Quasi uniformity for the abstract Neumann antimaximum principle and applications with a priori estimates

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Abstract

In this paper we prove a general result giving the maximum and the antimaximum principles in a unitary way for linear operators of the form $L + \lambda I$, provided that 0 is an eigenvalue of $L$ with associated constant eigenfunctions. To this purpose, we introduce a new notion of “quasi”–uniform maximum principle, named $k$-uniform maximum principle: it holds for $\lambda$ belonging to certain neighborhoods of 0 depending on the fixed positive multiplier $k > 0$ which selects the good class of right–hand–sides. Our approach is based on a $L^\infty - L^p$ estimate for some related problems. As an application, we prove some generalization and new results for elliptic problems and for time periodic parabolic problems under Neumann boundary conditions.

Keywords: Maximum principle, antimaximum principle, uniform maximum principle with fixed multiplier, polyharmonic operators, periodic parabolic problems

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1 Introduction and abstract setting

Let us start considering the easy problem

\[
\begin{aligned}
\Delta u + \lambda u &= f(x) \quad \text{in } \Omega, \\
Bu &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]  

(1.1)

where \( \Omega \) is a bounded domain of \( \mathbb{R}^N \), \( N \geq 1 \), \( B \) denotes Dirichlet or Neumann boundary conditions and \( f \in L^p(\Omega) \). Denoting by \( \lambda_1 \) the first eigenvalue of \( -\Delta \) under the corresponding homogeneous boundary conditions \( Bu = 0 \), a classical result is the following maximum principle:

\((MP)\): if \( \lambda < \lambda_1 \), then for any \( f \geq 0 \) the associated solution \( u \leq 0 \) in \( \Omega \).

On the other hand, a related stronger version, namely the strong maximum principle, holds:

\((SMP)\): if in addition \( f \neq 0 \), then \( u < 0 \) in \( \Omega \).

We immediately observe that these results present two main features: they are \textit{uniform} in \( \lambda \) and no special functional space of right hand sides \( f \) is involved.

Moreover, it is now well know that jumping after \( \lambda_1 \) changes the situation a lot: indeed, Clément and Peletier in [8] proved the following antimaximum principle:

\((AMP)\): for any \( f \geq 0 \) in \( L^p(\Omega) \), \( p > N \), there exists \( \delta = \delta(f) > 0 \) such that if \( u \) solves (1.1) with \( \lambda \in (\lambda_1, \lambda_1 + \delta) \), then \( u \geq 0 \) in \( \Omega \).

They also showed that under Neumann boundary conditions a uniform antimaximum principle, i.e.

\((UAMP)\): in (AMP) it is possible to take \( \delta \) independent of \( f \)

holds only when \( N = 1 \).

Therefore, it is clear that, in general, (AMP) is \textit{not} uniform, while (MP) is. Moreover, (MP) needs no special requirements on \( p \), while (AMP) \textit{needs} \( p > N \), as showed in [30]: under Dirichlet boundary conditions, there exists \( f \in L^N(\Omega) \) with \( f > 0 \) such that for all \( \lambda > \lambda_1 \) different from an eigenvalue of \( -\Delta \) in \( H^1_0(\Omega) \), the solution of (1.1) changes sign.

After [8] a large number of refinements of (UAMP) have been established. We recall [6] for higher order ODE’s with periodic boundary conditions, while for general second order PDE’s with Neumann or Robin boundary conditions we mention [20] (where again it is proved that (UAMP) holds only if \( N = 1 \)), while polyharmonic operators in low dimensions (say for all those dimensions for which the natural Sobolev space containing weak solutions is embedded in \( C^0(\Omega) \)) are considered in [9], [10], [23]. In [30] it is showed that the condition \( p > N \) in [8] is sharp for the validity of an antimaximum principle when \( L = \Delta \) under Dirichlet boundary conditions: in this context, taking a right–hand side \( f \in L^2(\Omega) \), forces to assume \( N = 1 \) also for the validity of (AMP) without any
uniformity. Finally, we recall a local extension in \( \mathbb{R}^N \) proved in [12] and some extensions in presence of the \( p \)-Laplace operator in [5] and [21].

Recently, Campos–Mawhin–Ortega in [7] show that it is possible to state maximum and antimaximum principles in a unitary way, at least in some cases. Roughly speaking, having in mind (1.1) with \( Lu = u'' \) and Neumann boundary conditions, so that \( \lambda_1 = 0 \), they start with the following definition of maximum principle:

**Definition 1.1.** Given \( \lambda \in \mathbb{R} \setminus \{0\} \), we say that the operator \( L + \lambda I \) satisfies a maximum principle if for every \( f \in L^1(\Omega) \) the equation

\[
Lu + \lambda u = f, \quad u \in \text{Dom}(L) \subset C^0(\overline{\Omega})
\]

has a unique solution with \( \lambda u \geq 0 \) for any \( f \geq 0 \). Moreover, the maximum principle is said to be strong if \( \lambda u(x) > 0 \) for any \( x \in \Omega \) whenever \( f \geq 0 \) and \( f(x) > 0 \) in a subset of \( \Omega \) with positive measure.

It is clear that the definition above covers the case of a “classical” maximum principle when \( \lambda < \lambda_1 = 0 \) and a “classical” antimaximum principle when \( \lambda > 0 \); more precisely, we remark that for \( \lambda > 0 \) Definition 1.1 includes a (UAMP) tout court.

The main result in [7] sounds as the following

**Theorem 1.2 ([7]).** There exist \( \lambda_- \) and \( \lambda_+ \) such that

\[
-\infty \leq \lambda_- < 0 < \lambda_+ \leq +\infty
\]

and \( L + \lambda I \) satisfies a maximum principle if and only if \( \lambda \in [\lambda_-, 0) \cup (0, \lambda_+] \). Moreover the maximum principle is strong if \( \lambda \in (\lambda_-, 0) \cup (0, \lambda_+] \).

