Critical Point Theory 0 and applications
Introduction

This notes are the result of two Ph.D. courses I held at the University of Florence in Spring 2006 and in Spring 2010 on Critical Point Theory, as well as of the course Superior Analysis I have been delivering for the Master Degree in Mathematics at the University of Perugia since 2003. They are a basic introduction to the subject, and this is the reason for the title.

The list of papers cited in the Bibliography reflects other subjects treated in the courses, such as Schroedinger-Poisson systems, Klein-Gordon-Maxwell systems, Born-Infeld-Maxwell type systems, reversed variational inequalities, ...  

Many thanks to some friends and students for a careful reading of the manuscript; among them it is a pleasure to thank Matteo Rinaldi, and Alessio Fiscella.
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Chapter 1

Some preliminary notions

In this chapter we recall some notions which will be used throughout these notes.

1.1 Variational tools

Let us start with classical notions (see [110], [5]). In the following \( X \) are \( Y \) Banach spaces, \( A \subseteq X \) is an open set and \( f : A \to Y \) is a given function. Here, and from now on, \( \mathcal{L}(X,Y) \) denotes the set of linear and continuous operator between \( X \) and \( Y \).

Definition 1.1.1. We say that \( f \) is Gateaux–differentiable at \( u_0 \in A \) along \( v \in X \) if

\[
\lim_{t \to 0} \frac{f(u_0 + tv) - f(u_0)}{t}
\]

exists. We say that \( f \) is Fréchet–differentiable (or simply differentiable) at \( u_0 \in A \) if there exists \( df(u_0) \in \mathcal{L}(X,Y) \) such that

\[
\lim_{h \to 0} \frac{f(u_0 + h) - f(u_0) - df(u_0)(v)}{\|h\|} = 0.
\]

If \( f \) is differentiable at any \( u \in A \), then it is simply said differentiable in \( A \).

The element \( df(u_0) \), also denoted by \( f'(u_0) \) or \( Df(u_0) \), is called the differential of \( f \) at \( u_0 \).

As for the finite dimensional case, the following properties hold:

- if \( f \) is Fréchet–differentiable at \( u_0 \), it is also Gateaux–differentiable at \( u_0 \) along any vector;
- if \( f \) is differentiable at \( u_0 \), then it is continuous at \( u_0 \);
- if \( f \) is differentiable at \( u_0 \), then \( df(u_0) \) is unique.

Particularly useful is the following classical result.

Theorem 1.1.2 (Total Differential Theorem). If \( f : A \to Y \) is such that \( f'(u)(v) \) exists \( \forall u \in A, \forall v \in X \), \( f'(u) \in \mathcal{L}(X,Y) \) and the map \( u \in A \mapsto f'(u) \in \mathcal{L}(X,Y) \) is continuous in \( u_0 \in A \), then \( f \) is differentiable at \( u_0 \) and \( df(u_0) = f'(u_0) \).

Example 1.1.3.

1. \( f(u) \equiv c \in \mathbb{R} \Rightarrow df(u) = 0 \ \forall u; \)

2. \( f(u) = Lu \), for some \( L \in \mathcal{L}(X,Y) \) \( \Rightarrow df(u) = L \ \forall u; \)
3. Let $B : X \times Y \rightarrow Z$ be a bilinear and continuous form with values in a Banach space $Z$, $\Rightarrow dB(u,v)(h,k) = B(h,v) + B(u,k) \quad \forall (u,v), (h,k) \in X \times Y$. In particular, if $X = Y$ and $B$ is symmetric, we get $dB(u,v)(v,v) = 2B(u,v);

4. $X$ is a Hilbert space and $f(u) = \|u\|^2 \Rightarrow df(u) = 2u \forall u$.

Definition 1.1.4. Let $f : A \rightarrow Y$ be a differentiable map. Denote by $f' : A \rightarrow \mathcal{L}(X,Y)$ its differential map, called the (Fréchet) differential map of $f$. If $f'$ is continuous, then $f$ is said to be of class $\mathscr{C}^1$.

Notations. From now on, with no other specification, if $H$ is a Hilbert space we denote by $\langle \, , \, \rangle$ its scalar product and by $\| \cdot \|$ its norm.

If $H$ is a Hilbert space and $f : H \rightarrow \mathbb{R}$ is differentiable at $u$, we denote by $\nabla f(u)$ or $\text{grad} f(u)$ the gradient of $f$ at $u$, i.e. the unique element $D \in H$ such that $\langle df(u), v \rangle_H = \langle D, v \rangle \forall v \in X$, which exists by Riesz Representation Theorem.

Remark 1.1.5. 1. Of course, if one changes the norms, the differential remains the same, but the gradient changes.

2. A scalar function $f$ defined on an open set $A$ of a Hilbert space $H$ is of class $\mathscr{C}^1$ if and only if the map $\nabla f : A \rightarrow H$ is continuous.

3. In the case 2 above, it is well known that (see [31])

$$\|f'(u)\|_{H'} = \max_{\|v\|=1} \langle f'(u), v \rangle = \|\nabla f(u)\|,$$

so that $v = \frac{\nabla f(u)}{\|\nabla f(u)\|}$ is the vector in $H$ which maximizes the scalar product $\langle f'(u), v \rangle$ on the unit sphere.

Example 1.1.6. Let $L \in \mathcal{L}(H,H)$ and consider the quadratic form $f(u) = \langle Lu, u \rangle$. Then, for every $u, v \in H$ we have $f'(u)(v) = \langle Lu, v \rangle + \langle Lv, u \rangle$. By the Total Differential Theorem $f$ is differentiable at any point and $df(u)(v) = \langle (L + L^*)u, v \rangle$, where $L^*$ is the adjoint of $L$, so that $\nabla f(u) = (L + L^*)u$. In particular, if $L$ is self–adjoint, then $df = 2L$ and $\nabla f(u) = 2Lu \forall u \in H$, a generalization of point 4 in Remark 1.1.3.

As we will see in the following pages, there is a strict relation between the set of solutions of a PDE with given boundary conditions and set of the critical points of a certain functional defined on a suitable Hilbert (or Banach) space. For this reason it is essential to investigate the existence of critical points. On the other hand there is no easy way to see if a given functional has critical points when working in infinite–dimensional spaces. It turns out that a good way consists in investigating the topological properties of the sublevels of the functional itself. More precisely, the differences between a sublevel and a lower one will imply the existence of critical points in between.

Let us start with some notations. For any $a < b \in \mathbb{R}$ we set $f^b_a := \{u \in X : a \leq f(u) \leq b\}$, $f^a := \{u \in X : f(u) \leq a\}$ and if $f$ is also differentiable, we also set $K_a = \{u \in X: f(u) = a\}$ and $f'(u) = 0\}$. A fundamental concept is introduced in the following definition introduced in [32].
**Definition 1.1.7.** Let $X$ be a Banach space, $f : X \to \mathbb{R}$ be a $C^1$ function and $c \in \mathbb{R}$. We say that a sequence $(u_n)_n$ in $X$ is a Palais-Smale sequence at level $c$, or a $(PS)_c$ sequence, if

$$
\lim_{n \to \infty} f(u_n) = c \quad \text{and} \quad \lim_{n \to \infty} f'(u_n) = 0,
$$

We say that $f$ satisfies the Palais-Smale condition at level $c$, or that $(PS)_c$ holds, if every Palais-Smale sequence at level $c$ has a (strongly) converging subsequence.

It is clear that, if the subsequence converges to $u$, then $f(u) = c$ and $f'(u) = 0$.

Actually, a slightly more general condition was given before $(PS)_c$ was introduced and it is known as $(PS)_c$ condition (so no level is considered). It states that if $(u_n)_n \in \mathbb{N}$ in $X$ is such that the sequence $(f(u_n))_n$ is bounded and $\lim_{n \to \infty} f'(u_n) = 0$, then there exists a converging subsequence. This version is actually a slight generalization of the compactness condition $(C)$ originally introduced by Palais and Smale in [113], sounding as follows: if $V$ is a subset of $X$ on which $|f|$ is bounded and on which $\|df\|$ is not bounded away from zero, then there is a critical point of $f$ in $V$. It is clear that $(PS)$ implies $(C)$, but they are not equivalent, since constant functions satisfy $(C)$ but not $(PS)$.

As one can easily see, $(PS)$ implies $(PS)_c$ at any level $c$, but not viceversa. On the other hand, in most standard cases both of them hold true. Let us also remark that this notion can be compared with that of an a priori estimate: indeed, it states that the set of (quasi)--critical points (i.e. of (quasi)--solutions of differential equations) is not too large.

**Example 1.1.8.** The function $f(x) = e^x$ satisfies $(PS)_c \forall c \neq 0$ (since there are no $(PS)_c$ sequences), but it doesn’t verify $(PS)_0$: in fact $x_n = -n$ is an unbounded $(PS)_0$ sequence. With this example it is clear that $f$ doesn’t satisfy $(PS)$.

We observe that the $(PS)$ condition provides a sort of compactness for particular sequences of functions. In particular it is a sufficient condition for the existence of a minimizer for a differentiable functional which is bounded from below on $X$, thanks to Eckland’s Variational Principle. Note also that $(PS)$ implies that any set of critical points with uniformly bounded energy is relatively compact, as one can easily prove.

We premise the following definition to the main ingredient of a classical critical point theory.

**Definition 1.1.9.** Let $X$ be a topological space and $Y, Z \subset X$. We say that $Z$ is a strong deformation retract of $Y$ (in $X$) if

- $Z \subset Y$;
- there exists a continuous map $\phi : [0,1] \times Y \to X$ such that $\phi(0,u) = u \forall u \in Y$, $\phi(1,u) \in Z \forall u \in Y$ and $\phi(t,u) = u \forall (t,u) \in [0,1] \times Z$.

**Example 1.1.10.** The closed ball $\overline{B}(0,1) = \{ x \in X : \|x\| \leq 1 \}$ is a strong deformation retract of $X$ by the application

$$
\phi(t,x) = \begin{cases} 
x & \text{if } x \in \overline{B}(0,1), \\
(1-t)x + t \frac{x}{\|x\|} & \text{if } \|x\| > 1.
\end{cases}
$$

The basic tools for variational theorems are the following results.
Lemma 1.1.11 (First Deformation Lemma). Let $X$ be a real Banach space, $f \in \mathcal{C}^1(X, \mathbb{R})$ and $-\infty < a \leq b \leq \infty$, possibly $b = \infty$ if $a \in \mathbb{R}$. If
\begin{equation}
\inf \{ \| f'(u) \| : a \leq f(u) \leq b \} > 0,
\end{equation}
then $f^a$ is a strong deformation retract of $f^b$.

Proof. We will not give the complete proof of this Lemma, which is a special case of the following Theorem 1.1.13 below. We present here a simple proof in the special case that $f$ is of class $\mathcal{C}^{1,1}_{\text{loc}}$ and $X$ is a Hilbert space.

Take $u \in f^b$ and consider the evolution problem
\begin{equation}
\begin{cases}
\dot{\mathcal{U}}(t) = - \frac{\nabla f(\mathcal{U}(t))}{\| \nabla f(\mathcal{U}(t)) \|^2}, \\
\mathcal{U}(0) = u.
\end{cases}
\end{equation}
Since $\nabla f$ is locally Lipschitz continuous, by the Cauchy Uniqueness Theorem, there exists $\delta = \delta(u) > 0$ and a unique solution $\mathcal{U} : (-\delta, \delta) \rightarrow X$ of problem (1.2). Let us look for the largest domain of $\mathcal{U}$ in the future, i.e. for $t > 0$.

