
(1 + \|b_1\| + \|a_2 + b_2\|^{1/p})^\nu, & \text{if } p > 1, \\
(1 + \|a_2 + b\|)^\nu, & \text{if } p = 1,
\end{array} \right.
\]  \hspace{1cm} (4.44)

where \( C = C(n, p, \varepsilon, |\Omega|) > 0 \) and \( \nu = n/\varepsilon p \) if \( 1 \leq p < n \) and \( \nu = 4/\varepsilon \) if \( p \geq n \).

We recall that by \( \|a\|, \|b\|, \|b_1\| \) and \( \|a_2 + b_2\| \) we mean the norms of \( a, b, a_1, \) \( a_2 + b_2 \) in the respective Lebesgue spaces (3.7).

Proof. We follow the proof of Lemma 4.2, but now with the coefficients of (3.3) in the Lebesgue spaces indicated in (3.7), and with \( k = \|a\| + \|b\| \). We first consider the case \( p > 1 \).

Step 1’. With \( \varphi \) and \( v \) defined as in the proof of Lemma 4.2, and proceeding exactly as before, we obtain again inequality (4.33). Step 2 is then replaced by

Step 2’. We claim the following statement: Let \( \theta = 1 \) and \( s = p^*/(n-p) \) if \( 1 < p < n \), while \( \theta = 2 \) and \( s = 2p/\varepsilon \) if \( p \geq n \); then there exists \( C = C(|\Omega|) > 0 \) such that

\[
\begin{align*}
\int_{\Omega} b_1 v|\nabla v|^{p-1} \, d\mathcal{H} & \leq C\|b_1\|_{\Omega^p}\|v\|_{p,\alpha}^{1/\theta}\|v\|_{s}^{1-\epsilon/\theta}\|\nabla v\|_{p}^{p-1}, \\
\int_{\Omega} (a_2 + b_2)v^p \, d\mathcal{H} & \leq C\|a_2 + b_2\|_{\Omega}\|v\|_{p,\alpha}^{p/\theta}\|v\|_{s}^{p(1-\epsilon/\theta)}, \\
\int_{\Omega} a^p v^p \, d\mathcal{H} & \leq C\|a\|_{\Omega^p}\|v\|_{p,\alpha}^{p/\theta}\|v\|_{s}^{p(1-\epsilon/\theta)}, \\
\int_{\Omega} b^{p-1}v^{p} \, d\mathcal{H} & \leq C\|b\|_{(p-1)\alpha}\|v\|_{p,\alpha}^{p/\theta}\|v\|_{s}^{p(1-\epsilon/\theta)}.
\end{align*}
\]  \hspace{1cm} (4.45)

For the proof we first recall the following inequality, see [2, Section 7.1 of Chapter 3]:

\[
\text{if } \sum_{i=1}^{s} \gamma_i/p_i = 1, \text{ then } \int_{\Omega} \Pi_i |f_i|^{\gamma_i} \, d\mathcal{H} \leq \Pi_i \|f_i\|_{p_i}^{\gamma_i}. \]  \hspace{1cm} (4.46)

When \( p < n \) the first inequality in (4.45) is a direct consequence of (4.46) applied to the four-fold product \( b_1v^{\theta/\alpha}v^{1-\varepsilon/\theta}\|Dv\|^{p-1} \) with \( \alpha = n/p(1-\varepsilon) \). The remaining inequalities for \( p < n \) follow in the same way.

When \( p \geq n \), one has \( \alpha = 1/(1-\varepsilon) \), \( \theta = 2 \), \( s = 2p/\varepsilon \). Then for the first line we use (4.46) with the five-fold product \( b_1v^{\theta/\alpha}v^{1-\varepsilon/\theta}\|Dv\|^{p-1} \cdot 1 \) and the exponent relation

\[
\frac{1}{p\alpha} + \frac{\varepsilon}{p\theta} + \frac{1-\varepsilon/\theta}{s} + \frac{p-1}{p} + \frac{1}{\kappa} = 1, \quad \kappa = \frac{4p}{\varepsilon^2}.
\]

In this way, the term \(|\Omega|^{1/\kappa}\) appears in the inequality. The remaining inequalities follow in the same way, however with \( \kappa = 4/\varepsilon^2 \), so that we can take \( C = \max\{1, |\Omega|^{1/\kappa}\} = \max\{1, |\Omega|^{\varepsilon^2/4p}, |\Omega|^{\varepsilon^2/4}\} \).
From the last three inequalities in (4.45), we obtain as in the proof of Lemma 4.2
\[
\int_{\Omega} (a_2 + b_2 + c_2) v^p \, d\mathcal{H} \leq c \|v\|^{|p/\theta|} \|v\|^{p(1-\varepsilon/\theta)},
\]
(4.47)
where \(c = C(||a_2 + b_2|| + 2)\) and, as in the previous case, \(c_2\) is given in (4.34).