The basic ingredient of [7] is an \( L^\infty - L^1 \) estimate for solution–datum of the form

\[
\|u\|_{L^\infty(\Omega)} \leq M\|f\|_{L^1(\Omega)},
\]

which is common and natural for ODE’s and for the wave equation in 1D. On the other hand, for the classical theory of elliptic problems like (1.1), by the classical Agmon–Douglis–Nirenberg result ([1]), if \( \Omega \) is smooth enough, say of class \( C^2 \), a more natural setting would be a \( W^{2,p} - L^p \) estimate of the form

\[
\|u\|_{W^{2,p}(\Omega)} \leq M\|f\|_{L^p(\Omega)},
\]

since data belonging to \( L^1 \) are not the good ones to perform a standard variational approach (we refer to [3] for this case). For example, the easy problem

\[
\begin{cases}
\Delta u + \lambda u = f & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \) is a bounded domain of \( \mathbb{R}^N \), \( N > 1 \), cannot be handled by Theorem 1.2 if \( f \in L^1(\Omega) \), since \( L^1 \) is not contained in the dual of the Sobolev space where weak solutions are sought.
On the other hand, by Morrey’s Theorem, if \( N < 2p \), then \( W^{2,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}) \), for some \( \alpha \in (0,1) \), so that (1.3) implies

\[
\|u\|_{C^{0,\alpha}(\Omega)} \leq M\|f\|_{L^p(\Omega)},
\]

where, of course, \( \|u\|_{C^{0,\alpha}(\Omega)} = \max_{\Omega} |u| = \|u\|_{L^\infty(\Omega)}. \)

In this paper we want to combine the spirit of [7] with the result stated in the other papers cited above showing two possible generalizations of Theorem 1.2. To this purpose, we first prove that, although a (UAMP) cannot hold (as for the Laplace operator with Neumann conditions in dimension greater than 1), a “quasi–(UAMP)” does (in the sense of \( k-(UMP) \) introduced in Definition 1.4 below), thus extending to any \( p \in (1,\infty) \) the sufficient condition proved for the case \( p = 2 \) in [16]. Moreover, with an additional hypothesis, we characterize completely the set of \( \lambda \)'s for which a strong \( k-(UMP) \) holds. To do this, we will also show some \textit{a priori} estimates of independent interest, and some of which are in the spirit of bifurcation from 0. Finally, we also show the sharpness of the exponent \( p > N/2 \) for problem (1.4). In this context, we believe that our results can help understanding better all the situation.

We start introducing the abstract framework. Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain with \( C^2 \) boundary \( \partial \Omega \); a positive and finite measure \( \mu \) is given in \( \Omega \), and we write \( L^q(\Omega) := L^q(\Omega,\mu) \) for any \( q \in [1,\infty] \).

From now on, we fix \( p \in (1,\infty) \) and for any \( f \in L^p(\Omega) \) we set

\[
\overline{f} := \frac{1}{\mu(\Omega)} \int_{\Omega} f \, d\mu \quad \text{and} \quad \tilde{f} := f - \overline{f}.
\]

We also set

\[
\mathcal{L}^p := \left\{ f \in L^p(\Omega) : \overline{f} = 0 \right\} \quad \text{and} \quad \mathcal{C} := C^0(\overline{\Omega}) \cap \mathcal{L}^p;
\]

finally, fixed \( k > 0 \), we introduce the cone

\[
\mathcal{F}^p_k := \left\{ f \in L^p(\Omega) : \|	ilde{f}\|_{L^p(\Omega)} \leq k\|f\|_{L^1(\Omega)} \right\}
\]

and

\[
\mathcal{P}^p_k := \left\{ f \in \mathcal{F}^p_k : f \geq 0 \right\},
\]

i.e. \( \mathcal{P}^p_k \) is the sub–cone of positive functions in \( \mathcal{F}^p_k \).

\textbf{Remark 1.3.} It is clear that any \( f \in L^p(\Omega) \) belongs to a suitable \( \mathcal{F}^p_k \) and to \( \mathcal{F}^p_\ell \) for any \( \ell \geq k \). Moreover, \( \cup_k \mathcal{F}^p_k \subset L^1(\Omega) \) with strict inclusion.

We now consider a linear operator \( L : \text{Dom}(L) \subset C^0(\overline{\Omega}) \rightarrow L^p(\Omega) \) satisfying the following properties:

\[
\text{Ker}(L) = \left\{ \text{constant functions} \right\}, \quad \text{Im}(L) = \mathcal{L}^p, \tag{1.5}
\]
the problem $Lu = \tilde{f}$ has a unique solution $\tilde{u} \in \tilde{C}$ and $\exists M = M(L) > 0$ such that $\|\tilde{u}\|_{C^0(\Omega)} \leq M\|\tilde{f}\|_{L^p(\Omega)}$. \hspace{1cm} (1.6)

We remark that these requirements are the natural extensions to our setting of the assumptions made in [7]. Therefore, having in mind Definition 1.1, we give the following uniform maximum principle with fixed multiplier $k > 0$.

**Definition 1.4.** Fix $k > 0$. Given $\lambda \in \mathbb{R} \setminus \{0\}$, we say that the operator $L + \lambda I$ satisfies a uniform maximum principle with multiplier $k$, $k$–(UMP) for short, if for every $f \in F^p_k$ equation (1.2) has a unique solution $u \in \text{Dom}(L) \subset C^0(\Omega)$ with $\lambda u \geq 0$ when $f \geq 0$. We say that a strong $k$–(UMP) holds if $\lambda u(x) > 0$ for any $x \in \Omega$ whenever $f \geq 0$ and $f(x) > 0$ in a subset of $\Omega$ having positive measure.

**Remark 1.5.** Again as in Definition 1.1, the case $\lambda < 0$ corresponds to a classical maximum principle, while the case $\lambda > 0$ corresponds to an antimaximum principle, which is “almost” uniform, since $f$ belongs to $F^p_k$ and not to the whole of $L^p(\Omega)$.

Our first result concerns the existence of a neighborhood $U_k$ of 0 such that $U_k \setminus \{0\} \subset \{ \lambda \in \mathbb{R} : L + \lambda I \text{ satisfies a } k$–(UMP) $\}$.

More precisely:

**Theorem 1.6.** Assume that (1.5) and (1.6) hold and fix $k > 0$. Then there exists $\Lambda = \Lambda(k) > 0$ such that $L + \lambda I$ satisfies a $k$–(UMP) if $\lambda \in [-\Lambda, \Lambda] \setminus \{0\}$. Moreover a strong $k$–(UMP) holds if $\lambda \in (-\Lambda, \Lambda) \setminus \{0\}$.

**Remark 1.7.** As it will be clear from the proof, $\Lambda = \Lambda(k)$ does not depend on $p$, so that the result is uniform also with respect to the integrability exponent.

As a corollary of the previous result, we obtain the validity of an (AMP), also in a quasi–uniform way, for problem (1.4) with $f$ in $L^p(\Omega)$, $p > \frac{N}{2}$, already established in [7] for $f$ in $L^2(\Omega)$ and $N = 1, 2, 3$. On the other hand, we underline the fact that our somehow “standard” everywhere approach is quite natural for antimaximum principles, and so we cannot extend our results to an almost everywhere fashion, typical of a (MP) also for more general PDE’s and differential inequalities, possibly on Riemannian manifolds (see the recent [2], [27]) or in anisotropic settings ([13]).