First, let us note that
\begin{equation}
\frac{d}{dt} f(\mathcal{U}(t)) = \langle \nabla f(\mathcal{U}(t)), \dot{\mathcal{U}}(t) \rangle = -1
\end{equation}
by (1.2), i.e. $\mathcal{U}$ is parameterized by the values of $f$. Let us prove that $\mathcal{U}$ can be extended to $[0, f(u) - a]$. In fact, if the maximal interval is $[0, T)$ with $T \leq f(u) - a$, then from (1.3) we get
\[ f(\mathcal{U}(t)) = f(u) - t > a \quad \forall t \in [0, T). \]
But in the strip $f^a$ by assumption we have $\| \nabla f \| \geq \varepsilon > 0$ for a suitable $\varepsilon$, so that (1.2) implies
\[ \| \dot{\mathcal{U}}'(t) \| = \frac{1}{\| \nabla f(\mathcal{U}(t)) \|} \leq \frac{1}{\varepsilon}, \]
so that $\mathcal{U}$ is Lipschitz continuous in $[0, T)$, and thus uniformly continuous. Then, since $X$ is complete, there exists $u_0 \in X$ such that $\lim_{t \to T^-} \mathcal{U}(t) = u_0$. But then we can solve the problem
\begin{equation}
\begin{cases}
\dot{\mathcal{V}}(t) = - \frac{\nabla f(\mathcal{V}(t))}{\| \nabla f(\mathcal{V}(t)) \|^2}, \\
\mathcal{V}(T) = u_0,
\end{cases}
\end{equation}
which provides a solution $\mathcal{V}$ that is an extension of $\mathcal{U}$, and this is absurd.

Let us denote by $\Psi(t, u)$ the solution of (1.2), which depends continuously on the initial datum $u$ and is defined in the set $\{ (t, u) : a \leq f(u) \leq b, t \in [0, f(u) - a] \}$. Finally, let us consider the function
\[ \phi(t, u) = \begin{cases} 
\Psi(\overline{f(u) - a}, u) & \text{if } a \leq f(u) \leq b, \\
u & \text{if } f(u) \leq a.
\end{cases} \]
It turns out that $\phi$ is the desired strong deformation. \hfill \Box

Of course, it is not clear how to get assumption (1.1) of Lemma 1.1.11. The easiest case is when $M$ is compact, so that it is sufficient to assume that $f'(u) \neq 0$ for all $u$ such that $a \leq f(u) \leq b$. In the noncompact case we have the following
Proposition 1.1.12. Let $X$ be a real Banach space, $f \in \mathcal{C}^1(X, \mathbb{R})$ and $a \leq b$ be real numbers. If $f'(u) \neq 0$ for every $u$ such that $a \leq f(u) \leq b$ and $(PS)_c$ holds for any $c \in [a, b]$, then there exists $\varepsilon > 0$ such that
\[
\inf \{ \|f'(u)\|_{X'} : a - \varepsilon \leq f(u) \leq b + \varepsilon \} > 0.
\]

Proof. Assume by contradiction that there exists a sequence $(u_k)_k$ in $X$ such that $a - \frac{1}{k} \leq f(u_k) \leq b + \frac{1}{k}$ and $f'(u_k) \to 0$. Up to a subsequence, we can assume that $f(u_k) \to c \in [a, b]$, so that we have a $(PS)_c$-sequence. Then we can extract a subsequence which converges to a point $u$ with $f(u) = c$ and $f'(u) = 0$, which is absurd. \hfill $\square$

The proof of the previous Lemma in general Banach spaces with $f$ of class $\mathcal{C}^1$ needs the notion of pseudo-gradient introduced by Palais (see [112]). Moreover, more precise result can be proved, in which the presence of critical points in the strip $f_{\alpha}$ can be handled, and this is the content of the classical Deformation Lemma.

Theorem 1.1.13 (Deformation Lemma). Let $X$ be a real Banach space and $f \in \mathcal{C}^1(X, \mathbb{R})$. Let $c \in \mathbb{R}$ and assume that $f$ satisfies $(PS)_c$. Then for every $\varepsilon > 0$ and for every neighborhood $O$ of $K_c$ there exist $\varepsilon \in (0, \varepsilon)$ and a continuous one-parameter family of homeomorphisms $\eta(t, \cdot)$ of $X$ onto $X$ for every $t \in [0, 1]$ such that
1. $\eta(0, u) = u$ for all $u$ in $X$;
2. $\eta(t, u) = u$ for all $t$ in $[0, 1]$ and for all $u$ such that $f(u) \notin [c - \varepsilon, c + \varepsilon]$;
3. $\|\eta(t, u) - u\| \leq 1$ for all $t \in [0, 1]$ and $u \in X$;
4. $f(\eta(t, u)) \leq f(u)$ for all $t \in [0, 1]$ and $u \in X$;
5. $\eta(1, f^{-\varepsilon} \setminus O) \subset f^{-\varepsilon}$ and $\eta(1, f^{+\varepsilon}) \subset f^{+\varepsilon} \cup O$;
6. if $K_c = \emptyset$, $\eta(1, f^{+\varepsilon}) \subset f^{+\varepsilon}$;
7. $\eta$ has the semigroup property: $\eta(s, \cdot) \circ \eta(t, \cdot) = \eta(s + t, \cdot)$ for all $s, t \geq 0$.

See [124] for a proof.

Remark 1.1.14. The same result holds if $X$ is replaced by a complete manifold $M$.

Another difficulty one can meet is to check the validity of $(PS)$. An easy way to prove it is with the help of the following theorem.

Theorem 1.1.15. Let $H$ be a Hilbert space and $f \in \mathcal{C}^1(H, \mathbb{R})$ such that $\nabla f = T + K$, where $T : H \to H$ is a linear, continuous, invertible operator with $T^{-1}$ continuous, and $K : H \to H$ is a compact operator.

If a bounded sequence $(u_n)_n$ in $H$ is such that $\nabla f(u_n) \to 0$ as $n \to \infty$, then $(u_n)_n$ is relatively compact.

Proof. Since $\nabla f(u_n) = Tu_n + K(u_n) \to 0$ and $T^{-1}$ is continuous, we get that $u_n + T^{-1}K(u_n) \to 0$. Since $K$ is compact, up to a subsequence, $K(u_n) \to u$ in $H$, so that $T^{-1}K(u_n) \to T^{-1}u$. Then $u_n \to -T^{-1}u$. \hfill $\square$

Remark 1.1.16. It is clear that we can always consider $T$ as the identity map on $H$, so that $\nabla f = Id + K$, i.e. $\nabla f$ is a compact perturbation of the identity, a typical request in order to apply the Leray–Schauder topological degree theory.
1.2 Nemytskij operator

In this section we denote by Ω a measurable subset of \( \mathbb{R}^N \) and by \( \|u\|_p \) the norm of a function \( u \) in \( L^p(\Omega) \), \( p \in [1, \infty] \).

**Definition 1.2.1.**
1. We say that \( g : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is a Carathéodory function if
   - the function \( x \mapsto g(x, s) \) is measurable \( \forall s \in \mathbb{R} \);
   - the function \( s \mapsto g(x, s) \) is continuous for a.e. \( x \in \Omega \).
2. The Nemytskij operator associated to \( g \) is the operator defined as \( N_g(u)(x) := g(x, u(x)) \).

For the ease of notation, for every function \( u : \Omega \rightarrow \mathbb{R} \) we shall denote simply by \( g(x, u) \) the function \( x \mapsto g(x, u(x)) \), i.e. the function \( N_g(u)(x) \).

**Proposition 1.2.2.** Let \( g : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) be a Carathéodory function. We have:

(a) if \( u_k, u : \Omega \rightarrow \mathbb{R} \) and \( u_k \rightarrow u \) a.e. in \( \Omega \), then \( g(x, u_k) \rightarrow g(x, u) \) a.e. in \( \Omega \);

(b) if \( u : \Omega \rightarrow \mathbb{R} \) is measurable, then \( g(x, u) \) is measurable;

(c) if \( u, v : \Omega \rightarrow \mathbb{R} \) coincide a.e. in \( \Omega \), then \( g(x, u) = g(x, v) \) a.e. in \( \Omega \).

**Proof.** (a): Since the map \( s \mapsto g(x, s) \) is continuous a.e. in \( \Omega \) and \( u_k \rightarrow u \) a.e., it is immediate that \( g(x, u_k(x)) \rightarrow g(x, u(x)) \) a.e., as claimed.

(b): First assume that \( u \) is a simple function of the form \( u = \sum_{i=1}^{k} \alpha_i \chi_{A_i} \), where \( A_i \) is a measurable subset of \( \Omega \) for any \( i = 1, \ldots, k \) such that \( \bigcup_{i=1}^{k} A_i = \Omega \) and \( A_i \cap A_j = \emptyset \) if \( i \neq j \). Then \( g(x, u) = \sum_{i=1}^{k} g(x, \alpha_i) \chi_{A_i} \). But \( g(x, \alpha_i) \) is measurable for every \( i = 1, \ldots, k \), \( \chi_{A_i} \) is measurable, and then \( g(x, u) \) is measurable as well.

If \( u \) is a measurable function, there exists a sequence \( (u_k)_k \) of step functions such that \( u_k \rightarrow u \) a.e. in \( \Omega \). By (a), \( g(x, u_k) \rightarrow g(x, u) \) a.e. in \( \Omega \), so that \( g(x, u) \) is measurable, since it is limit of the sequence \( (g(x, u_k))_k \), which is measurable by the step above.

(c) Straightforward. \( \square \)

**Remark 1.2.3.** Statement (b) of Proposition 1.2.2 can be read as: "the Nemytskij operator maps measurable functions in measurable functions".

**Theorem 1.2.4.** Let \( g : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) be a Carathéodory function. Suppose there exist \( p_1, p_2 \in [1, \infty) \), \( a \in L^{p_2}(\Omega) \) and \( b \in \mathbb{R} \) such that

\[
|g(x, s)| \leq a(x) + b|s|^{p_1/p_2} \quad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega.
\]

Then \( g(x, u) \in L^{p_2}(\Omega) \) for any \( u \in L^{p_1}(\Omega) \), i.e. \( N_g(L^{p_1}(\Omega)) \subset L^{p_2}(\Omega) \). Moreover, the injection \( N_g : L^{p_1}(\Omega) \rightarrow L^{p_2}(\Omega) \) is continuous.

**Proof.** First note that if \( u \in L^{p_1} \), then \( g(x, u) \) is measurable by (b) of Proposition 1.2.2 and by assumption \( |g(x, u)| \leq a(x) + b|u|^{p_1/p_2} \in L^{p_2}(\Omega) \).