**Step 3’.** Set
\[
z = \|\nabla v\|/\|v\|, \quad y = \|v\|/\|v\|s,
\]
where \(s\) is given in Step 2’. Then from (4.33), using the first inequality of (4.45) together with (4.47), we find
\[
z^p \leq C\|b_1\| y^{\varepsilon/\theta} z^{p-1} + c r^p y^{p\varepsilon/\theta}.
\]
The rest of the proof is essentially the same as before, with (4.37) replaced by
\[
z \leq \{C\|b_1\| + (pc)^{1/p}r\} y^{\varepsilon/\theta}.
\]
Rewriting Lemma 4.4 with \(p^*\) replaced by \(s\), and so with \(S = \bar{S}\) now denoting the Sobolev constant for the embedding \(H^{1,p}_0(\Omega) \hookrightarrow L^s(\Omega)\), we get
\[
\|v\|_s \leq \bar{S}\|\nabla v\| + \bar{C}\|v\|_p,
\]
where \(\bar{C} = C(n,p,|\Omega|) > 0\). This leads to the analogue of (4.38), which in fact is now replaced by
\[
1 \leq \bar{S} d y^{\varepsilon/\theta} + \bar{C} y,
\]
with \(d = C\|b_1\| + (pc)^{1/p}\).

In turn, from Lemma 4.3 in the case \(z = 1\) and exponent \(\theta/\varepsilon\geq 1\), one gets
\[
1 \leq \left[ (\bar{S} r)^{\theta/\varepsilon} + C_{\varepsilon}^\theta \right] y.
\]
By definition of \(y\), it now follows that
\[
\|v\|_s \leq (\bar{K} \varepsilon)^{\theta/\varepsilon} \|v\|_p,
\]
where
\[
\bar{K} = \bar{S} d + (C\theta/\varepsilon)^{1/\theta} \leq \bar{S} d + C\varepsilon/\varepsilon, \bar{K}^{1/\varepsilon}.
\]
Reverting to the variable \(w\), we get
\[
\|w\|_{\kappa pr} \leq (\bar{K} \varepsilon)^{\theta/(\varepsilon r)} \|w\|_{pr}, \quad \kappa = s/p,
\]
(4.48)
and we proceed as in the case of constant coefficients.

Now the proof can be concluded by iteration, as in the case of Lemma 4.2. In fact, when \(1 < p < n\) we have \(\theta = 1, \kappa = n/p^* = n/(n - p)\), so the same calculation used in the derivation of (4.43) gives now
\[
\|w\|_\infty \leq (\bar{K} \varepsilon)^{n/p^*} \|w\|_p.
\]
(4.49)
Starting from (4.35), by (4.32) we get

\[ \|w\|_{\infty} \leq (\bar{K}^{2/\varepsilon}) \| \left( (2/\varepsilon)^{2/\varepsilon} \right) w \|_p \leq (\bar{K} e^{2/\varepsilon})^{4/\varepsilon} \|w\|_p, \]  

(4.50)

since \( \bar{K} > 1 \) and \((2/\varepsilon)^{2/\varepsilon} \leq e^{2(2-\varepsilon)^2} \leq e^{2/\varepsilon} \).

In conclusion, (4.44) follows with the stated constants from (4.49) when \( 1 < p < n \) and from (4.50) when \( p \geq n \).

For the case \( p = 1 \), when \( b_1 = b_2 = 0 \), the calculation is made in a different way. Starting from (4.35), by (4.32) we get

\[ \int_{\Omega} |\nabla v| \, d\mathcal{M} \leq r \int_{\Omega} \left[ (a_2 + b) v + \frac{a}{\bar{K}} \right] v \, d\mathcal{M}. \]

Moreover, inequalities (4.45) are replaced by

\[ \int_{\Omega} (a_2 + b) v \, d\mathcal{M} \leq C \|a_2 + b\|_a \|v\|_{1/\varepsilon}^{\varepsilon/\theta} \|v\|_s^{(1-\varepsilon/\theta)}, \]
\[ \int_{\Omega} a v \, d\mathcal{M} \leq C \|a\|_a \|v\|_{1/\varepsilon}^{\varepsilon/\theta} \|v\|_s^{(1-\varepsilon/\theta)}, \]
\[ \int_{\Omega} b v \, d\mathcal{M} \leq C \|b\|_a \|v\|_{1/\varepsilon}^{\varepsilon/\theta} \|v\|_s^{(1-\varepsilon/\theta)}, \]

with \( \theta = 1 \) and \( s = n/(n-1) \) if \( n > 1 \) and \( \theta = 2 \) and \( s = 2/\varepsilon \) if \( p = n = 1 \).