Let us note that, in some sense, any function $f$ defines a whole set of “good” right–hand–sides, in the sense that a crucial rôle is played by the ratio $\|\tilde{f}\|_{L^p(\Omega)}/\|f\|_{L^1(\Omega)}$. Moreover, as a direct consequence of Theorem 1.6, in contrast to the Dirichlet case considered in [30], we get the following result:
Corollary 1.8. Concerning problem (1.4), $L^N$ is not sharp for the validity of the antimaximum principle.

Actually, this fact was already noticed in [20, Remark 4.5] and in [10, Proposition 2], where the sufficiency of the condition “$p > N/2$” was stated for the validity of the (AMP), but not for any form of uniformity, as we give in Theorem 1.6. Moreover, the following result holds:

Proposition 1.9. In problem (1.4), $L^{N/2}(\Omega)$ is sharp for the validity of the antimaximum principle.

Proof. It follows the lines of [30] with obvious adaptations, so we will be sketchy, though we seize this opportunity to make precise a small step in the proof of the Dirichlet case which contained a misleading missprint. Assume $\Omega = B_1$, the unit ball, and take

$$f(x) = \frac{1}{|x|^2(2 - \log |x|)} \left( N - 2 + \frac{1}{2} - \log |x| \right) \in L^{N/2}(\Omega).$$

Calculations show that $\nu(x) = \log(2 - \log |x|)$ solves $\Delta \nu = -f$. Now set $w_\lambda = u_\lambda + \nu v$, where $\chi$ is a smooth function with $\chi(x) = 1$ if $|x| \leq 1/2$, $\chi(x) = 0$ if $|x| \geq 7/8$, and $u_\lambda$ solves problem (1.4). Thus $w_\lambda$ solves the problem

$$\begin{cases} \Delta w = -\lambda w + (1 - \chi)f + v\Delta \chi + 2D\chi \cdot Dv + \lambda \nu v & \text{in } B_1 \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial B_1. \end{cases}$$

Since $(1 - \chi)f + v\Delta \chi + 2D\chi \cdot Dv + \lambda \nu v \in L^q(B_1)$ for any $q \geq 1$, choosing $\lambda$ different from an eigenvalue implies that $w_\lambda \in C(\bar{B}_1)$. Hence

$$\lim_{x \to 0} \frac{u_\lambda(x)}{v_\lambda(x)} = 0,$$

so that near 0 we have $u_\lambda(x) = -\log(2 - \log |x|) + w_\lambda < 0$. 

Our final results, together with Theorem 1.6, give an almost complete characterization of the set of $\lambda$’s for which $L + \lambda I$ satisfies a $k$–(UMP), in the spirit of Theorem 1.2. Of course, due to the validity of a classical maximum principle for $\lambda < 0$, in the following we concentrate on $\lambda > 0$. As usual, we denote by

$$R_\lambda = (L + \lambda I)^{-1} : L^p(\Omega) \to C^0(\bar{\Omega})$$

the resolvent of $L$, whenever it exists. Finally, let us set

$$\Lambda_+(k) := \left\{ \lambda > 0 : L + \lambda I \text{ satisfies a } k-(\text{UMP}) \right\}$$

and $\lambda_+ = \lambda_+(k) := \sup \Lambda_+(k)$. In the next result we show that the set of $\lambda$’s for which a $k$–(UMP) holds is an interval, in whose interior a strong $k$–(UMP) is satisfied; moreover, such a result characterizes completely such an interval. We focus on $\lambda_+$, an analogous result being valid for $\lambda_-$. 

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Theorem 1.10. Assume (1.5) and (1.6), and suppose that for all \( \lambda < \lambda_+ \) the following condition holds:

\[
\text{if } L + \lambda I \text{ satisfies a } k-\text{(UMP), then } R_\lambda(P_k^p) \subseteq P_k^p.
\]

(1.8)

Then \( L + \lambda I \) satisfies a \( k-\text{(UMP)} \) if and only if \( \lambda \in (0, \lambda_+] \). Moreover, the \( k-\text{(UMP)} \) is strong if \( \lambda \in (0, \lambda_+] \).

In general, it is not clear if condition (1.8) can be verified, and without it, we cannot prove that \( L + \lambda_+ I \) satisfies a \( k-\text{(UMP)} \). For this, a useful tool is the following result on a priori estimates for \( R_\lambda(P_k^p) \):

Proposition 1.11 (A priori estimates). Take \( \lambda > 0 \) and suppose that \( L + \lambda I \) satisfies a \( k-\text{(UMP)} \). Then, for any nonconstant \( f \in P_k^p \), setting \( u = R_\lambda f \), we have

\[
\text{if } \| \tilde{f} \|_{L^p(\Omega)} < \lambda \| \tilde{u} \|_{L^p(\Omega)}, \text{ then } 2M\mu(\Omega)^{1/p}\lambda > 1.
\]

(1.9)

Moreover, if \( \lambda \| \tilde{u} \|_{L^p(\Omega)} \leq \| \tilde{f} \|_{L^p(\Omega)} \), then \( u \in P_k^p \), where

\[
h = \min \left\{ 2M\mu(\Omega)^{1/p}\lambda k, k \right\} \leq k.
\]

(1.10)

If \( M\mu(\Omega)^{1/p}\lambda < 1 \), then \( u \in P_k^p \), where

\[
h = \max \left\{ \min \left\{ 2M\mu(\Omega)^{1/p}\lambda k, k \right\}, \frac{M\mu(\Omega)^{1/p}\lambda}{1 - M\mu(\Omega)^{1/p}\lambda} \right\}.
\]

(1.11)

As a consequence, we have the

Corollary 1.12. Suppose that \( L + \lambda I \) satisfies a \( k-\text{(UMP)} \) and \( 2M\mu(\Omega)^{1/p}\lambda \leq 1 \). Then, for every \( f \in P_k^p \), also \( R_\lambda f \in P_k^p \).

Proof. It is an immediate consequence of (1.10), since \( h \leq k \). \qed

Remark 1.13. This corollary implies that condition (1.8) is satisfied if \( \lambda \) is small, in the spirit of bifurcation.

As a straightforward consequence, in addition to (1.10), we have the following sufficient condition for Theorem 1.10:

Proposition 1.14. Assume (1.5) and (1.6); moreover, suppose that \( \lambda_+ \in \mathbb{R} \) and that \( 2M\mu(\Omega)^{1/p}\lambda_+ \leq 1 \). Then the conclusion of Theorem 1.10 holds.

2 Proof of Theorem 1.6

We introduce the operator \( \tilde{R}_0 : L^p \to \tilde{C} \) defined by

\[
\tilde{u} = \tilde{R}_0 \tilde{f} \iff L\tilde{u} = \tilde{f},
\]

which is well defined by assumption (1.6).