Now consider a sequence \( (u_k)_k \) such that \( u_k \rightarrow u \) in \( L^{p_1}(\Omega) \) and take any subsequence (still labelled by \( (u_k)_k \)). Up to a further subsequence, we can assume that \( u_k \rightarrow u \) a.e. in \( \Omega \). Then (a) of Proposition 1.2.2 implies that \( g(x, u_k) \rightarrow g(x, u) \) a.e. in \( \Omega \). Moreover, \( |g(x, u_k)| \leq a(x) + b|u_k|^{p_1/p_2} \) and \( |u_k|^{p_1/p_2} \) converges to \( |u|^{p_1/p_2} \) in \( L^{p_2}(\Omega) \), so that the generalized Lebesgue Theorem of Dominated Convergence implies that \( g(x, u_k) \rightarrow g(x, u) \) in \( L^{p_2}(\Omega) \). Since this happens for any subsequence, the result follows. \( \square \)
There is a sort of *viceversa* of the previous result:

**Theorem 1.2.5** ([7]). Let \( g : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) be a Carathéodory function and \( p_1, p_2 \in [1, \infty) \). Suppose that \( \mathcal{N}_g(L^{p_1}(\Omega)) \subseteq L^{p_2}(\Omega) \). Then there exist \( a \in L^{p_2}(\Omega) \) and \( b \in \mathbb{R} \) such that (1.4) holds.

Note that the result is true without any continuity assumption on \( \mathcal{N} \), which is however re-obtained by Theorem 1.2.4.

**Proposition 1.2.6.** Let \( g : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) be a linear Carathéodory function, \( p_1, p_2 \in [1, \infty] \) be such that \( p_2 > p_1 \) and \( \mathcal{N}_g : L^{p_1}(\Omega) \rightarrow L^{p_2}(\Omega) \) is defined. Then \( g \) is independent of \( s \).

**Proof.** By assumption there exist two measurable functions \( a, b : \Omega \rightarrow \mathbb{R} \) such that \( g(x, s) = a(x) + b(x)s \). Since \( \mathcal{N}_g(L^{p_1}(\Omega)) \subseteq L^{p_2}(\Omega) \), we get that \( a = \mathcal{N}_g(0) \in L^{p_2}(\Omega) \), so that \( b(x)u \in L^{p_2}(\Omega) \) for any \( u \in L^{p_1}(\Omega) \). Then \( b \equiv 0 \). Indeed, if there exist \( \varepsilon > 0 \) and \( \Omega' \subset \Omega \) with \( |\Omega'| > 0 \) and \( |b(x)| \geq \varepsilon \) a.e. \( x \in \Omega' \), we could take \( u \in L^{p_1}(\Omega) \setminus L^{p_2}(\Omega') \) so that

\[
L^{p_2}(\Omega') \not\ni \varepsilon |u| \leq |b(x)u| \in L^{p_2}(\Omega)
\]

and a contradiction arises. \( \square \)

**Remark 1.2.7.** The previous Proposition does not states that there are not linear continuous operators \( L^{p_1} \rightarrow L^{p_2} \) with \( p_2 > p_1 \) of course there are many, but they are not Nemyskij operators. Consider, for example, the operator \( I : L^1(0,1) \rightarrow L^\infty(0,1) \) defined as \( Iu(x) = \int_0^x u(t) \, dt \).

Up to now we only considered \( L^p \) with \( p \neq \infty \). Now we show a continuity result in \( L^\infty \).

**Proposition 1.2.8.** Let \( g : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) be a Carathéodory function and suppose that there exist \( a \in L^p(\Omega), \ p \geq 1, \ g_0 : \mathbb{R} \rightarrow \mathbb{R} \) locally bounded such that \( |g(x,s)| \leq a(x)g_0(s) \ \forall s \in \mathbb{R}, \ \text{a.e.} \ x \in \Omega \). Then \( \mathcal{N}_g : L^\infty(\Omega) \rightarrow L^p(\Omega) \) is continuous.

**Proof.** By (b) of Proposition 1.2.2, \( g(x,u) \) is measurable, and from the growth assumption on \( g \), we get that \( |g(x,u)| \leq a(x)g_0(u) \in L^p(\Omega) \ \forall u \in L^\infty(\Omega) \). Moreover, if \( u_k \rightharpoonup u \) in \( L^\infty(\Omega) \), then by (a) of Proposition 1.2.2, \( g(x,u_k) \rightarrow g(x,u) \) a.e. in \( \Omega \) and if \( |u_k| \leq M \) a.e. in \( \Omega \ \forall n \), then \( |g(x,u_k) - g(x,u)|^p \leq 2^{p-1}a(x)^p\max_{[-M,M]} g_0^p \), and from the Lebesgue Theorem of Dominated Convergence we get that \( g(x,u_k) \rightarrow g(x,u) \) in \( L^p(\Omega) \). \( \square \)

It is immediate to prove the following result:

**Proposition 1.2.9.** Let \( g : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) be a Carathéodory function and set \( G(x,s) := \int_0^s g(x,t) \, dt \). Then \( G : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is a Carathéodory function such that the map \( s \mapsto G(x,s) \) is of class \( \mathcal{C}^1 \) for every \( s \in \mathbb{R} \) and a.e. \( x \in \Omega \).

From now on, though not stated all the times, we understand that \( g : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is a Carathéodory function and \( G(x,s) := \int_0^s g(x,t) \, dt \).

**Theorem 1.2.10.** Let \( 1 \leq p_1 < p_2 < \infty \) and assume that there exist \( a \in L^q(\Omega), \ q = \frac{p_1 p_2}{p_2 - p_1} \) and \( b \in \mathbb{R} \) such that

\[
|g(x,s)| \leq a(x) + b|s|^{\frac{p_2}{p_1} - 1} \ \forall s \in \mathbb{R}, \ \text{a.e. in} \ \Omega.
\]

Then \( \mathcal{N}_G : L^{p_2}(\Omega) \rightarrow L^{p_1}(\Omega) \) is of class \( \mathcal{C}^1 \) and \( d\mathcal{N}_G(u)(v) = \mathcal{N}_g(u)(v) \ \forall u, v \in L^{p_2}(\Omega) \).
Proof. The growth condition on \( g \) implies that \( N_G : L^{p_2}(\Omega) \to L^{p_1}(\Omega) \) is well defined, as well as \( N_g : L^{p_2}(\Omega) \to L^q(\Omega) \), while Proposition 1.2.9 ensures that the map \( s \mapsto G(x, s) \) is of class \( \mathcal{C}^1 \).

Take \( u \in L^{p_2}(\Omega) \). The growth condition on \( g \) also implies that the map \( L^{p_2}(\Omega) \to L^{p_1}(\Omega) \), \( v \mapsto g(x, u)v \) is a well defined linear continuous operator. By the Total Differential Theorem we get that \( N_G : L^{p_2}(\Omega) \to L^{p_1}(\Omega) \) is Fréchet differentiable at any point. The fact that \( dN_G(u)(v) = N_g(u)(v) \forall u, v \in L^{p_2}(\Omega) \) is a simple calculation.

Finally, let us note that for any \( u_1, u_2, v \in L^{p_2}(\Omega) \) by the Hölder inequality

\[
\|g(x, u_1)v - g(x, u_2)v\|_{p_1} \leq \|g(x, u_1) - g(x, u_2)\|_q \|v\|_{p_2},
\]

that is

\[
\|dN_G(u_1) - dN_G(u_2)\|_{\mathcal{L}(L^{p_2}(\Omega), L^{p_1}(\Omega))} \leq \|g(x, u_1) - g(x, u_2)\|_q,
\]

so that \( (N_G)' \) is continuous and \( N_G \) is of class \( \mathcal{C}^1 \).

A special case of Theorem 1.2.10 is when, setting \( p_2 = p > 2 \), we choose \( p_1 = p' = \frac{P}{p-1} < 2 \), so that we get the following particular version.

**Theorem 1.2.11.** Assume that there exist \( p > 2 \) and \( a \in L^q(\Omega) \), \( q = \frac{P}{p-2} \), such that

\[
|g(x, s)| \leq a(x) + |s|^{p-2} \quad \forall s \in \mathbb{R} \text{ a.e. in } \Omega.
\]

Then \( N_G : L^p(\Omega) \to L^{p'}(\Omega) \) is Fréchet differentiable and \( dN_G(u)(v) = N_g(u)(v) \forall u, v \in L^p(\Omega) \).

**Remark 1.2.12.** The extreme case \( p_1 = p_2 \) in Theorem 1.2.10, that is the case \( p = 2 \) in Theorem 1.2.11, are special ones. In this case, in general, \( N_G \) is only Gateaux–differentiable, but not Fréchet differentiable.

An essential tool in our future investigation is the following result about the differentiability of the Nemytskij operator defined in Sobolev spaces.

**Theorem 1.2.13.** Let \( 1 \leq p < N \) and \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function such that

\[
|g(x, s)| \leq a(x) + |s|^{q} \quad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega,
\]

where \( a \in L^{\frac{pN}{N(p-1)}}(\Omega) \), \( b \in \mathbb{R} \), \( q \in [1, p^*-1] \), and \( p^*= \frac{pN}{N-p} \) is the critical Sobolev exponent.

Then the function \( G : W^{1,p}(\Omega) \to \mathbb{R} \) defined as \( G(u) = \int_{\Omega} G(x, u) \, dx \) is a \( \mathcal{C}^1 \) function such that

\[
G'(u)(v) = \int_{\Omega} g(x, u)v \, dx \quad \forall u, v \in W^{1,p}(\Omega).
\]

**Proof.** It is a consequence of the Theorems on the differentiability of the Nemytskij operator and of Lebesgue’s Theorem. See also [124, Theorem C.1] for more general integral operators in the case \( p = 2 \).
1.3 Weak solutions

Let us consider the following Dirichlet problem in presence of a nonlinear Poisson equation

\begin{equation}
-\Delta u = g(x, u) \quad \text{in } \Omega,
\end{equation}
\begin{equation}
u = 0 \quad \text{on } \partial \Omega,
\end{equation}

where \( \Omega \) is a domain (open and connected set) of \( \mathbb{R}^N, N \geq 1 \), and \( g : \Omega \times \mathbb{R} \) is a Carathéodory function. If \( u \in H^2(\Omega) \cap H^1_0(\Omega) \) satisfies (1.5) a.e. in \( \Omega \), take any \( v \in H^1_0(\Omega) \) and multiply both sides of (1.5) by \( v \). By the Gauss Green formula we finally end up with

\begin{equation}
\int_\Omega Du \cdot Dv \, dx = \int_\Omega g(x, u)v \, dx \quad \forall v \in H^1_0(\Omega).
\end{equation}

Going the other way, we give the following

Definition 1.3.1. We say that \( u \in H^1_0(\Omega) \) is a weak solution of problem (1.5) if (1.6) holds.

Remark 1.3.2. One can define weak solutions requiring that (1.6) holds for any \( v \in C^1_0(\Omega) \) or for any \( v \in C^\infty_0(\Omega) \), but this is equivalent to our choice since both \( C^1_0(\Omega) \) and \( C^\infty_0(\Omega) \) are dense in \( H^1_0(\Omega) \).

It is clear that if \( g \) has non supercritical growth in the sense of the Sobolev exponent, then problem (1.5) is variational: denoting by \( f : H^1_0(\Omega) \to \mathbb{R} \) the functional defined as

\begin{equation}
f(u) = \frac{1}{2} \int_\Omega |Dv|^2 \, dx - \int_\Omega G(x, u) \, dx,
\end{equation}

where as usual \( G(x, s) = \int_0^s g(x, t) \, dt \), then \( f \) is differentiable by Theorem 1.2.13 and \( u \) is a weak solution of problem (1.5) if and only if \( u \) is a critical point of \( f \). Indeed, \( u \) is a critical point for \( f \) if and only if \( f'(u)(v) = 0 \) for any \( v \in H^1_0(\Omega) \), that is (1.6).