Proceeding as above, we can take

\[ \bar{K} = C_1 (\|a_2 + b\| + 1), \]

(4.51)

with \( C_1 \) depending only on \( n, |\Omega| \) and \( \varepsilon \), to get

\[ \|w\|_{\infty} \leq (\bar{K} e^{2/\varepsilon}) \|w\|_1 \quad \text{and} \quad \|w\|_{\infty} \leq (\bar{K} e^{2/\varepsilon})^{4/\varepsilon} \|w\|_1, \]

(4.52)

which are the analogues of (4.49) and (4.50).

\( \square \)

**Proof of Theorem 3.2.** First set \( \tilde{u} = u - M \), so that \( \tilde{u} \leq 0 \) on \( \partial \Omega \).

**Step 1.** Consider first the case \( p > 1 \). Using the notations of (4.19), if \( k = ||\tilde{u}|| + ||\tilde{b}|| \) and \( \tilde{w} = \tilde{u}^+ + k \), Lemma 4.5 gives

\[ \tilde{w} \leq C (1 + \|b_1\| + \|\tilde{a}_2 + \tilde{b}_2\|^{1/p})^{\nu} \|\tilde{w}\|_p, \]

where the constant \( C \) depends only on \( n, p, \varepsilon \) and \( |\Omega| \). Going back to the variable \( u \), since \( ||u^+|| \leq ||u^+||_p \), with the aid of (4.19), we get

\[ u - M \leq C (1 + \|b_1\| + \|\tilde{a}_2 + \tilde{b}_2\|^{1/p})^{\nu} (||u^+||_p + K), \]

(4.53)

that is (3.8) when \( p > 1 \).

If \( p = 1 \), by (4.52) we obtain an analogous inequality, but slightly different, to (4.53), that is

\[ u - M \leq C (1 + \|a_2 + b\|^{1/p})^{\nu} (||u^+||_1 + ||a|| + M ||a_2||), \]

with the constant \( C > 0 \) depending again only on \( n, \varepsilon \) and \( |\Omega|. \)

\( \square \)
5 Proofs of Theorems 3.3, 3.4 and 3.5 for the case $p > 1$

We begin the section with some lemmas.

**Lemma 5.1** Let $A$ and $B$ satisfy (3.3), with $b_1 = b_2 = 0$, while $a, b, a_2$ are in the respective Lebesgue spaces as in (3.7). Let $1 < p \leq n$ and $u$ be a $p$–regular solution of (3.2) such that $u^+ \in L^\infty(\Omega)$ and $u \leq 0$ on $\partial\Omega$. Assume that $k = \|a\| + \|b\| > 0$ and $\|w\|_p \geq 2k|\Omega|^{1/p}$, where $w = u^+ + k$. Then there exists a constant $Q = Q(\Omega) > 0$ such that

$$\log \frac{W}{k} \leq 2\left(\frac{|\Omega|^{1/p} + Q(\|a_2\| + p'/p)}{\|w\|_p} - 1\right),$$

where $W = \|w\|_\infty$.

**Proof.** Of course, it is enough to treat only the non–trivial case $k < W$. Let $\ell$ be an arbitrary constant, with $k < \ell < W$, and set

$$\psi(t) = \begin{cases} 0, & \text{if } k \leq t \leq \ell, \\
\ell^{1-p} - t^{1-p}, & \text{if } \ell < t \leq W. \end{cases}$$

We choose $\varphi = \psi(w)$ as test function for (3.2). Putting $\Gamma = \{x \in \Omega : \ell < w(x) \leq W\}$, then $\varphi = 0, \nabla \varphi = 0$ in $\Omega \setminus \Gamma$ and $\nabla \varphi = (p-1)w^{-p}\nabla w$ in $\Gamma$. Therefore, from (4.20) and (3.3), we get

$$(p-1) \int_{\Gamma} w^{-p}(|\nabla w|^p - a_2w^p - a^p)dM \leq \int_{\Gamma}(b/\ell)^{p-1}dM. \tag{5.54}$$