The first lemma we prove gives a condition that ensures the existence of \( R_\lambda \).
Lemma 2.1. There exists $\Lambda_1 > 0$ such that for all $\lambda \in [-\Lambda_1, \Lambda_1] \setminus \{0\}$ the resolvent $R_{\lambda} : L^p(\Omega) \to C^0(\overline{\Omega})$ of $L$ is well defined. Moreover, there exists $C > 0$ such that, if $\tilde{f} \in L^p$ and $\lambda \in [-\Lambda_1, \Lambda_1] \setminus \{0\}$ then
\[
\|R_{\lambda}\tilde{f}\|_{C^0(\overline{\Omega})} \leq C\|	ilde{f}\|_{L^p(\Omega)}.
\]
More precisely, $C := \frac{M}{1 - \Lambda_1\|R_{\lambda}\|_{C_{-\infty}^{\infty}}}$, where $M$ is the constant appearing in (1.6).

Here $\|R_0\|_{\mathcal{C}_{-\infty}^{\infty}}$ denotes the norm of the restriction of the operator $R_0$ from $\mathcal{C}$ to $\mathcal{C}$, which is well defined, since $\mathcal{C} \subset L^p$.

Proof. Rewrite equation (1.2) as the system
\[
\begin{aligned}
\{L\tilde{u} + \lambda \tilde{u} &= \tilde{f}, \\
\lambda \pi &= \tilde{f}.
\end{aligned}
\]
(2.1)

Applying $R_0$, the first equation in (2.1) can be rewritten as
\[
(I + \lambda R_0)\tilde{u} = R_0\tilde{f}.
\]
(2.2)

Now, observe that $R_0\tilde{f} \in \mathcal{C}$; take $|\lambda| < 1/\|R_0\|_{\mathcal{C}_{-\infty}^{\infty}}$, so that $I + \lambda R_0$ is invertible from $\mathcal{C}$ to $\mathcal{C}$, and equation (2.2) is solved by $\tilde{u} = (I + \lambda R_0)^{-1}R_0\tilde{f}$.

Finally, take $\Lambda_1 \in (0, 1/\|R_0\|_{\mathcal{C}_{-\infty}^{\infty}})$; therefore, from the triangle inequality, (2.2) and (1.6), for all $\lambda$ with $|\lambda| \leq \Lambda_1$ we have that
\[
\begin{aligned}
\|\tilde{u}\|_{L^\infty(\Omega)} - \Lambda_1\|R_0\|_{\mathcal{C}_{-\infty}^{\infty}}\|\tilde{u}\|_{L^\infty(\Omega)} &\leq \|\tilde{u}\|_{L^\infty(\Omega)} - |\lambda|\|R_0\|_{\mathcal{C}_{-\infty}^{\infty}}\|\tilde{u}\|_{L^\infty(\Omega)} \\
&\leq \|(I + \lambda R_0)\tilde{u}\|_{L^\infty(\Omega)} = \|R_0\tilde{f}\|_{L^\infty(\Omega)} \\
&= \|\tilde{u}\|_{L^\infty(\Omega)} \leq M\|	ilde{f}\|_{L^p(\Omega)}.
\end{aligned}
\]

The thesis follows.

Note that the proof above provides the estimate
\[
\Lambda_1 < \frac{1}{\|R_0\|_{\mathcal{C}_{-\infty}^{\infty}}}.
\]

After proving that $R_{\lambda}$ is well defined, we have our second

Lemma 2.2. For any $k > 0$ there exists $\Lambda_2 := \Lambda_2(k) \in (0, \Lambda_1]$ such that for all $\lambda \in [-\Lambda_2, \Lambda_2] \setminus \{0\}$ the operator $L + \lambda I$ has a $k$–(UMP). Moreover, a strong $k$–(UMP) holds if $\lambda \in (-\Lambda_2, \Lambda_2) \setminus \{0\}$.

Proof. If $f \in \mathcal{F}_k^p$, $f \geq 0$, then $\tilde{f} = \frac{1}{\mu(\Omega)}\|f\|_{L^1(\Omega)}$. Thus, using the second equation in (2.1), one has
\[
\lambda u = \lambda R_{\lambda}(\tilde{f} + \tilde{f}) = \lambda R_{\lambda}(\tilde{f}) + \tilde{f} = \lambda R_{\lambda}(\tilde{f}) + \frac{1}{\mu(\Omega)}\|f\|_{L^1(\Omega)}
\]
\[
\geq \frac{1}{\mu(\Omega)}\|f\|_{L^1(\Omega)} - |\lambda|\|R_{\lambda}\tilde{f}\|_{L^\infty(\Omega)}.
\]

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By the previous Lemma it results that if \( \lambda \in [-\Lambda_1, \Lambda_1] \setminus \{0\} \), then
\[
\lambda u \geq \frac{1}{\mu(\Omega)} \|f\|_{L^1(\Omega)} - |\lambda| \frac{M}{1 - \Lambda_1 ||R_0||_{C \to C} - |\lambda|} \|\tilde{f}\|_{L^p(\Omega)}
\geq \left( \frac{1}{\mu(\Omega)} - k|\lambda| \frac{M}{1 - \Lambda_1 ||R_0||_{C \to C}} \right) \|f\|_{L^1(\Omega)},
\]
since \( f \in \mathcal{F}_k^p \). The Lemma is thus completely proved, provided that
\[
\Lambda_2 = \min \left\{ \Lambda_1, \frac{1 - \Lambda_1 ||R_0||_{C \to C}}{kM\mu(\Omega)} \right\}.
\]

**Proof of Theorem 1.6.** Fixed \( k > 0 \), the result follows by taking \( \Lambda = \Lambda_2(k) \) as given in Lemma 2.2.

3 Proof of Theorem 1.10

In this section we prove Theorem 1.10, so that we assume all the conditions of the previous section. Let us fix \( k > 0 \).

**Lemma 3.1.** Assume that there exists \( \lambda_0 > 0 \) such that \( L + \lambda_0 I \) has a \( k \)-(UMP) and \( R_{\lambda_0}(\mathcal{P}_k^p) \subseteq \mathcal{P}_k^p \). Then \( R_\lambda : \mathcal{P}_k^p \to C^0(\Omega) \) is defined for all \( \lambda \in (0, \lambda_0] \) and

1. the function \( \lambda \mapsto R_\lambda f \) is analytic in \( (0, \lambda_0] \) for all \( f \in \mathcal{P}_k^p \);
2. \( L + \lambda I \) satisfies a \( k \)-(UMP) for all \( \lambda \in (0, \lambda_0] \);
3. \( L + \lambda I \) satisfies a strong \( k \)-(UMP) for all \( \lambda \in (0, \lambda_0) \).