Then, in order to find solutions of problem (1.5), we will look for critical points of \( f \).

Historically, problem (1.5) is called the strong Euler–Lagrange equation associate to \( f \), while (1.6) is called the weak Euler–Lagrange equation associate to \( f \). We have seen that (strong) solutions of the strong Euler–Lagrange equation (1.5) are also (weak) solutions of the weak Euler–Lagrange equation (1.6). Viceversa, by regularity results it is possible to show that in most essential cases weak solutions are also strong solutions; this is the case, for example, when \( g \) has subcritical growth in the sense of the Sobolev exponent.

Theorem 1.3.3. Let \( \Omega \) be of class \( \mathcal{C}^2 \) and let \( u \in H^1_0(\Omega) \) be a weak solution of (1.5), where \( g : \Omega \times \mathbb{R} \) is a Carathéodory function such that there exist \( C > 0 \) and \( p > 0 \) for which

\[ |g(x, s)| \leq C(1 + |s|^{p-1}) \quad \forall s \in \mathbb{R} \text{ a.e. } x \in \Omega. \]

Moreover, assume that \( p \leq \frac{2N}{N-2} \) if \( N \geq 3 \).

Then \( u \in \mathcal{C}^{1+\alpha}(\overline{\Omega}) \) for some \( \alpha \in (0, 1) \). If in addition \( g \) is Hölder continuous, then \( u \) is of class \( \mathcal{C}^2(\overline{\Omega}) \).

Finally, if \( \Omega \) is of class \( \mathcal{C}^{k+2} \) and \( g \) is of class \( \mathcal{C}^{k,\alpha}(\overline{\Omega} \times \mathbb{R}) \), then \( u \in W^{k+2,2}(\Omega) \) and if \( \Omega \) is of class \( \mathcal{C}^\infty \) and \( g \) is of class \( \mathcal{C}^\infty(\overline{\Omega} \times \mathbb{R}) \), then \( u \in \mathcal{C}^\infty(\overline{\Omega}) \).
We underline the fact that such a regularity result requires that $g$ is not supercritical, but it lets $g$ be sublinear. The proof of this theorem is based on Moser’s iteration technique (see [104]) and its development due to Brezis and Kato (see [33]): first one shows that $u$ belongs to $W^{2,q}(\Omega) \cap H^1_0(\Omega)$ $\forall q < \infty$, so that by the Morrey Theorem $u \in C^{1,\alpha}(\Omega)$ (see [124], Lemma B.3 and the following statements). If in addition $g$ is Hölder continuous, the $C^2(\Omega)$ regularity is proved with the help of Schauder techniques, see [73, Chapter 6]. See also [73, Theorem 8.13] for some estimates of the solution in terms of the data and [105] for some $L^\infty - H^1$ estimates.

Throughout these notes we will consider only elliptic problems where the governing operator is the Laplacian, but everything can be immediately extended to other problems where the principal part is replaced by a more general uniformly elliptic operator in divergence form and the boundary condition is not 0. Let us see how: consider the problem

\begin{equation}
\begin{cases}
- \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = g(x,u) & \text{in } \Omega, \\
u = u_0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}

where $\Omega$ is a domain of $\mathbb{R}^N$, $N \geq 1$, $g : \Omega \times \mathbb{R}$ is a Carathéodory function and the principal part is uniformly elliptic according to the following

**Definition 1.3.4.** The operator

\[ \mathcal{L} = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right) \]

is called uniformly elliptic if $a_{ij} = a_{ij} \in L^\infty(\Omega) \forall i,j = 1, \ldots, N$, and there exists $\lambda_0 > 0$ (the ellipticity constant) such that

\[ \sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j \geq \lambda_0 |\xi|^2 \quad \forall \xi \in \mathbb{R}^N \text{ a.e. } x \in \Omega. \]

First, let us note that in problem (1.8) we can assume $u_0 = 0$, provided $u_0$ is sufficiently regular, say $u_0$ is the restriction to $\partial \Omega$ of a function $\tilde{u}_0 \in H^1(\Omega)$. In fact, consider the solution $v \in H^1(\Omega)$ to the problem

\begin{equation}
\begin{cases}
- \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial v}{\partial x_j} \right) = 0 & \text{in } \Omega, \\
v = u_0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}

(which exists, for example, by [73, Theorem 8.3]); then the function $z := u - v$ solves (both weakly and strongly) the problem

\begin{equation}
\begin{cases}
- \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial z}{\partial x_j} \right) = g(x,u) = g(x, z + v) := \tilde{g}(x, z) & \text{in } \Omega, \\
z = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}

and $\tilde{g}$ satisfies all the same growth condition satisfied by $g$ in the $s$ variable (if there are any). Moreover, we can apply Theorem 1.3.3 to the function $v$, so that higher regularity on $\Omega$ and $g$ imply higher regularity on $v$ itself.
Now let us show that problem (1.8) with \( u_0 = 0 \) is variational and then it can be treated as (1.5). With an obvious generalization of Definition 1.3.1, we say that \( u \in H^1_0(\Omega) \) is a weak solution of

\[
\begin{aligned}
- \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) &= g(x,u) \quad \text{in} \ \Omega, \\
 u &= 0 \quad \text{on} \ \partial \Omega,
\end{aligned}
\]

if

\[
\int_{\Omega} \sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx = \int_{\Omega} g(x,u) v \, dx \quad \forall v \in H^1_0(\Omega),
\]

so that weak solutions of (1.11) are critical points of the functional \( F : H^1_0(\Omega) \rightarrow \mathbb{R} \) defined as

\[
F(u) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} \, dx - \int_{\Omega} G(x,u) \, dx.
\]

We remark that the previous statement is not true any more if we drop the symmetry assumption \( a_{ij} = a_{ji} \forall i, j = 1, \ldots, N \).

Then we can proceed in a standard way: the operator

\[
\mathcal{L} = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right)
\]

has a diverging sequence of eigenvalues \( 0 < \mu_1 \leq \mu_2 \leq \ldots \) on \( H^1_0(\Omega) \) which play the role of the eigenvalues \( 0 < \mu_1 \leq \mu_2 \leq \ldots \) of \(-\Delta\) on \( H^1_0(\Omega)\). Moreover it provides \( H^1_0(\Omega) \) with the new scalar product

\[
\langle u, v \rangle_* = \int_{\Omega} \sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx + \int_{\Omega} uv \, dx,
\]

which is equivalent to the usual one, and which induces the new norm

\[
\| u \|_*^2 = \int_{\Omega} \sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} \, dx + \int_{\Omega} u^2 \, dx.
\]

As usual, the second integral in the two definitions above can be removed if \( \Omega \) is bounded. In this last case the gradient of \( F \) with respect to the new scalar product is again given by \( \nabla_* F(u) = u + \mathcal{L}^{-1}(g(x,u)) \), in the same way as the gradient of the functional \( f \) defined in (1.7) with respect the standard scalar product is \( \nabla f(u) = u + \Delta^{-1}(g(x,u)) \), as we will see in Section 2.1.

**Remark 1.3.5.** Concerning the definition of ellipticity, there are other cases considered in literature, which are slightly different from Definition 1.3.4. For example we can say that the operator

\[
\mathcal{L} = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right)
\]
is uniformly elliptic if $a_{i,j}$ is a measurable function $\forall i, j = 1, \ldots, N$, and there exist $\Lambda_0 > \lambda_0 > 0$ such that

\[
\lambda_0 |\xi|^2 \leq \sum_{i,j=1}^{N} a_{i,j}(x)\xi_i \xi_j \leq \Lambda_0 |\xi|^2 \quad \forall \xi \in \mathbb{R}^N \text{ a.e. } x \in \Omega.
\]

Surprisingly, this new definition and the one above coincide only under the symmetry assumption $a_{ij} = a_{ji} \forall i,j = 1, \ldots, N$. Indeed, if $\mathcal{L}$ satisfies Definition 1.3.4, then

\[
\sum_{i,j=1}^{N} a_{i,j}(x)\xi_i \xi_j \leq \max_{i,j} \{\|a_{i,j}\|_{\infty}\} \sum_{i,j=1}^{N} |\xi_i||\xi_j| \leq \frac{M}{2} \sum_{i,j=1}^{N} (|\xi_i|^2 + |\xi_j|^2) = MN|\xi|^2,
\]

where we have set $M = \max_{i,j} \{\|a_{i,j}\|_{\infty}\}$. Conversely, if $\mathcal{L}$ satisfies this new definition of ellipticity, then $a_{ij} \in L^\infty(\Omega) \forall i,j = 1, \ldots, N$. Otherwise if $a_{ij} \notin L^\infty(\Omega)$ and for example $a_{12} > 0$ and $a_{i,j} \notin L^\infty(\Omega')$ with $\Omega' \subset \Omega$, take $\xi$ such that $\xi_i = \xi_j = 1$ and $\xi_i = 0 \forall i \neq \bar{i}, \bar{j}$, so that

\[
2\Lambda_0 = \Lambda_0 |\xi|^2 \geq \sum_{i,j=1}^{N} a_{i,j}(x)\xi_i \xi_j = 2a_{12} \notin L^\infty(\Omega).
\]

Note that we used the symmetry $a_{ij} = a_{ji} \forall i,j = 1, \ldots, N$ precisely in this last calculation. If this condition is removed, the equivalence fails: it is enough to consider $a_{12} = \log x$, $a_{21} = -\log x + 1$ and $a_{ij} = \delta_{ij}$ in the other cases. Then

\[
\sum_{i,j=1}^{N} a_{i,j}(x)\xi_i \xi_j = \xi_1 \xi_2 + |\xi|^2,
\]

so that (1.12) is satisfied, but of course $a_{12}$ and $a_{21}$ are not bounded.
Chapter 2

Mountain pass theorems

In [6] a fundamental result in critical point theory was proved. Due to the geometrical structure involved therein, it is called the Mountain Pass Theorem. We present the following version of the theorem, where the geometrical situation is referred to the point 0, but if the situation is translated somewhere else, of course the result still holds.

**Theorem 2.0.6** (Mountain Pass Theorem, [6]). Let $X$ be a real Banach space and $f \in C^1(X, \mathbb{R})$ be such that $f(0) = 0$. Assume that there exist $\rho, \alpha > 0$ such that

$$\inf_{\|u\| = \rho} f(u) = \alpha$$

and there exists $e$ in $X \setminus B_\rho$ such that $f(e) \leq 0$.

Set $\Gamma := \{ \gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e \}$ and

$$\beta = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)).$$

Finally, assume that $(PS)_\beta$ holds. Then $\beta$ is a critical value.

**Proof.** First, take $\gamma \in \Gamma$, so that $\gamma([0,1]) \cap S_\rho \neq \emptyset$, where $S_\rho = \{ u \in X : \|u\| = \rho \}$. In this way

$$\max_{t \in [0,1]} f(\gamma(t)) \geq \alpha,$$

and then $\beta \geq \alpha$. Moreover, taking $\gamma(t) = te$, the function $(\gamma(t))$ is continuous on $[0,1]$, so that it admits a maximum $M < \infty$. Then $\beta \leq M$.