In addition

$$\int_{\Gamma} \left[(p-1)\left(\frac{a}{k}\right)^p + \left(\frac{b}{k}\right)^{p-1}\right]dM \leq (p-1)\left(\frac{\|a\|}{k}\right)^p + \left(\frac{\|b\|}{k}\right)^{p-1} \leq p.$$ 

Since $w > \ell > k$ in $\Gamma$, then (5.54) gives

$$(p-1) \int_{\Gamma} |\nabla \log w|^p dM \leq (p-1)\|a_2\|_1 + p. \tag{5.55}$$

Note that $\log(w/\ell) \in H_0^{1,p}(\Gamma)$, so that by Poincaré’s inequality in $\Gamma$

$$\|\log(w/\ell)\|_{p,\Gamma} \leq Q_\Gamma \|\nabla \log(w/\ell)\|_{p,\Gamma} \leq Q_\Gamma(\|a_2\| + p'/p)^{1/p},$$

where $Q_\Gamma$ is the Poincaré constant in $\Gamma$.

However, $1 < w/\ell \leq W/\ell$ in $\Gamma$, and the function $t \mapsto \frac{t}{1 + \log(t/\ell)}$ is increasing if $t > \ell$, so that

$$w \leq \frac{W}{1 + \log(W/\ell)} \left(1 + \log \frac{w}{\ell}\right) \text{ in } \Gamma.$$
By integration and Minkowski’s inequality
\[ \|w\|_{p, \Gamma} \leq \frac{W}{1 + \log(W/\ell)} \left( 1 + \left\| \log \frac{w}{\ell} \right\|_{p, \Gamma} \right) \leq \left\{ 1 + Qr(|a_2| + p')^{1/p} \right\} \frac{W}{1 + \log(W/\ell)}. \] (5.56)

Now \( \|w\|_p \leq \|w\|_{p, \tilde{\Omega}} + k|\Omega|^{1/p} \), where \( \tilde{\Omega} = \{ x \in \Omega : w(x) > k \} \). Thus, since \( \|w\|_p \geq 2k|\Omega|^{1/p} \) by assumption, we get \( \|w\|_p \leq \|w\|_{p, \tilde{\Omega}} + \frac{1}{2}\|w\|_p \), that is \( \|w\|_p \leq 2\|w\|_{p, \tilde{\Omega}} \). Letting \( \ell \to k \), so that \( \|w\|_{p, \Gamma} \to \|w\|_{p, \tilde{\Omega}} \), by (5.56) it follows
\[ \|w\|_p \leq 2\|w\|_{p, \tilde{\Omega}} \leq 2\left\{ 1 + Q(|a_2| + p')^{1/p} \right\} \frac{W}{1 + \log(W/k)}, \]
where \( Q \) is a positive constant which bounds all Poincaré’s constants of the sets \( \Gamma \)’s.

Rearranging proves the lemma.

**Lemma 5.2** Let the hypotheses of Lemma 5.1 be satisfied, with the only exceptions that \( p > n \) and \( u^+ \) is no longer assumed a priori to be of class \( L^\infty(\Omega) \). If \( k > 0 \), then \( w \in L^\infty(\Omega) \) and
\[ \log \frac{W}{k} \leq Q_\infty(|a_2| + p')^{1/p}, \]
where \( Q_\infty \) denotes Morrey’s constant, depending only on \( n \) and \( p \).

**Proof.** Inequality (5.55) holds in the same way when \( p > n \). The lemma is then an immediate consequence of Morrey’s inequality. \( \square \)

**Proof of Theorem 3.5 when \( b_1 = b_2 = 0 \).** As usual, set \( \tilde{u} = u - M \), so that \( \tilde{u} \leq 0 \) on \( \partial\Omega \). With the notations of (4.19), set \( k = \|\tilde{a}\| + \|\tilde{b}\| \) and \( \tilde{w} = \tilde{u}^+ + k \). First suppose \( k > 0 \).

**Case 1:** \( \|\tilde{w}\|_p < 2k|\Omega|^{1/p} \). From Lemma 4.5 in the case \( 1 < p < n \) we get
\[ \tilde{u} \leq \tilde{w} \leq C(1 + \|\tilde{a}_2\|^{1/p})^{n/\varepsilon p} \|\tilde{w}\|_p \leq C_1(1 + \|\tilde{a}_2\|^{n/\varepsilon p})k, \] (5.57)
where the constant \( C_1 \) depends only on \( p, n, \varepsilon \) and \( |\Omega| \). Using again (4.19), we get
\[ u \leq M + C_2(1 + \|a_2\|^{n/\varepsilon p})(\|a\| + \|b\| + M\|a_2\|^{1/p}), \] (5.58)
with \( C_2 \) depending only on \( n, p, \varepsilon \) and \( |\Omega| \), that is inequality (3.15) in this case.