**Proof.** First, by (1.7), we immediately obtain \( R_{\lambda_0}1 = \frac{1}{\lambda_0} \), and \( 1 \in \mathcal{P}_k^p \) for any \( k > 0 \), since \( \tilde{1} = 0 \).

Now, take \( f \in \mathcal{P}_k^p \cap C^0(\Omega) \); using the \( k \)-(UMP), the inclusion \( R_{\lambda_0}(\mathcal{P}_k^p) \subseteq \mathcal{P}_k^p \), assumption (1.6) and system (2.1) for \( \lambda = \lambda_0 \), we immediately get
\[
0 \leq R_{\lambda_0}f = \tilde{u} + \tilde{v} \leq M\|\tilde{f} - \lambda_0 \tilde{u}\|_{L^p(\Omega)} + \frac{T}{\lambda_0} \leq Mk(\|f\|_{L^1(\Omega)} + \lambda_0 ||u||_{L^1(\Omega)}) + \frac{T}{\lambda_0}
\]
\[
= 2Mk\|f\|_{L^1(\Omega)} + \frac{T}{\lambda_0} \leq \left( 2Mk\mu(\Omega) + \frac{1}{\lambda_0} \right) \|f\|_{C^0(\Omega)}.
\]
Thus
\[
\|R_{\lambda_0}\| = \sup_{f \in \mathcal{P}_k^p \cap C^0(\Omega), ||f||_{C^0(\Omega)} \leq 1} \|R_{\lambda_0}f\|_{C^0(\Omega)} \leq \frac{\eta + 1}{\lambda_0},
\]
where we have set \( \eta := 2Mk\mu(\Omega)\lambda_0 \).
Now, for any $\lambda \in \mathbb{R}$ we rewrite $Lu + \lambda u = f$ as
\[ Lu + \lambda_0 u = f + (\lambda_0 - \lambda)u, \]
and applying $R_{\lambda_0}$, we get
\[ u - (\lambda_0 - \lambda)R_{\lambda_0}u = R_{\lambda_0}f. \tag{3.2} \]
If $(\lambda_0 - \lambda)\|R_{\lambda_0}\| < 1$, then $I - (\lambda_0 - \lambda)R_{\lambda_0}$ is invertible from the cone $\mathcal{P}_k \cap C^0(\Omega)$ to itself and (3.2) is solvable, since $R_{\lambda_0}f \in \mathcal{P}_k \cap C^0(\Omega)$.

Moreover, since $\lambda \in \mathbb{R}$, we get
\[ R_{\lambda_0}f \quad \text{for operators we get} \]
If $(\lambda_0 - \lambda)\|R_{\lambda_0}\| < 1$, then $I - (\lambda_0 - \lambda)R_{\lambda_0}$ is invertible from the cone $\mathcal{P}_k \cap C^0(\Omega)$

Now, if $\|R_{\lambda_0}\| = 1/\lambda_0$, we find
\[ (\lambda_0 - \lambda)\|R_{\lambda_0}\| < 1 \iff \frac{\lambda_0 - \lambda}{\lambda_0} < 1 \iff \lambda > 0, \]
which implies that (3.2) is solvable for any $\lambda \in (0, \lambda_0)$.

On the other hand, if $\|R_{\lambda_0}\| > 1/\lambda_0$, since $\|R_{\lambda_0}\| \leq \frac{\eta + 1}{\lambda_0}$, we have
\[ \lambda > \frac{\eta}{\lambda_0} \lambda_0 \iff (\lambda_0 - \lambda)\frac{\eta + 1}{\lambda_0} < 1 \iff (\lambda_0 - \lambda)\|R_{\lambda_0}\| < 1. \]
This implies that if $\lambda \in \left(\frac{\eta}{\eta + 1}\lambda_0, \lambda_0\right)$, then (3.2) is solvable.

If $f \in \mathcal{P}_k^p$ then $R_{\lambda_0}f \in \mathcal{P}_k^p \cap C^0(\Omega)$ by assumption and (3.2) is solvable as well, provided that $\lambda \in \left(\frac{\eta}{\eta + 1}\lambda_0, \lambda_0\right)$, or $\lambda \in (0, \lambda_0)$ if $\|R_{\lambda_0}\| = 1/\lambda_0$, the easier case that we don’t discuss.

Now, since $u = [I - (\lambda_0 - \lambda)R_{\lambda_0}]^{-1}R_{\lambda_0}f$ by (3.2), from Neumann’s formula for operators we get
\[ R_{\lambda}f = [I - (\lambda_0 - \lambda)R_{\lambda_0}]^{-1}R_{\lambda_0}f = \left(\sum_{n=0}^{\infty}(\lambda_0 - \lambda)^nR_{\lambda_0}^n\right)R_{\lambda_0}f, \tag{3.3} \]
and thus the function $\lambda \mapsto R_{\lambda}f$ is analytic.

Since $f \geq 0$, by assumption we have $R_{\lambda_0}f \geq 0$. Moreover, $R_{\lambda_0}f$ belongs to $\mathcal{P}_k^p$ by hypothesis; so we can iterate obtaining $R_{\lambda}f \geq 0$ whenever $\lambda \in \left(\frac{\eta}{\eta + 1}\lambda_0, \lambda_0\right)$ and $f \geq 0$.

In order to conclude the proof of claims 1 and 2, let us show that it is possible to extend the results above also for $\lambda \in \left(0, \frac{\eta}{\eta + 1}\lambda_0\right]$. Indeed, by assumption (1.6) and system (2.1) for $\lambda_1 = \frac{\eta}{\eta + 2}\lambda_0$, operating as for (3.1), we find
\[ \|R_{\lambda_1}f\|_{C^0(\Omega)} \leq \|\tilde{u}\|_{C^0(\Omega)} + \|\tilde{v}\|_{C^0(\Omega)} \leq M\|\tilde{f}\|_{L^1(\Omega)} + \frac{T}{\lambda_1} \leq Mk\|f\|_{L^1(\Omega)} + \frac{T}{\lambda_1} \leq \left(\frac{\eta}{\lambda_0} + \frac{1}{\lambda_1}\right)\|f\|_{C^0(\Omega)} \leq \frac{\eta + 1}{\lambda_1}\|f\|_{C^0(\Omega)}. \]
Moreover, since $\lambda_1 \in \left(\frac{\eta}{\eta + 1}\lambda_0, \lambda_0\right)$, by the previous part we know that $R_{\lambda_1}(\mathcal{P}_k^p) \subseteq \mathcal{P}_k^p$, so that we can proceed as above, extending the results for $\lambda$ belonging to the
interval \( \left( \frac{n}{\gamma + 1} \lambda_1, \lambda_0 \right) \). Now, we repeat the procedure above starting from \( \frac{n}{\gamma + 1} \frac{\gamma}{\gamma + 2} \lambda_0 \), and by induction we can extend the results above for all those \( \lambda \)'s belonging to the whole interval \( \left( 0, \frac{2 + \eta}{1 + \eta} \lambda_0 \right) \), so that claims 1 and 2 follow.