Assume by contradiction that $\beta$ is not a critical value. Then by Proposition 1.1.12, there exists $\varepsilon > 0$ such that

$$\inf \{ \|f'(u)\| : \beta - \varepsilon \leq f(u) \leq \beta + \varepsilon \} > 0,$$

and of course we can choose $\varepsilon$ such that $\beta - \varepsilon > 0$. By the First Deformation Lemma (Theorem 1.1.11), there exists strong deformation $\mathcal{U}(t, \cdot)$ of $f^{\beta + \varepsilon}$ on $f^{\beta - \varepsilon}$. By definition of $\beta$ there exists $\gamma \in \Gamma$ such that $\max f(\gamma) < \beta + \varepsilon$.

Now consider the function $h(t) := \mathcal{U}(1, \gamma(t))$, which is of class $C([0,1], X)$. First note that $h(0) = \mathcal{U}(1, \gamma(0)) = \mathcal{U}(1, 0) = 0$, since $0 \in f^0 \subset f^{\beta - \varepsilon}$, and $f^{\beta - \varepsilon}$ is kept fixed by $\mathcal{U}$. In the same way $h(1) = e$, so that $h \in \Gamma$ and so

$$\beta \leq \max_{t \in [0,1]} f(h(t)).$$
On the other hand, $\gamma([0, 1]) \subset f^{\beta+\varepsilon}$, so that $\mathcal{W}(1, \gamma([0, 1])) \subset f^{\beta-\varepsilon}$, that is
\[
\max_{t \in [0, 1]} f(h(t)) \leq \beta - \varepsilon,
\]
and a contradiction arises.

**Definition 2.0.7** (See [75]). Let $X$ be a real Banach space, $f \in C^1(X, \mathbb{R})$ and $u \in X$ be such that $f'(u) = 0$. We say that $u$ is a mountain pass point

- **in the sense of Katriel** if for any neighborhood $U$ of $u$ the set $U \cap \{x \in X : f(x) < f(u)\}$ is non empty and disconnected;

- **in the sense of Hofer** if for any neighborhood $U$ of $u$ the set $U \cap \{x \in X : f(x) < f(u)\}$ is non empty and not path-connected.

It may happen that critical points found with the help of the Mountain Pass Theorem are not mountain pass points:

**Example 2.0.8.** Consider the function $f(x) = \|x\|^2 - \|x\|^4$ on $X$. Then all the critical points found with the Mountain Pass Theorem are maxima points.

As it is natural, symmetries in the functional lead to the existence of several critical points: if $f$ is even and $u$ is a critical point at level $c$, then also $-u$ is a critical point at level $c$. We recall some classical theorems from Ljusternik and Schnirelmann theory.

**Theorem 2.0.9** (Ljusternik–Schnirelmann, [89]). Let $f \in C^1(\mathbb{R}^N, \mathbb{R})$ be a even function. Then $f|_{S^{N-1}}$ admits at least $N$ couples $(u, -u)$ of critical points.

In infinite dimension the compactness property of the sphere is lost, so that an additional assumption is needed. Again, the $(PS)$–condition turns out to be a good one, so that we get the following extension of Theorem 2.0.9.
Let $f$ be an even $C^1$ function defined on an infinite dimensional Banach space $X$. If there exists $\alpha \in \mathbb{R}$ such that $f|_{S_\alpha}$ satisfies $(PS)_c$ for all $c \in \mathbb{R}$ and $f|_{S_\alpha}$ is bounded from below, then $f|_{S_\alpha}$ has infinitely many couples $(u, -u)$ of distinct critical points.

Verifying $(PS)$ for $f$ constrained on spheres is not so immediate, and it is more natural to work with unconstrained functionals. In this setting the following result is extremely useful (see also [16, Theorem 9.12]).

Let $f$ be an even $C^1$ function defined on an infinite dimensional Banach space $X$ such that $f(0) = 0$. Assume that $X$ is decomposable as direct sum of two closed subspaces $X = X_1 \oplus X_2$ with $\dim X_1 < \infty$. Suppose that

(i) there exist $\alpha, \rho > 0$ such that $\inf_{S_\rho \cap X_2} f \geq \alpha$;

(ii) for any finite dimensional subspace $Y \subset X$ there exists $R = R(Y) > 0$ such that $f(u) \leq 0$ for all $u \in Y$ with $\|u\| \geq R$;

(iii) $(PS)_c$ holds for all $c \geq \alpha$.

Then $f$ has an unbounded sequence of positive critical values.

**Remark 2.0.12.** If $X_1 = \{0\}$, condition (i) of Theorem 2.0.11 is nothing but the first condition in Theorem 2.0.6, and, indeed, a first version of Theorem 2.0.11 was proved in [6] in the case $X = X_2$.

### 2.1 Applications: one solution

Let us consider the problem

$$
(P) \quad \begin{cases} 
-\Delta u - \lambda u = g(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
$$

where $\Omega$ is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, $\lambda \in \mathbb{R}$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$. First of all we assume the standard assumptions for a superlinear and subcritical nonlinearity, whose prototype is the function $g(x, s) = |s|^{p-2} s$ (such assumptions are quite natural and appear quite often while studying nonlinear subcritical problems, even in less general forms, see [6], [81], [116]):

(1) $g : \Omega \times \mathbb{R}$ is a Carathéodory function;

(2) there exist constants $a_1, a_2 > 0$ and $p > 2$ such that $\forall s \in \mathbb{R}$ and for a.e. $x \in \Omega$

$$
|g(x, s)| \leq a_1 + a_2 |s|^{p-1},
$$

where $p < \frac{2N}{N-2}$ if $N \geq 3$;

(3) $g(x, s) = o(|s|)$ as $s \to 0$ uniformly in $\Omega$;

(4) $\exists \mu > 2$ and $R \geq 0$ such that $\forall s$ with $|s| > R$ and for a.e. $x \in \Omega$

$$
0 < \mu G(x, s) \leq g(x, s)s,
$$

and there exist $a_3 > 0$ and a positive function $a_4 \in L^1(\Omega)$ such that $\forall s \in \mathbb{R}$ and for a.e. $x \in \Omega$, $G(x, s) \geq a_3 |s|^\mu - a_4(x)$.
Note that \((g_3)\) implies that \(G(x, |s|) = o(|s|^2)\) as \(s \to 0\) uniformly in \(\Omega\) and that \(u = 0\) is a solution of problem \((P)\). Condition \((g_4)\) is known as the Ambrosetti–Rabinowitz condition.

**Remark 2.1.1.** Condition \((g_2)\) implies that \(|G(x, s)| \leq a_1|s| + \frac{a_2}{p}|s|^p\ \forall s \in \mathbb{R} \) and for a.e. \(x \in \Omega\), so that it is a subcritical assumption on \(G\), since \(p\) is strictly less than \(2^* = \frac{2N}{N-2}\), the Sobolev embedding exponent.

**Remark 2.1.2.** As seen in \((g_2)\), no upper bound on the exponent \(p\) is given when \(N = 1, 2\), since \(H^1_0(\Omega) \hookrightarrow L^\infty(\Omega)\) if \(N = 1\) and \(H^1_0(\Omega) \hookrightarrow L^q(\Omega)\) for any \(q < \infty\) if \(N = 2\). Actually, by the Trudinger–Moser inequality (see [73, Theorem 7.15]) in the case \(N = 2\) we could replace \((2.1)\) with

\[|g(x, s)| \leq a_1e^{\phi(s)} + a_2, \quad \text{where } \lim_{s \to \infty} \frac{\phi(s)}{s^2} = 0.\]

**Remark 2.1.3.** Condition \((g_4)\) says that \(G\) is superquadratic, or that \(g\) is superlinear. We also note that the final request in \((g_4)\) is useless if \(\Omega\) is smooth and \(g : \overline{\Omega} \times \mathbb{R}\) is continuous. In fact, when \(R > 0\) integrating \((2.2)\) gives \(G(x, s) \geq a_3|s|^{\mu} - a_4(x)\), where

\[a_3 = \min \left\{ \min_{x \in \Omega} \frac{G(x, R)}{R^{\mu}}, \min_{x \in \Omega} \frac{G(x, -R)}{R^{\mu}} \right\}\]

and

\[a_4(x) = \max \left\{ \max_{\sigma \in [0, R]} G(x, \sigma), \max_{\sigma \in [-R, 0]} G(x, \sigma) \right\}.\]

If \(R = 0\) the value \(R\) in the definitions above must be replaced by limits.

Let us denote by \(0 < \lambda_1 < \lambda_2 \leq \ldots\) the diverging sequence of the eigenvalues of \(-\Delta\) in \(H^1_0(\Omega)\). If \(i \geq 1\) set \(H_i = \text{Span}(e_1, \ldots, e_i)\), where \(e_j\) is the eigenfunction associated to \(\lambda_j\), and define \(H_i^\perp\) as the orthogonal complement of \(H_i\) in \(H^1_0(\Omega)\).

We want to prove the following

**Theorem 2.1.4.** Assume \((g_1) - (g_4)\) and \(\lambda < \lambda_1\). Then problem \((P)\) admits a nontrivial solution.

**Proof.** The proof will be made with the help of the Mountain Pass Theorem, for which the case \(\lambda \geq \lambda_1\) cannot be handled (for this case, see Chapter 3).

First of all let us recall that \(H^1_0(\Omega)\), as usual, is endowed with the scalar product \(\langle u, v \rangle = \int Du \cdot Dv\), which induces the usual norm \(\|u\|^2 = \int |Du|^2\).

Now consider the functional \(f : H^1_0(\Omega) \rightarrow \mathbb{R}\) defined as

\[f(u) = \frac{1}{2} \int_{\Omega} |Du|^2 \, dx - \lambda \frac{1}{2} \int_{\Omega} u^2 \, dx - \int_{\Omega} G(x, u) \, dx.\]

It is easy to see that \(f\) is a \(C^1\) functional on \(H^1_0(\Omega)\) whose critical points solve problem \((P)\).

Thus Theorem 2.1.4 will be proved if we show the existence of a nontrivial critical point for \(f\). To this aim, we want to apply Theorem 2.0.6 to \(f\).

First, \(f(0) = 0\) and by \((g_3)\), fixed \(\varepsilon > 0\), there exists \(\delta > 0\) such that \(|G(x, s)| \leq \frac{\varepsilon}{2}|s|^2\) for any \(s\) such that \(|s| \leq \delta\) and for a.e. \(x \in \Omega\). Moreover, if \(|s| \geq \delta\), by Remark 2.1.1 we get

\[|G(x, s)| \leq a_1|s| + \frac{a_2}{p}|s|^p \leq \left(\frac{a_1}{\delta p - 1} + \frac{a_2}{p}\right)|s|^p = c_3|s|^p,
\]
so that
\[
|G(x, s)| \leq \frac{\varepsilon}{2} |s|^2 + c_3 |s|^p \quad \forall s \in \mathbb{R} \text{ and for a.e. } x \in \Omega.
\]
By the Poincaré, Hölder and Sobolev inequalities there exist \( c_1 = c_1(\delta, |\Omega|) > 0 \) such that
\[
(2.4) \quad \left| \int_{\Omega} G(x, s) \, dx \right| \leq \varepsilon \frac{1}{2\lambda_1} \|u\|^2 + c_1 \|u\|^p.
\]
In this way (we can assume \( \lambda > 0 \), the case \( \lambda \leq 0 \) being easier)
\[
f(u) \geq \left[ \frac{1}{2} \left( 1 - \frac{\lambda + \varepsilon}{\lambda_1} \right) - c_1 \|u\|^{p-2} \right] \|u\|^2.
\]
Let us choose \( \varepsilon \) so small that \( \lambda + \varepsilon < \lambda_1 \). Now take
\[
\inf_{S_p} f > 0.
\]
Now take \( u \neq 0 \) in \( H_0^1(\Omega) \) and \( t > 0 \). Then by \( (g_4) \)
\[
f(tu) \leq \frac{t^2}{2} \int_{\Omega} |Du|^2 \, dx - \frac{\lambda t^2}{2} \int_{\Omega} u^2 \, dx - a_3 t^p \int_{\Omega} |u|^p \, dx + \int_{\Omega} a_4(x) \, dx,
\]
so that
\[
\lim_{t \to \infty} f(tu) = -\infty,
\]
and the last geometrical condition of the Theorem 2.0.6 follows.