If \( p \geq n \) in (5.58) the only change is in the exponent \( n/\varepsilon p^2 \), which is replaced by \( 4/\varepsilon p^2 \).

**Case 2:** \( \|\tilde{w}\|_p \geq 2k|\Omega|^{1/p} \). If \( 1 < p \leq n \), Lemma 5.1 and Lemma 4.5 give
\[ \log \frac{\tilde{w}}{k} \leq C(1 + \|\tilde{a}_2\|)^{1/p + n/\varepsilon p^2}, \]
for a universal constant $C$, and so
\[
\tilde{u} \leq k \exp \left( C(1 + \|a_2\|^p) \right).
\]
By (4.19), the inequality above gives
\[
u \leq M + (\|a\| + \|b\| + M\|a_2\|^p)^{1/p} \exp \left( \tilde{C}(1 + \|a_2\|^p) \right),
\]
for a universal constant $\tilde{C}$, that is (3.15) holds true.

When $p > n$ we apply Lemma 5.2 and the conclusion follows as before using (4.50), with the exponent $(n + \varepsilon p)/\varepsilon^2$ replaced by $(\varepsilon + 4)/\varepsilon p$. If $k = 0$ replace $k$ by $\ell > 0$, then let $\ell$ go to zero and the assertion follows.

Theorem 3.3 is obtained from a very special case when $\varepsilon = 1$.

**Remark 5.1** That the coefficients are in different Lebesgue spaces in (3.13) when $1 < p \leq n$ and $p > n$ is due to the use of Lemma 4.5 in obtaining (5.57).

**Lemma 5.3** Let the hypotheses of Lemma 5.1 be satisfied, with the exception that now $a_2 = b_2 = 0$. Suppose $k = \|a\| + \|b\| > 0$, set $w = u + k$, $W = \|w\|_\infty$, and define
\[
v = \log \frac{W}{W - w + k}.
\]
Then $v \in H^1_0(\Omega) \cap L^\infty(\Omega)$ and there exists a universal constant $Q = Q(n, p, \Omega) > 0$ such that
\[
\|v\|_p \leq \frac{Q}{p - 1} \left( \|b_1\|_p + 1 \right).
\]
Moreover, $v$ satisfies an inequality of the form
\[
\text{div} \tilde{A}(x, v, \nabla v) + \tilde{B}(x, v, \nabla v) \geq 0 \quad \text{in } \Omega,
\]
for suitable $\tilde{A}$ and $\tilde{B}$ and with condition (3.3) now valid with $A$, $B$, $a_2$, $b_2$, $a$ and $b$ replaced respectively by $\tilde{A}$, $\tilde{B}$, $0$, $0$, $\tilde{a}$, $\tilde{b}$, where
\[
\tilde{a} = a/k, \quad \tilde{b}^{p-1} = (p - 1)(a/k)^p + (b/k)^{p-1}.
\]

**Proof.** **Step 1.** As in the proof of Lemma 5.1, we can assume $k < W$. Let $\ell \in (k, W)$, and define
\[
\psi(t) = \begin{cases} 0, & \text{if } k \leq t < \ell, \\ (W - t + \ell)^{1-p} W^{1-p}, & \text{if } \ell \leq t \leq W. \end{cases}
\]
Set $w = u + k$. Then $\varphi = \psi(w) \in H^1_0(\Omega)$, and it can be used as a test function for (3.2). Moreover,
\[
\nabla \varphi = \begin{cases} 0, & \text{in } \Omega \setminus \Gamma, \\ (p - 1)(W - w + \ell)^{-p} \nabla w, & \text{in } \Gamma, \end{cases}
\]
where $\Gamma = \{ x \in \Omega : \ell < w \leq W \}$. Starting from Lemma 4.1, using (3.3) with $a_2 = b_2 = 0$, since $u^+ = u$ and $\nabla w = \nabla u$ in $\Gamma$, we have

\[
(p - 1) \int_{\Gamma} (W - w + \ell)^{-p} (|\nabla u|^p - a^p) \, d\mathcal{M} \leq \int_{\Gamma} (W - w + \ell)^{1-p} (b_1 |\nabla w|^{p-1} + b^{p-1}) \, d\mathcal{M}.
\]