Now, take \( f \in \mathcal{P}_k^\gamma \), \( f > 0 \) in a subset of \( \Omega \) having positive measure, and let \( \lambda \in (0, \lambda_0) \). First note that by (3.3) the following monotonicity property is immediate:

\[
\text{if } 0 < \lambda_2 < \lambda_1 \leq \lambda_0 \text{ then } R_{\lambda_2} f \geq R_{\lambda_1} f. \tag{3.4}
\]

Then, fix \( x \in \Omega \) and consider the function \( g : (0, \lambda_0) \to C^0(\overline{\Omega}) \) defined as \( g(\lambda)(x) = R_{\lambda_0} f(x) \). Then, by (3.3) and (3.4), it is clear that \( g(\lambda) \) is nonnegative and non increasing.

By the first statement of this Lemma, we can assume that \( \Lambda_1 < \frac{2 + \eta}{1 + \eta} \lambda_0 \). By Lemma 2.2 there exists \( \Lambda_2 = \Lambda_2(k) \in (0, \Lambda_1) \) such that for all \( \lambda \in (0, \Lambda_2) \) the operator \( L + \lambda I \) satisfies a strong \( k-\text{(UMP)} \), i.e. \( g(\lambda) > 0 \) for all \( \lambda \in (0, \Lambda_2) \), so that \( g \) is not identically trivial. On the other hand, \( g \) is analytic in \( (0, \lambda_0) \), so by analytic continuation we also get that \( g(\lambda) > 0 \) for all \( \lambda \in (0, \lambda_0) \).

Now we are able to give the

**Proof of Theorem 1.10.** From (1.8), we can apply Lemmas 2.2 and 3.1, and thus we infer that the set

\[
\Lambda_+(k) := \left\{ \lambda > 0 : L + \lambda I \text{ satisfies a } k-\text{(UMP)} \right\}
\]

is a nonempty interval. By Lemma 3.1 we also know that a strong \( k-\text{(UMP)} \) holds if \( \lambda \in (0, \lambda_+) \).

Letting \( \lambda_+ \in \mathbb{R} \), we now show that \( \lambda_+ \in \Lambda_+ \). Indeed, take a strictly increasing sequence \( \lambda_j \) such that \( \lim_{j \to +\infty} \lambda_j = \lambda_+ \) and let \( j_0 \) be such that \( \lambda_+ < \frac{2 + \eta}{1 + \eta} \lambda_0 \). By assumption \( R_{\lambda_0} \) satisfies a \( k-\text{(UMP)} \); thus, from the first statement of Lemma 3.1 we know that \( R_{\lambda_0} \) is well defined, and by (3.3) applied with \( \lambda_0 = \lambda_+ \) we can conclude that \( u_j = R_{\lambda_0} f \to u = R_{\lambda_+} f \) uniformly in \( \Omega \) as \( j \to \infty \). Since \( f \in \mathcal{P}_k^\gamma \), then \( u_j \geq 0 \), and hence \( u \geq 0 \) as well.

**Proof of Proposition 1.11.** By (2.1) we get \( \lambda \pi = \tilde{f} \) and \( L \tilde{u} = \tilde{f} - \lambda \tilde{u} \), so that \( \|f\|_{L^1(\Omega)} = \lambda \|u\|_{L^1(\Omega)} \), since \( f \in \mathcal{P}_k^\gamma \) and \( u \geq 0 \) by assumption. Moreover, \( f \) is nonconstant if and only if \( u \) is nonconstant as well.

Hence, from hypothesis (1.6), we have

\[
\| \tilde{u} \|_{L^p(\Omega)} \leq \mu(\Omega)^{1/p} \| \tilde{u} \|_{L^\infty(\Omega)} \leq M \mu(\Omega)^{1/p} \| \tilde{f} - \lambda \tilde{u} \|_{L^p(\Omega)} \\
\leq M \mu(\Omega)^{1/p} \| \tilde{f} \|_{L^p(\Omega)} + \lambda \| \tilde{u} \|_{L^p(\Omega)}. \tag{3.5}
\]

Now, if \( \| \tilde{f} \|_{L^p(\Omega)} < \lambda \| \tilde{u} \|_{L^p(\Omega)} \), we get

\[
\| \tilde{u} \|_{L^p(\Omega)} < 2M \mu(\Omega)^{1/p} \lambda \| \tilde{u} \|_{L^p(\Omega)},
\]

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and if $\tilde{u} \neq 0$, i.e. $u$ is not a constant, we find $2M\mu(\Omega)^{1/p}\lambda > 1$, and (1.9) is proved.

Now, let us prove (1.10): if $\lambda \|\tilde{u}\|_{L^p(\Omega)} \leq \|\tilde{f}\|_{L^p(\Omega)}$, we have two estimates:

first, $\lambda \|\tilde{u}\|_{L^p(\Omega)} \leq \|\tilde{f}\|_{L^p(\Omega)} \leq k\|f\|_{L^1(\Omega)} = \lambda k\|u\|_{L^1(\Omega)},$

so that $\|\tilde{u}\|_{L^p(\Omega)} \leq k\|u\|_{L^1(\Omega)}$.

Moreover, starting from (3.5), we have

$\|\tilde{u}\|_{L^p(\Omega)} \leq 2M\mu(\Omega)^{1/p}\|\tilde{f}\|_{L^p(\Omega)} \leq 2M\mu(\Omega)^{1/p}k\|f\|_{L^1(\Omega)} = 2M\mu(\Omega)^{1/p}\lambda k\|u\|_{L^1(\Omega)}$.

Hence (1.10) holds.

Finally, suppose that $M\mu(\Omega)^{1/p}\lambda < 1$; then from (3.5), we obtain

$\|\tilde{u}\|_{L^p(\Omega)} \leq \frac{M\mu(\Omega)^{1/p}}{1 - M\mu(\Omega)^{1/p}\lambda}\|\tilde{f}\|_{L^p(\Omega)} \leq \frac{M\mu(\Omega)^{1/p}}{1 - M\mu(\Omega)^{1/p}\lambda}k\|f\|_{L^1(\Omega)}$

which, together with (1.10), concludes (1.11).

4 Applications

In this section we present some differential problems where Theorems 1.6 and 1.10 can be applied. The first two examples are almost straightforward, after the considerations made in Section 1.