Now, let us prove the Palais–Smale condition. By Theorem 1.1.15 it is enough to prove that for every \( c \in \mathbb{R} \) (more than required, actually) any \( (PS)_c \) sequence is bounded and that \( \nabla f \) has the form \( Id + K \), where \( K \) is a compact operator.

First step: \( \nabla f = Id + K \). Indeed, we have
\[
\nabla f(u) = u + \Delta^{-1}(\lambda u + g(x, u)),
\]
where \( \Delta^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega) \) is the operator which associates to any element \( h \in H^{-1}(\Omega) = (H_0^1(\Omega))^\prime \) the unique solution in \( H_0^1(\Omega) \) of the problem
\[
\begin{cases}
\Delta u = h & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
which exists by the Lax–Milgram Theorem.

By \( (g_2) \), for any \( u \in H_0^1(\Omega) \), the function \( \lambda u + g(x, u) \) belongs to \( L^p(\Omega) \) and since \( p > 2 \), \( L^p(\Omega) \subset H^{-1}(\Omega) \), so that \( \Delta^{-1}(\lambda u + g(x, u)) \) is well defined.

Claim: \( \Delta^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega) \) is compact. Indeed, consider the Nemytskij operator
\[
N_g : \quad H_0^1(\Omega) \to L^p(\Omega) \to L^p(\Omega) \to H^{-1}(\Omega)
\]
\[
u \quad \mapsto \quad u \quad \mapsto \quad g(\cdot, u) \quad \mapsto \quad g(\cdot, u);
\]
since the embedding $H^1_0(\Omega) \hookrightarrow L^p(\Omega)$ is compact, we get that $\mathcal{N}_g$ is compact as well.

**Second step: any $(PS)_c$ sequence is bounded.** Take $c \in \mathbb{R}$ and a $(PS)_c$ sequence $(u_n)_n$, that is $\nabla f(u_n) \to 0$ and $f(u_n) \to c$ as $n \to \infty$. Then there exist $M, N > 0$ such that

$$
(2.6) \quad \mu f(u_n) - \langle \nabla f(u_n), u_n \rangle \leq M + N \|u_n\|.
$$

On the other hand,

$$
(2.7) \quad \mu f(u_n) - \langle \nabla f(u_n), u_n \rangle = \left( \frac{\mu}{2} - 1 \right) \|u_n\|^2 - \lambda \left( \frac{\mu}{2} - 1 \right) \int_\Omega u_n^2 \, dx + \int_\Omega [g(x, u_n)u_n - \mu G(x, u_n)] \, dx.
$$

Now, let us write

$$
\int_\Omega = \int_{\{x \in \Omega: |u_n(x)| \leq R\}} + \int_{\{x \in \Omega: |u_n(x)| > R\}},
$$

so that, by $(g_2)$ and $(g_4)$ there exists $c_R > 0$ such that

$$
(2.8) \quad \left( \frac{\mu}{2} - 1 \right) \|u_n\|^2 - \lambda \left( \frac{\mu}{2} - 1 \right) \int_\Omega u_n^2 \, dx + \int_\Omega [g(x, u_n)u_n - \mu G(x, u_n)] \, dx 
\geq \left( \frac{\mu}{2} - 1 \right) \left( 1 - \frac{\lambda}{\lambda_1} \right) \|u_n\|^2 - c_R.
$$

Of course, here we used the Poincaré inequality and again we only considered the case $\lambda > 0$. By (2.6), (2.7) and (2.8) we finally get

$$
\left( 1 - \frac{\lambda}{\lambda_1} \right) \left( \frac{\mu}{2} - 1 \right) \|u_n\|^2 - c_R \leq M + N \|u_n\|,
$$

that is $(u_n)_n$ is bounded.

Since all the assumptions of Theorem 2.0.6 are fulfilled, we can conclude that problem $(P)$ has a solution $u$, which is not the trivial one, since $f(u) > 0$, and so Theorem 2.1.4 is proved. □

### 2.2 Applications: infinitely many solutions

In the assumptions of the previous section, if $g$ is odd in the $s$ variable, we expect many solutions. Indeed we have:

**Theorem 2.2.1.** Assume $(g_1) - (g_4)$, $g(x, s) = -g(x, -s)$ $\forall s \in \mathbb{R}$ and for a.e. $x \in \Omega$, and $\lambda < \lambda_1$. Then problem $(P)$ admits an unbounded sequence $(u_n, -u_n)_n$ of couples of solutions.

**Proof.** We apply Theorem 2.0.11 to the already introduced functional

$$
f(u) = \frac{1}{2} \int_\Omega |Du|^2 \, dx - \frac{\lambda}{2} \int_\Omega u^2 \, dx - \int_\Omega G(x, u) \, dx,
$$

and we only need to verify the geometrical conditions $(i)$ and $(ii)$, since all the other request are verified, because this is a special case in which in addition $f$ is even.

**Condition (i).** Proceeding as in the proof of Theorem 2.1.4, we obtain (2.4), so that with the same choice of (2.5), we prove $(i)$ of Theorem 2.0.11 with the choice $X_1 = \{0\}$ and $X_2 = H^1_0(\Omega)$.

**Condition (ii).** Let $Y \subset H^1_0(\Omega)$ be a finite dimensional subspace. We already observed that

$$
f(u) \leq \frac{1}{2} \int_\Omega |Du|^2 \, dx - \frac{\lambda}{2} \int_\Omega u^2 \, dx - a_3 \int_\Omega |u|^p \, dx + \int a_4(x) \, dx,
$$
and by the Poincaré inequality (again we consider only $\lambda > 0$)

$$f(u) \leq \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_1}\right) \|u\|^2 - a_3 \int_\Omega |u|^\mu \, dx + \int_\Omega a_4(x) \, dx.$$ 

Since all norms are equivalent in $Y$, then $f(u) \to -\infty$ when $u \in Y$ and $\|u\| \to \infty$.

Then by Theorem 2.0.11 there exists a sequence $(u_n)_n$ such that $f(u_n) \to \infty$. But by $(g_4)$

$$\frac{\|u_n\|^2}{2} = f(u_n) + \int G(x, u_n) \, dx \geq f(u_n) - c_R \to \infty,$$

where $c_R$ is the same constant appearing in (2.8). The theorem is now completely proved.

\[\square\]

### 2.3 Unbounded domains

In the previous section we saw that an essential tool in proving $(PS)$ is that the embedding $i : H^1_0(\Omega) \hookrightarrow L^p(\Omega)$ is compact for any $p < 2^*$ when $\Omega$ is bounded, so that the adjoint $i^*$ is compact as well. When $\Omega$ is not bounded, say for example $\Omega = \mathbb{R}^N$, this compactness cannot hold any more, since it is possible to translate functions in every direction. For example, take $u \neq 0$ in $H^1(\mathbb{R}^N)$, then $u_h(\cdot + h)$ has the same norm of $u$, but it is impossible to extract any converging sequence from $(u_h)_h$.

In some cases, however, some compactness can still be recovered, like in the case of radial functions. Let us denote by $H^1_r(\mathbb{R}^N)$ the set of radial Sobolev functions in $\mathbb{R}^N$, i.e. $u \in H^1(\mathbb{R}^N)$ if and only if $u(x) = u(|x|)$. It holds:

**Theorem 2.3.1** ([27], [121]). *The immersions $H^1_r(\mathbb{R}^N) \hookrightarrow L^p_r(\mathbb{R}^N)$ is compact for any $p \in (2, 2^*)$.***

Of course, by duality, $L^p_r(\mathbb{R}^N)$ is compactly embedded in $(H^1_r(\mathbb{R}^N))'$, so that it suggests to follow the approach of Section 2.1 in order to show solutions for superlinear and subcritical problems in $\mathbb{R}^N$.

The first result in this spirit is the one of [27], where the authors prove the following.

**Theorem 2.3.2.** If $g$ satisfies $(g_1) - (g_4)$, then there exists a radial nontrivial solution of problem

$$\begin{cases}
-\Delta u + u = g(x, u) & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N).
\end{cases}$$

Actually $g$ was supposed to be a bit more regular, but the proof is essentially the same in this case and it is modelled on the proof of Theorem 2.1.4: denote by $f : H^1_r(\mathbb{R}^N) \to \mathbb{R}$ the functional

$$f(u) = \frac{1}{2} \int_{\mathbb{R}^N} |Du|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} u^2 \, dx - \frac{\lambda}{\lambda_1} \int_{\mathbb{R}^N} u^2 \, dx - \int_{\mathbb{R}^N} G(x, u) \, dx,$$

and then prove that $f$ satisfies the geometrical conditions of Theorem 2.0.6. Finally, in order to prove $(PS)$, again it is enough to show that every $(PS)$ sequence is bounded, since

$$\nabla f(u) = u + \Delta^{-1}(g(x, u)),$$
where $\Delta^{-1} : H^{-1}_r(\mathbb{R}^N) \to H^1_r(\mathbb{R}^N)$ is the operator which associates to any element $h \in H^{-1}_r(\mathbb{R}^N) = (H^1_r(\mathbb{R}^N))^\prime$ the unique solution in $H^1(\mathbb{R}^N)$ of the problem

$$
\begin{cases}
\Delta u - u = h & \text{in } \Omega, \\
u \in H^1_r(\mathbb{R}^N),
\end{cases}
$$

which exists by the Lax–Milgram Theorem. We leave the details to the reader, while in the following chapters we present three systems of differential equations arising in quantum mechanics which can be studied with the tools presented above.
Chapter 3

The linking theorem

**Theorem 3.0.3** (Linking Theorem, [26]). Let $X$ be a Banach space, with $X = X_1 \oplus X_2$, where $X_1$ and $X_2$ are closed subspaces with $\dim X_1 < \infty$. Let $f : X \to \mathbb{R}$ be of class $C^1$ and assume that

(i) there exist $\rho, \alpha > 0$ such that
   $$\inf f(S_\rho \cap X_2) = \alpha;$$

(ii) there exists $e$ in $X_2 \setminus \{0\}$ and $R > \rho$ such that $\sup f(\Sigma) \leq 0$, where, denoting $(e) = \text{Span}(e)$, we have set
   $$\Delta = (\overline{B}_R \cap X_1) \oplus \{te : t \in [0, R]\} \text{ and } \Sigma = \partial X_1 \oplus (e) \Delta;$$

(iii) setting $\mathcal{H} = \{h \in C(\Delta, X) : h = \text{Id} \text{ on } \Sigma\}$ and
   $$\beta = \inf_{h \in \mathcal{H}} \max_{u \in \Delta} f(h(u)),$$
   assume that $(PS)_\beta$ holds.