Recalling that $W - w + \ell \geq \ell > k$, this leads to

\[
(p - 1) \left\| \nabla \log(W - w + \ell) \right\|_{p, \Gamma} \leq \int_{\Omega} \left( b_1 |\nabla \log(W - w + \ell)|^{p-1} + \tilde{b}^{p-1} \right) \, d\mathcal{M}. \tag{5.63}
\]

For convenience, let $\tilde{v}$ be the function defined as $v$, but with $k$ replaced by $\ell$ in (5.59). Then (5.63) takes the form

\[
(p - 1) \left\| \nabla \log(W - w + \ell) \right\|_{p, \Gamma} \leq \int_{\Omega} \left( b_1 |\nabla \log(W - w + \ell)|^{p-1} + \tilde{b}^{p-1} \right) \, d\mathcal{M}. \tag{5.64}
\]

But \[
\left\| \tilde{b}^{p-1} \right\|_1 \leq |\Omega|^{1-1/\beta} |\tilde{b}^{p-1}|_\beta \leq C \left( \left\| (a/k)^p \right\|_\beta + \left\| (b/k)^{p-1} \right\|_\beta \right)
\]
\[
\leq C \left( \left\| a \right\|/k \right)^p + \left( \left\| b \right\|/k \right)^{p-1} \leq C,
\]

for some $C = C(n, p, \beta, |\Omega|)$, since $\left\| a \right\| \leq k$ and $\left\| b \right\| \leq k$. Moreover,

\[
\int_{\Omega} b_1 |\nabla \tilde{v}|^{p-1} \, d\mathcal{M} \leq \left\| b_1 \right\|_p \left\| \nabla \tilde{v} \right\|_{p}^{p-1}. \tag{5.66}
\]

If $\ell \to k$, then $\tilde{v} \to v$; from (5.64)-(5.66), and noting that $v = 0$ when $w = k$, we obtain

\[
\left\| \nabla v \right\|_p \leq \frac{\left\| b_1 \right\|_p}{p-1} \left\| \nabla \tilde{v} \right\|_{p}^{p-1} + \frac{|\Omega|}{p-1}.
\]

By Lemma 4.3

\[
\left\| \nabla v \right\|_p \leq \frac{\left\| b_1 \right\|_p}{p-1} + \left( \frac{p}{p-1} \right)^{1/p} \left[ \left\| \tilde{b}_1 \right\|_p + p|\Omega|^{1/p} \right],
\]

being

\[
\left( \frac{p}{p-1} \right)^{1/p} = \frac{p}{p-1} \left( \frac{p-1}{p} \right)^{-1/p} \quad \text{and} \quad \left( 1 - \frac{1}{p} \right)^{-1/p} < 1.
\]

Now $v = \log(W/(W - w + \ell)) \in H_0^{1,p}(\Omega)$, so that, by Poincaré’s inequality, there exists $Q > 0$ such that

\[
\left\| v \right\|_p \leq \frac{Q}{p-1} \left( \left\| b_1 \right\|_p + 1 \right),
\]

that is (5.60).
Step 2. We use an idea of Gilbarg and Trudinger in [7], already used in [17, Chapter 6]: let $\eta$ be a non-negative test function for (3.2) in $\Omega$. Set

$$\psi(t) = (W - t + k)^{1-p}, \quad k \leq t \leq W,$$

and take $\varphi = \eta \psi(w)$. Then, since $\psi$ and $\psi'$ are bounded in $[k, W]$, by Lemma 2.1 it follows that $\varphi \in H^{1,p}(\Omega) \cap L^\infty(\Omega)$, with $\varphi = 0$ near $\partial \Omega$. Hence, by a simple extension of Lemma 4.1, one can take $\varphi$ as a test function for (3.2). Now write

$$\mu = W - w + k$$

and observe that $\nabla \psi(w) = (p - 1)\mu^{-p}\nabla w$. Then, since $\varphi = 0$ and $\nabla \varphi = 0$ a.e. in the set where $w = k$, i.e. where $u \leq 0$, for a.e. $x \in \Omega$ we have

$$\langle A(x,u,\nabla u), \nabla \varphi \rangle - \langle B(x,u,\nabla u) + \varphi \rangle = \langle A(x,w - k,\nabla w), \nabla \varphi \rangle - \langle B(x,w - k,\nabla w) + \varphi \rangle$$

$$= \mu^{1-p} \langle A(x,w - k,\nabla w), \nabla \eta \rangle$$

$$+ (p - 1)\mu^{-p} \langle A(x,w - k,\nabla w), \nabla w \rangle \eta$$

$$- \mu^{1-p} [B(x,w - k,\nabla w)]^+ \eta$$

$$\geq \mu^{1-p} \langle A(x,w - k,\nabla w), \nabla \eta \rangle - [b_1(\nabla w/\mu)^{p-1} + b^{p-1}] \eta,$$