The third application concerns time–periodic parabolic problems, which have raised a growing interest in the last years, especially in their nonlinear versions, mainly for the large number of biological applications they describe (see [14], [15], [18], [24], [29],[31] and also [17] and [28] for non periodic applications), and for which a general approach for the validity of an antimaximum principle seemed to miss so far, except for [11] and [22].

4.1 Laplace operator

Let us go back to the classical Neumann problem given in (1.4)

$$\begin{align*}
\Delta u + \lambda u &= f & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 & \text{on } \partial \Omega,
\end{align*}$$

where $\Omega$ is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$, and $f \in L^p(\Omega)$, $p \in (N/2, \infty)$. Then it is well known that problem (1.4) with $\lambda = 0$ has a solution if and only if $\int_{\Omega} f = 0$. On the other hand, setting $L = \Delta$, it is clear that $\text{Ker}(L) = \{0\}$.
constant functions} and that $\lambda_1 = 0$. In addition, by elliptic regularity, the problem
\[
\begin{cases}
L\bar{u} = \bar{f} & \text{in } \Omega, \\
\frac{\partial \bar{u}}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\]
has a unique solution
\[
\bar{u} \in W^{2,p}(\Omega) \hookrightarrow C^0(\overline{\Omega}) \quad \text{(since } p > N/2)\]
satisfying the additional condition $\int_{\Omega} \bar{u} = 0$, that is $\bar{u} \in \mathcal{C}$, according to the notations introduced in Section 1. Moreover, as already remarked at the beginning, by the classical estimates in [1], there exists $\tilde{M} > 0$ such that
\[
\|\bar{u}\|_{W^{2,p}(\Omega)} \leq \tilde{M} \|\bar{f}\|_{L^p(\Omega)};
\]
note that, by uniqueness, the usual term $\|\bar{u}\|_{L^p(\Omega)}$ that would appear on the right hand side of the previous inequality can be deleted. Hence, all the abstract requirements (1.5) and (1.6) for $L$ are fulfilled, with the underlying measure $\mu$ equal to Lebesgue’s measure in $\Omega$.

Applying Theorem 1.6 we immediately get the following

**Proposition 4.1.** Let $p > N/2$; for any $k > 0$ there exists $\Lambda = \Lambda(k) > 0$ such that $\Delta + \lambda I$ under homogeneous Neumann boundary conditions satisfies a $k$–(UMP) if $\lambda \in (0, \Lambda]$. Moreover, a strong $k$–(UMP) holds if $\lambda \in (0, \Lambda)$.

In [7] it was proved that this result was valid for $N = p = 1$, and that the analogous of Theorem 1.10 holds, while in general we cannot consider this characterization, unless Proposition 1.14 can be applied (see Section 4.4). However, the authors underlined the fact that they could not prove this result for $N > 1$, so that they were naturally turned to consider polyharmonic operators in low dimensions. We consider the same operators in the following section.

### 4.2 Polyharmonic operator

Let us now consider an elliptic Neumann problem in presence of the $m$–polyharmonic operator $\Delta^m$, $m \in \mathbb{N}$,
\[
\begin{cases}
\Delta^m u + \lambda u = f & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} \cdots = \frac{\partial \Delta^{m-1} u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\]  
(4.1)

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $f \in L^p(\Omega)$ with $p \in (N/2m, \infty)$. As before, $\text{Ker}(L) = \{\text{constant functions}\}$, with $L = \Delta^m$, and as already remarked in [7], the assumption (1.5) for $L$ is satisfied. Moreover, by [1], the weak solution $u \in W^{m,p}(\Omega)$ of (4.1) actually belongs to $W^{2m,p}(\Omega)$ and is such that the estimate
\[
\|\bar{u}\|_{W^{2m,p}(\Omega)} \leq \tilde{M} \|\bar{f}\|_{L^p(\Omega)}
\]  
(4.2)

\[13\]
holds for a suitable constant $\tilde{M}$. Indeed, $\tilde{u}$ solves
\[
\begin{cases}
\Delta^n \tilde{u} = \tilde{f} & \text{in } \Omega, \\
\frac{\partial \tilde{u}}{\partial \nu} = \frac{\partial \Delta \tilde{u}}{\partial \nu} = \ldots = \frac{\partial \Delta^{m-1} \tilde{u}}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\]
and it belongs to $W^{2m,p}(\Omega)$; multiplying both sides of the equation above by $|\Delta^m \tilde{u}|^{p-2} \Delta^m \tilde{u}$, integrating and applying the Hölder inequality, we obtain
\[
\|\Delta^m \tilde{u}\|_{L^p(\Omega)} \leq \|\tilde{f}\|_{L^p(\Omega)}.
\]
By the Open Mapping Theorem we can prove, in the spirit of [26, Remark 2.1], that $\|\Delta^m \cdot\|_p$ is a norm (equivalent to the usual one) for functions in $W^{2m,p}(\Omega)$ with zero mean. Thus, being $p > N/2m$, we have $W^{2m,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$, so that (4.2) holds, i.e. also assumption (1.6) is verified.

Now, Theorem 1.6 can be immediately applied, extending to higher dimensions the result showed in [7] when $N \leq 2m - 1$ and $p = 1$. As in the previous case, we remark that in [7] there was a complete characterization of the “good” $\lambda$’s, while here we give a sufficient condition for the validity of a $k$–(UMP):

**Proposition 4.2.** Let $m \in \mathbb{N}$ and $p > N/2m$; for any $k > 0$ there exists $\Lambda = \Lambda(k) > 0$ such that $\Delta^m + \lambda I$ under homogeneous Neumann boundary conditions satisfies a $k$–(UMP) if $\lambda \in (0, \Lambda]$. Moreover, a strong $k$–(UMP) holds if $\lambda \in (0, \Lambda)$.