Then $\beta$ is a critical value.

**Proof.** The proof follows the one given in [116].

First, let us note that $\alpha \leq \beta < \infty$. Indeed, take $h = I_d \in \mathcal{H}$, so that $\beta \leq \sup f(\Delta) < \infty$ since $\Delta$ is compact. In order to prove rigorously that $\beta \geq \alpha$ we would need an intersection theory, but it goes beyond air purposes, so that we only underline that intuitively it is clear that $(S_\rho \cap X_2) \cap h(\Delta) \neq \emptyset \forall h \in \mathcal{H}$, so that there exists $u_h \in (S_\rho \cap X_2) \cap h(\Delta) \forall h \in \mathcal{H}$. Of course $f(u_h) \geq \alpha \forall h \in \mathcal{H}$ and then $\beta \geq \alpha$.

Now assume by contradiction that $\beta$ is not critical for $f$. Then by Proposition 1.1.12 there exists $\varepsilon > 0$ such that
   $$\inf \{\|f'(u)\|_{X'} : a - \varepsilon \leq f(u) \leq a + \varepsilon\} > 0.$$

By Lemma 1.1.11 there exists $U \in C([0, 1], X)$ which deforms $f^{\beta+\varepsilon}$ on $f^{\beta-\varepsilon}$ strongly, and we can assume $\varepsilon$ so small that $\beta - \varepsilon > 0$.

By definition of $\beta$ there exists $h \in \mathcal{H}$ such that

$$\sup_{u \in \Delta} f(h(u)) < \beta + \varepsilon.$$  (3.1)
Since \( f(\Sigma) \leq 0 \), for our choice of \( \varepsilon \) we get \( \Sigma \subset f^{\beta - \varepsilon} \), so that \( \mathcal{U}(t, h(u)) = u \) for all \( t \in [0, 1] \) and for all \( u \in \Sigma \). In particular, if \( u \in \Delta \), \( \mathcal{U}(1, h(u)) \in \mathcal{H} \), so that \( \beta \leq \sup_{\Delta} f(\mathcal{U}(1, h(u))) \).

On the other hand, \( f(h(u)) \leq \beta + \varepsilon \) \( \forall u \in \Delta \). Since \( \mathcal{U} \) is a strong deformation, we get \( \mathcal{U}(1, h(u)) \in f^{\beta - \varepsilon} \) and then \( \sup_{\Delta} f(\mathcal{U}(1, h(u))) \leq \beta - \varepsilon \). Absurd.

**Remark 3.0.4.** 1. If \( X_1 = \{0\} \), then Theorem 3.0.3 is nothing but Theorem 2.0.6.
2. If \( \Sigma \) and \( \Delta \) are replaced by homeomorphic manifolds not centered at 0 and the topological situation is translated in another point, the theorem still holds.

For a more complete version of this theorem, where the existence of another critical point is proved, see [92].

### 3.1 An application

Let us go back to problem

\[
(P) \quad \begin{cases} 
-\Delta u - \lambda u = g(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

where \( \Omega \) is a bounded domain of \( \mathbb{R}^N \), \( N \geq 1 \). In Theorem 2.1.4 we showed that, under conditions \( (g_1) - (g_4) \) introduced in Section 2.1, problem \( (P) \) has a nontrivial solution when \( \lambda < \lambda_1 \). With the help of Theorem 3.0.3 we will remove this bound on \( \lambda \). For simplicity, we will make an extra assumption on \( G \).
Theorem 3.1.1. Assume \((g_1) - (g_4)\) of Section 2.1 and if \(R > 0\) in \((g_4)\), assume also that \(G(x,s) \geq 0\ \forall s \in \mathbb{R}\) and for a.e. \(x \in \Omega\). Then problem \((P)\) has a nontrivial solution for any \(\lambda \in \mathbb{R}\).

Proof. Of course, we consider only the case \(\lambda \geq \lambda_1\), the other case being already considered in Theorem 2.1.4. Therefore we assume that there exists a natural number \(i \geq 1\) such that \(\lambda_i \leq \lambda < \lambda_i\).

Set \(H_i = \text{Span}(e_1, \ldots, e_i)\) and denote by \(H_i^\perp\) the orthogonal complement of \(H_i\) in \(H_0^\perp(\Omega)\), so that \(H_0^\perp(\Omega) = H_i \oplus H_i^\perp\). We apply Theorem 3.0.3 with \(X = H_0^\perp(\Omega)\), \(X_1 = H_i\) and \(X_2 = H_i^\perp\).

First of all, let us recall the following form of the Poincaré inequality, which can be easily proved by a spectral decomposition in \(H_i^\perp(\Omega)\),

\[
(3.2) \quad \lambda_{i+1} \int_\Omega u^2 \, dx \leq \int_\Omega |Du|^2 \, dx \quad \forall u \in H_i^\perp,
\]

and its counterpart

\[
(3.3) \quad \int_\Omega |Du|^2 \, dx \leq \lambda_i \int_\Omega u^2 \, dx \quad \forall u \in H_i.
\]

Take \(u \in H_i^\perp\). Proceeding as in the in the proof of Theorem 2.1.4 we get (2.3), so that by (3.2) we get (after relabelling \(\varepsilon\))

\[
f(u) = \frac{1}{2} \int_\Omega |Du|^2 \, dx - \lambda \int \Omega u^2 \, dx - \int_\Omega G(x,u) \, dx \geq \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_{i+1}} - \varepsilon\right) \|u\|^2 - C\|u\|^p,
\]

where \(C > 0\) is a constant depending only on \(\Omega\) and \(\varepsilon\). Of course we choose \(\varepsilon\) so small that \(1 - \frac{\lambda}{\lambda_{i+1}} - \varepsilon > 0\). Now take \(u \in H_i^\perp\) such that

\[
\|u\| = \rho < \left[\frac{1}{2C} \left(1 - \frac{\lambda}{\lambda_{i+1}} - \varepsilon\right)\right]^{1/(p-2)}.
\]

In this way condition \((i)\) of Theorem 3.0.3 holds.

By (3.3) and by the positivity of \(G\) it is readily seen that \(f(H_i) \leq 0\). Moreover, take \(u \in H_i\), \(t > 0\) and consider

\[
f(u + te_{i+1}) = \frac{1}{2}\|u\|^2 - \lambda \int \Omega u^2 \, dx + \frac{t^2}{2} \left(\|e_{i+1}\|^2 - \lambda \int \Omega e_{i+1}^2 \, dx\right) - \int_\Omega G(x,u + te_{i+1}) \, dx,
\]

since \(u\) and \(e_{i+1}\) are orthogonal. By (3.3) and \((g_4)\) we get

\[
f(u + te_{i+1}) \leq \frac{t^2}{2} (\lambda_{i+1} - \lambda) - a_3 t^\mu \int_\Omega \frac{u}{t^\mu} + e_{i+1} |^\mu dx + \|a_4\|_1 \rightarrow_\infty -\infty,
\]

so that condition \((ii)\) of Theorem 3.0.3 is satisfied with \(\epsilon = e_{i+1}\).

We now prove that \((PS)_c\) holds \(\forall c \in \mathbb{R}\). First we note that

\[
\nabla f(u) = u + \Delta^{-1}(\lambda u + g(x,u)),
\]

so that we only need to prove that \((PS)\) sequences are bounded, like in the proof of Theorem 2.1.4, since \(\Delta^{-1}\) is compact and the problem is subcritical. So, let \((u_n)_n\) be such that \(f(u_n) \rightarrow c\) and \(\nabla f(u_n) \rightarrow 0\). Take \(k \in (2, \mu)\) and note that

\[
(3.4) \quad kf(u_n) - (\nabla f(u_n), u_n) \leq M + N\|u_n\|
\]
for some $M, N > 0$. On the other hand

$$\begin{align*}
kf(u_n) - \langle \nabla f(u_n), u_n \rangle & \geq \left( \frac{k}{2} - 1 \right) \|u_n\|^2 - \left( \frac{k}{2} - 1 \right) \lambda \int_{\Omega} u_n^2 \, dx + \int_{\Omega} [g(x, u_n)u_n - kG(x, u_n)] \, dx \\
& \geq \left( \frac{k}{2} - 1 \right) \|u_n\|^2 - \left( \frac{k}{2} - 1 \right) \lambda \int_{\Omega} u_n^2 \, dx + (\mu - k) \int_{\Omega} G(x, u_n) \, dx - CR
\end{align*}$$

for some constant $C_R \geq 0$, as already found in (2.8) in the proof of Theorem 2.1.4. By $(g_4)$ we finally get

$$\begin{align*}
kf(u_n) - \langle \nabla f(u_n), u_n \rangle & \geq \left( \frac{k}{2} - 1 \right) \|u_n\|^2 - \left( \frac{k}{2} - 1 \right) \lambda \int_{\Omega} u_n^2 \, dx + (\mu - k)a_3 \int_{\Omega} |u_n|^\mu \, dx - C
\end{align*}$$

for some constant $C \geq 0$.

By the Hölder and Young inequalities we get that for any $u$ and for any $\varepsilon > 0$

$$\|u\|_2^2 \leq C(\Omega)\|u\|_\mu^2 \leq \varepsilon \|u\|_\mu^2 + C_\varepsilon,$$

where $C_\varepsilon \to \infty$ as $\varepsilon \to 0$. Then by (3.5) we get

$$\begin{align*}
kf(u_n) - \langle \nabla f(u_n), u_n \rangle & \geq \left( \frac{k}{2} - 1 \right) \|u_n\|^2 + \left[ (\mu - k)a_3 - \varepsilon \left( \frac{k}{2} - 1 \right) \lambda \right] \int_{\Omega} |u_n|^\mu \, dx - \tilde{C}_\varepsilon,
\end{align*}$$

where $\tilde{C}_\varepsilon$ is independent on $n$. Let us choose

$$\varepsilon < \frac{(\mu - k)a_3}{\left( \frac{k}{2} - 1 \right) \lambda},$$

so that (3.6) gives

$$kf(u_n) - \langle \nabla f(u_n), u_n \rangle \geq \left( \frac{k}{2} - 1 \right) \|u_n\|^2 - \tilde{C}_\varepsilon.$$

Together with (3.4) this gives that $(u_n)_n$ is bounded, as claimed.

**Remark 3.1.2.** An existence theorem for the Dirichlet problem

$$\begin{align*}
-\Delta u &= h(x, u) & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}$$

continues to hold if $h$ is such that $h(u) \sim \lambda u + g(x, u)$, where $g$ is as above, and the proof is based on Theorem 2.0.6 or Theorem 3.0.3 according to the cases $\lambda < \lambda_1$ or $\lambda \geq \lambda_1$, respectively.

With the aim of Theorem 2.0.11 one can prove the following multiplicity result.

**Theorem 3.1.3.** Assume $(g_1) - (g_4)$ of Section 2.1 and if $R > 0$ in $(g_4)$, assume also that $G(x, s) \geq 0 \ \forall \ s \in \mathbb{R}$ and for a.e. $x \in \Omega$. Suppose that $g(x, s) = -g(x, -s) \ \forall \ s \in \mathbb{R}$ and for a.e. $x \in \Omega$, and $\lambda < \lambda_1$. Then problem $(P)$ admits infinitely many couples $(u_n, -u_n)_n$ of solutions.