where we have used (3.3) at the last step with the inequality $\mu \geq k$ and the definition of $b$. From (5.59) we have $w - k = (1 - e^{-x})W$ and $\nabla w = We^{-x}\nabla v = \mu\nabla v$, so that integrating (5.67) over $\Omega$ and using (4.20) gives

$$\int_{\Omega} \{ \langle \bar{A}(x,v,\nabla v), \nabla \eta \rangle - B(x,v,\nabla v)\eta \} \, d\mathcal{H} \leq 0,$$

where

$$\bar{A}(x,v,\nabla v) = \mu^{1-p} A(x,w - k,\nabla w),$$

$$B(x,v,\nabla v) = b_1(\nabla w/\mu)^{p-1} + b^{p-1} = b_1|\nabla v|^{p-1} + b^{p-1}.$$  \hfill (5.68)

We claim that (3.3) holds for $\bar{A}$, $\bar{B}$ with $a_2$, $b_2$, $a$, $b$ respectively replaced by $0$, $0$, $a$, $b$. Indeed, since $a_2 = 0$, by (5.68),

$$\langle \bar{A}(x,v,\nabla v), \nabla v \rangle = \mu^{-p} \langle A(x,w - k,\nabla w), \nabla w \rangle \geq |\nabla v|^p - (a/k)^p,$$

proving the claim and the lemma. \hfill $\Box$

Lemma 5.4 Let the hypotheses of Lemma 5.2 be satisfied, with the exception that $a_2 = b_2 = 0$. Then $w \in L^\infty(\Omega)$ and there exists a universal constant $Q_\infty > 0$ such that

$$v = \log \frac{W}{W - w + k} \leq \frac{Q_\infty}{p - 1} (\|b_1\|_p + 2p), \quad W = \|w\|_\infty.$$
Proof. Inequality (5.60) holds equally when \( p > n \). The lemma is then an immediate consequence of Morrey’s inequality.

Proof of Theorem 3.5 when \( a_2 = b_2 = 0 \). As usual, put \( \tilde{u} = u - M \), so that \( \tilde{u} \leq 0 \) on \( \partial \Omega \). In this case \( \tilde{a} = a \) and \( \tilde{b} = b \).

First suppose \( 1 < p \leq n \) and \( k = \|a\| + \|b\| > 0 \). Put \( w = \tilde{u}^+ + k \), so that Lemma 5.3 applies to \( \tilde{v} = \log(W/(W - \tilde{w} + k)) \), where \( W = \|\tilde{w}\|_\infty \). Then \( \tilde{v} \in H_1^{1,p}(\Omega) \cap L^\infty(\Omega) \) and satisfies (5.61) with \( \tilde{a}_2 = \tilde{b}_2 = 0 \) and \( \tilde{a}, \tilde{b} \) given in (5.62). Therefore by Theorem 3.2 we get

\[
\tilde{v} \leq C(1 + \|b_1\|)^\nu(\|\tilde{w}\|_p + \tilde{k}), \tag{5.69}
\]

with \( C = C(n, p, \varepsilon, |\Omega|) > 0 \) and \( \tilde{k} = \|\tilde{a}\| + \|\tilde{b}\| \). On the other hand, from (5.65) one has \( \tilde{k} \leq C_1 \). Then by (5.69), together with (5.60) and the definition of \( \tilde{v} \), one obtains a.e. in \( \Omega \)

\[
\tilde{v} = \log\frac{W}{W - \tilde{w} + \tilde{k}} \leq C(1 + \|b_1\|)^\nu \left( \frac{\|b_1\| + 1}{p - 1} + 1 \right)
\]

\[
\leq \frac{C_1}{p - 1} (1 + \|b_1\|)^{\nu + 1} := D.
\]

Solving for \( \tilde{w} \) we have

\[
\tilde{w} \leq W(1 - e^{-D}) + k \quad \text{a.e. in } \Omega,
\]

and, since ess sup \( \tilde{w} = W \), we get \( W \leq k e^D \), which implies

\[
u \leq M + (\|a\| + \|b\|) e^{C(1 + \|b_1\|)^{\nu + 1}},
\]

that is (3.17) for this case.

When \( p > n \) and \( k > 0 \) we obtain directly from Lemma 5.4 that

\[
\log\frac{W}{W - \tilde{w} + \tilde{k}} \leq \frac{Q_\infty}{p - 1} (\|b_1\| + 1)
\]

and the conclusion again follows.