### 4.3 Periodic parabolic problems

Here we consider the periodic parabolic problem
\[
\begin{cases}
\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} + \lambda u = f & \text{in } \Omega \times (0, \infty), \\
\frac{\partial u}{\partial x} = 0 & \text{on } \partial \Omega \times (0, \infty), \\
u(0) = u(T),
\end{cases}
\]
where $\Omega$ is a bounded interval of $\mathbb{R}$, $\alpha > 0$, $T > 0$, $\lambda \in \mathbb{R}$ and $f \in L^p(Q_T)$, with $Q_T = \Omega \times (0,T)$ and $p > 1$. Using the notation of Section 1, we set
\[Lu := \frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2}\]
with
\[D(L) = \left\{ u \in H^1(0,T;W^{2,p}(\Omega)) : u_x = 0 \text{ on } \partial \Omega \text{ for a.e. } t \in (0,T) \right\}.
\]

First, by parabolic regularity (see [25]), any weak solution of (4.3) belongs to $C^0([0,T];W^{1,p}(\Omega))$. This fact, with an obvious adaptation of the proof in [16], lets us prove that $\text{Ker}(L) = \{ \text{constant functions} \}$ and that $f \in \text{Im}(L)$ if and only if
\[f \in \mathcal{L}^p = \left\{ f \in L^p(Q_T) : \int_{Q_T} f \, dx \, dt = 0 \right\}.
\]
Moreover, $\tilde{u} \in C^0([0,T];W^{1,p}(\Omega))$ and
\[\|\tilde{u}\|_{L^\infty(0,T;W^{1,p}(\Omega))} \leq C \left\{ \|\tilde{u}(0)\|_{W^{1,p}(\Omega)} + \|f\|_{L^p(Q_T)} \right\}
\]
for a universal constant $C = C(\alpha, T, \Omega)$.

We are not able to apply Theorem 1.6 to every problem of the form (4.3). Thus, at this point we assume to deal with data $(\alpha, T, \Omega)$ such that the related constant $C(\alpha, T, \Omega)$ is such that

$$C = C(\alpha, T, \Omega) < 1,$$

which holds, for example, if $\alpha$ is large enough.

Thus from (4.4) we easily get

$$\|\tilde{u}\|_{L^\infty(0,T;W^{1,p}(\Omega))} \leq \frac{C}{1-C}\|f\|_{L^p(Q_T)}; \quad (4.6)$$

By Poincaré–Wirtinger inequality (see, for example, [4, Chapter VIII]) we know that

$$\|\tilde{u}(t)\|_{L^\infty(\Omega)} \leq \sqrt{\|\Omega\|}\|\tilde{u}(t)\|_{W^{1,p}(\Omega)} \quad \forall t,$$

so that (4.6) implies

$$\|\tilde{u}\|_{L^\infty(Q_T)} \leq \frac{C\sqrt{\|\Omega\|}}{1-C}\|f\|_{L^p(Q_T)};$$

thus (1.6) is satisfied with $M = \frac{C\sqrt{\|\Omega\|}}{1-C}$.

Without other assumptions we can now apply Theorem 1.6 to problem (4.3) to get:

**Theorem 4.3.** Assume (4.5), fix $k > 0$ and set $Lu := u_t - \alpha u_{xx}$. Then there exists $\Lambda = \Lambda(k) > 0$ such that $L + \lambda I$ satisfies a $k-$ (UMP) if $\lambda \in (0, \Lambda]$. Moreover a strong $k-$ (UMP) holds if $\lambda \in (0, \Lambda)$.

To our best knowledge, there are not many results concerning (AMP) or (UAMP) for parabolic problems like (4.3). We quote [11], where an eventual antimaximum principle is proved; more precisely, the authors prove that solutions of Cauchy–Dirichlet problems are positive for large times, also when the data is negative.

In [22], a periodic parabolic problem under both homogeneous Dirichlet or Neumann conditions is considered, and an (AMP) also in presence of a weight is proved. But if $N = 1$ they assume that the right–hand–side of the parabolic equation belongs to $L^p(\Omega)$ with $p > 3$, and in addition their result is not uniform. On the contrary, we can handle the case $f \in L^p(\Omega)$, $p > 1$, and we can prove a $k-$ (UMP), so that a certain uniformity is guaranteed for the (AMP), at least if $\alpha$ is large enough.

Of course, a result analogous to Theorem 4.3 can be proved if $-\Delta$ in dimension 1 is replaced by a polyharmonic operator in a higher dimension $N$ with the natural Neumann boundary conditions:
with \( N \leq 2m - 1 \), so that \( C^0([0,T]; H^m(\Omega)) \subset C^0(\overline{Q_T}) \) and \( \alpha \) is large. The details are left to the reader.

### 4.4 The limit case: validity of Proposition 1.14

We give an example for the validity of Proposition 1.14, taking \( p = 2 \) and considering the problem

\[
\begin{aligned}
    u'' + \lambda u &= f & \text{in } (0, \pi), \\
    u'(0) &= u'(&) = 0.
\end{aligned}
\]

(4.7)

Let \( \tilde{u} \in W^{2,2}(0, \pi) \) solve

\[
\begin{aligned}
    \tilde{u}'' &= \tilde{f} & \text{in } (0, \pi), \\
    \tilde{u}'(0) &= \tilde{u}'(\pi) = 0.
\end{aligned}
\]

Then we claim that

\[
\| \tilde{u} \|_{L^\infty(0, \pi)} \leq \left( \frac{\pi}{3} \right)^{1/2} \| \tilde{f} \|_{L^2(0, \pi)}.
\]

(4.8)

Indeed, \( \tilde{u}' \in H^1_0(0, \pi) \), so that, being \( 1 \) the first eigenvalue of \(-d^2/dx^2\), by the Poincaré inequality, we have

\[
\| \tilde{u}' \|_{L^2(0, \pi)} \leq \| \tilde{u}'' \|_{L^2(0, \pi)} = \| \tilde{f} \|_{L^2(0, \pi)}.
\]

Moreover, after an even reflection of \( \tilde{u} \) around \( x = 0 \), we have that \( \tilde{u} \) is a \( 2\pi \)-periodic function such that \( \int_{-\pi}^\pi \tilde{u} = 0 \). Hence, by [19, Proposition 2.5.35 (b)], we have

\[
\| \tilde{u} \|_{L^\infty(0, \pi)} = \| \tilde{u} \|_{L^\infty(-\pi, \pi)} \leq \frac{\pi}{6} \| \tilde{u}' \|_{L^2(-\pi, \pi)} = \frac{\pi}{3} \| \tilde{u}' \|_{L^2(0, \pi)},
\]

since \( \| \tilde{u}' \|_{L^2(-\pi, \pi)} = 2 \| \tilde{u}' \|_{L^2(0, \pi)} \), and then (4.8) easily follows.

In conclusion, the condition in (1.6) holds with

\[
M = \left( \frac{\pi}{3} \right)^{1/2},
\]

so that the requirement in Proposition 1.14 holds if

\[
2 \frac{\pi}{\sqrt{3}} \lambda_+ \leq 1.
\]

But in [8] it was shown that for \( d^2/dx^2 + \lambda I \) with Neumann boundary conditions on \( (0, \pi) \), the associated Green function is nonnegative only if \( \lambda \leq 1/4 = \lambda_+ \), of course independendly of the Lebesgue space containing \( f \). Thus

\[
2M \mu(\Omega)^{1/2} \lambda_+ < 1,
\]

as required in Proposition 1.14.
References