To remove the condition \( k > 0 \) we proceed exactly as in the proof of Theorem 3.5 for the case \( b_1 = b_2 = 0 \). \( \square \)

Remark 5.2 In the case of constant coefficients the previous arguments can be simplified considering \( \varepsilon = 1 \), so that Theorem 3.4 follows.

6 Proofs of theorems 3.3 and 3.5 for the case \( p = 1 \)

In this section a slightly different version of Lemma 4.5 is required.
Lemma 6.1 Let $u \in H^{1,1}_\text{loc}(\Omega)$ be a $1$–regular solution of inequality (3.2) with $u \leq 0$ on $\partial \Omega$. Suppose the coefficients $a, b, a_2$ in (3.4) belong to $L^\sigma(\Omega)$ for some $\sigma > n \geq 1$. Take $\tau > 0$ and define $w = u^+ + k$, with $k = \tau \|a\|_\sigma$. Then $w \in L^\infty(\Omega)$ and

$$w \leq K^{n\sigma/(\sigma-n)}\|w\|_1, \quad (6.70)$$

where $K = C(|\Omega|, n)(1 + \|a_2 + b\|_\sigma)$.

Proof. We follow the proof of Lemma 4.5, but using more precise constants. Indeed, since $p = 1$ we have $\theta = 1$ and $s = 1^* = n/(n-1)$ if $n > 1$. Writing $\sigma = n/(1 - \varepsilon)$, that is $\varepsilon = (\sigma - n)/\sigma$, then (4.35) reads as

$$ \|w\|_{r\sigma/(n-1)} \leq (Kr)^{(\sigma-n)r}\|w\|_1, $$

where $K$ is given as in (4.51) by $K = C(\|a_2 + b\|_\sigma + \|b_1\| + 1)$. Finally, as for (4.52), replacing $\|a_2 + b\|$ with $\|a_2 + b\|_\sigma$, we obtain (6.70).

If $n = 1$, operating as in the proof of Lemma 4.5 to obtain (4.50), simpler calculations give again (6.70). $\square$

Lemma 6.2 Let the hypotheses of Lemma 6.1 hold true and moreover assume (3.14). Then

$$\|w\|_1 \leq \frac{|\Omega|^{1-1/n}}{\delta} (S\|a\|_1 + k|\Omega|^{1-1/n}).$$

Proof. In Step 1 of the proof of Lemma 4.2, in the case $p = 1 = q = r = 1$, we proved inequality (4.35), i.e.

$$\|\nabla v\|_1 \leq \int_{\Omega} [(a_2 + b)v] dH.$$

By Hölder’s inequality and (3.14),

$$\|\nabla v\|_1 \leq \|a_2 + b\|_n \|v\|_{n/(n-1)} + \|a\|_1 \leq \frac{1 - \delta}{S} \|v\|_{n/(n-1)} + \|a\|_1. \quad (6.71)$$

By Sobolev’s inequality, since $v - \ell \in H^{1,1}_0(\Omega),$

$$\|v\|_{n/(n-1)} \leq \|v - \ell\|_{n/(n-1)} + \|\ell\|_{n/(n-1)} \leq S\|\nabla v\|_1 + \|\ell\|_1.$$

Using (6.71) this implies

$$\|v\|_{n/(n-1)} \leq (1 - \delta)\|v\|_{n/(n-1)} + S\|a\|_1 + \|\ell\|_1.$$

Here one can take $\ell \to k, m \to \infty$. Then $v \to w$, since $v \leq w$ when $w > \ell$, so that

$$\|w\|_1 \leq \|w\|_{n/(n-1)}|\Omega|^{1-1/n} \leq \frac{|\Omega|^{1-1/n}}{\delta} (S\|a\|_1 + |\Omega|^{1-1/n}k)$$

by Hölder’s inequality. $\square$
Proof of Theorems 3.3 and 3.5. Define $w = u^+ + k$ and $k = \tau \|a\|$, $\tau > 0$. Then Lemmas 6.1 and 6.2 apply, so that

$$u^+ \leq w \leq K^{n\sigma/(\sigma-n)} \frac{|\Omega|^{1-1/n}}{\delta} (S\|a\|_1 + |\Omega|^{1-1/n}k).$$

Of course, we are free to choose $\tau$ as we wish and the most natural choice is letting $\tau \to 0$, so that (3.16) follows in this case.

In the general case, when $u \leq M$ on $\partial \Omega$, proceed as in all other proofs, to get (3.11) and (3.16). □

References