The proof is left to the reader.
Chapter 4

The saddle point theorem

As usual, though not stated explicitly, $X$ is a Banach space and $f \in C^1(X, \mathbb{R})$.

**Definition 4.0.4.** A point $u_0 \in X$ is called a **saddle point** if $f'(u_0) = 0$ and for any neighborhood $U$ of $u_0$, the function $f - f(u_0)$ is not definite positive either negative in $U$.

**Theorem 4.0.5** (Saddle Point Theorem). Suppose that there exist two closed subspaces $X_1$ and $X_2$ of $X$ with $X_1 \neq \{0\}$ and $\dim X_1 < \infty$. Moreover, assume that $\exists R > 0$ such that

\[ \sup_{S(0, R) \cap X_1} f < \inf_{X_2} f. \]

Set

\[ \Gamma = \{ h \in C(\bar{B}_R \cap X_1, X) : h = Id \text{ on } \partial B_R \} \]

and

\[ \beta = \inf_{h \in \Gamma} \sup_{u \in \bar{B}_R \cap X_1} f(h(u)) \]

and assume that $(PS)_\beta$ holds. Then $\beta \geq \inf f(X_2)$ is a critical value for $f$.

See [92] or [116] for a proof.

Let us only remark that again the Palais–Smale condition is essential. Indeed, the function $(x, y) = \arctan y - x^2$ defined in $\mathbb{R}^2$ is of class $C^\infty$, $f(0, y) \geq -\frac{\pi}{2}$ $\forall y \in \mathbb{R}$, $f(x, 0) \to -\infty$ as $|x| \to \infty$, but it has no critical points, since $\nabla f(x, y) = (-2x, 1/(1 + y^2))$.

**4.1 An application**

While the Mountain Pass Theorem and the Linking Theorem are useful in presence of superlinear problems, the natural way to face problems with linear growth is the Saddle Point Theorem.

In this section we assume that $\Omega$ is a bounded domain of $\mathbb{R}^N$, $N \geq 1$ and $g : \Omega \times \mathbb{R}$ is a Catathéodory function such that there exist $a \in L^2(\Omega)$ and $b \in \mathbb{R}$ such that

\[ |g(x, s)| \leq a(x) + b|s| \quad \forall s \in \mathbb{R} \quad \text{and a.e. } x \in \Omega. \]

We have the following.

**Theorem 4.1.1.** Assume that

\[ \liminf_{|s| \to -\infty} \frac{g(x, s)}{|s|} =: \alpha(x) \leq \limsup_{|s| \to -\infty} \frac{g(x, s)}{|s|} =: \overline{\alpha}(x) \]
are measurable functions. If \( \pi(x) < \lambda_1 \) or there exists \( i \in \mathbb{N} \) such that \( \lambda_i < \alpha(x) \leq \pi(x) < \lambda_{i+1} \), then problem

\[
(Q) \quad \begin{cases} 
-\Delta u = g(x, u) + h & \text{in } \Omega, \\
\quad u = 0 & \text{on } \partial \Omega
\end{cases}
\]

has a solution for any \( h \in L^2(\Omega) \).

**Remark 4.1.2.** Without loss of generality, we can assume \( h \equiv 0 \), since the function \( \tilde{g}(x, s) = g(x, s) + h(x) \) satisfies the same assumptions of \( g \).

**Proof of Theorem 4.1.1.** We look for critical points of the \( C^1 \) functional \( f : H^1_0(\Omega) \to \mathbb{R} \) given as usual by

\[
f(u) = \frac{1}{2} \int_\Omega |Du|^2 \, dx - \frac{\lambda}{2} \int_\Omega u^2 \, dx - \int_\Omega G(x, u) \, dx.
\]

**I:** \( \pi(x) < \lambda_1 \). For \( \varepsilon > 0 \) there exists \( K > 0 \) such that

\[
\frac{g(x, s)}{s} - \pi(x) < \varepsilon \quad \forall \ |s| > K.
\]

Integrating we get

\[
G(x, s) \leq \frac{\pi(x)}{2} + \frac{\varepsilon}{2} (s^2 - R^2) \quad \forall \ |s| > K,
\]

while (4.2) implies

\[
|G(x, s)| \leq \left( a(x) R + \frac{b}{2} R \right) R \quad \forall \ |s| \leq K.
\]

In this way we prove that

\[
\limsup_{|s| \to \infty} \frac{G(x, s)}{s^2} \leq \frac{\pi(x)}{2}.
\]

Indeed by (4.3) and (4.4) there exists \( C_K > 0 \) such that

\[
\frac{G(x, s)}{s^2} = \frac{f^R + \int_0^s {g(x, \sigma)} \, d\sigma}{s^2} < \frac{C_K + \frac{\pi(x) + \varepsilon}{2} (s^2 - R^2)}{s^2},
\]

so that

\[
\limsup_{|s| \to \infty} \frac{G(x, s)}{s^2} \leq \frac{\pi(x) + \varepsilon}{2}
\]

for every \( \varepsilon > 0 \), and then (4.5) follows.

Now let us show

\[
\lim_{\|u\| \to \infty} \frac{f(u)}{\|u\|^2} > 0.
\]

Take \( (u_n) \) such that \( \|u_n\| \to \infty \). Up to a subsequence we can assume that \( u_n := \frac{u_n}{\|u_n\|} \) converges to a function \( u \) weakly in \( H^1_0(\Omega) \), strongly in \( L^2(\Omega) \) and a.e. in \( \Omega \). Moreover \( \|u\| \leq 1 \). Then

\[
\frac{|G(x, u_n)|}{\|u_n\|^2} \leq \frac{a(x)\|u_n\| + b\|u_n\|^2/2}{\|u_n\|^2} \to \frac{b}{2} \|u\|^2 \text{ in } L^2(\Omega)
\]
an by the generalized Fatou Lemma

\[ (4.7) \limsup_{n \to \infty} \int_{\Omega} \frac{G(x, u_n)}{\|u_n\|^2} \, dx \leq \int_{\Omega} \limsup_{n \to \infty} \frac{G(x, u_n)}{\|u_n\|^2} \, dx. \]

But

\[ \Omega = \{ x \in \Omega : u_n(x) \text{ is bounded} \} \cup \{ x \in \Omega : |u_n(x)| \to \infty \}, \]

and \( \frac{G(x, u_n)}{\|u_n\|^2} \to 0 \) in \( \{ x \in \Omega : u_n(x) \text{ is bounded} \} \), while in \( \{ x \in \Omega : |u_n(x)| \to \infty \} \)

\[ \limsup_{n \to \infty} \frac{G(x, u_n)}{\|u_n\|^2} = \limsup_{n \to \infty} \frac{G(x, u_n)}{\|u_n\|^2} \frac{u_n^2}{\|u_n\|^2} \leq \frac{\alpha(x)}{2} u^2(x), \]

by (4.5). Therefore (4.7) gives

\[ \limsup_{n \to \infty} \int_{\Omega} \frac{G(x, u_n)}{\|u_n\|^2} \, dx \leq \int_{\Omega} \frac{\alpha(x) s^2}{2} \, dx \quad \text{if } u \neq 0, \]

and

\[ = 0 \quad \text{if } u = 0. \]

so that

\[ \lim_{n \to \infty} f(u_n) \geq \begin{cases} \frac{1}{2} - \frac{1}{2} \int_{\Omega} |Du|^2 \, dx \geq 0 & \text{if } u \neq 0, \\ \frac{1}{2} & \text{if } u = 0. \end{cases} \]

By the Poincaré inequality and the fact that \( \|u\| \leq 1 \), we get

\[ \lim_{n \to \infty} f(u_n) \geq \begin{cases} \frac{1}{2} - \frac{1}{2} \int_{\Omega} |Du|^2 \, dx \geq 0 & \text{if } u \neq 0, \\ \frac{1}{2} & \text{if } u = 0, \end{cases} \]

and (4.6) follows.

Finally, let us note that the map \( u \mapsto \|u\|^2 \) is lower semicontinuous in the weak topology of \( H^1_0(\Omega) \), while the map \( u \mapsto \int_{\Omega} G(x, u) \) is continuous, so that \( f \) is lower semicontinuous and by the Weierstrass Theorem there exists a minimum of \( f \) on \( H^1_0(\Omega) \), which is clearly a solution of problem (Q).

\( \text{II: } \exists i \in \mathbb{N} \text{ such that } \lambda_i < \alpha(x) \leq \alpha_i < \lambda_{i+1}. \) As usual, let us set \( H_i = \text{Span}(e_1, \ldots, e_i) \) and \( H_i^+ = \text{Span}(e_{i+1}, \ldots) \), so that \( H^1_0(\Omega) = H_1 \oplus H_i^+ \) and \( \text{dim} H_i = i < \infty. \)

Reasoning as above we can prove that

\[ \liminf_{|s| \to \infty} \frac{G(x, s)}{s^2} \geq \frac{\alpha(x)}{s^2}, \]

so that, by using (3.3)

\[ \limsup_{u \in H_i, \|u\| \to \infty} \frac{f(u)}{\|u\|^2} < 0, \]

which implies

\[ (4.8) \lim_{u \in H_i, \|u\| \to \infty} f(u) = -\infty. \]

Moreover, in an analogous way one can prove

\[ \liminf_{u \in H_i^+, \|u\| \to \infty} \frac{f(u)}{\|u\|^2} > 0. \]
This implies that \( f(u) \to \infty \) when \( u \in H^1_i \) and \( \|u\| \to \infty \), and then \( \forall M > 0 \exists K > 0 \) such that \( \forall u \in H^1_i \) with \( \|u\| \geq K \), \( f(u) \geq M \). If \( \|u\| \leq K \) we have

\[
f(u) \geq \frac{1}{2} \|v\|^2 - \int_{\Omega} a(x)|u|\,dx - \frac{b}{2} \int_{\Omega} u^2\,dx \geq -\|a\|_2 \|u\|_2 - \frac{b\lambda_{i+1}}{2} \|u\|^2 \geq C_K
\]

for some universal constant \( C_K > 0 \). Summing up,

\[
f(u) \geq M - C_K \quad \forall u \in H^1_i.
\]

By (4.8) we can choose \( R > 0 \) in such a way that \( \forall u \in H_i \) with \( \|u\| = R \) we have

\[
\sup f(u) < M - C_K \leq \inf f(H^1_i).
\]

We have thus proved that then topological situation of Theorem 4.0.5 holds.

We now only need to prove that \((PS)_c\) holds \( \forall c \in \mathbb{R} \). Again, it is enough to show that any \((PS)_c\) sequence is bounded, since

\[
\nabla f(u) = u + \Delta^{-1}g(x,u),
\]

and Theorem 1.1.15 can be applied, since \( g \) has linear growth and then \( \Delta^{-1} \) is well defined and compact by the Rellich–Kondrachov Theorem.

Thus, let \((u_n)\) be a \((PS)_c\) sequence and assume by contradiction that it is unbounded. Up to a subsequence we can assume \( \|u_n\| \to \infty \) as \( n \to \infty \) and that there exists \( u \in H^1_0(\Omega) \) such that \( v_n := u_n/\|u_n\| \) converges to \( u \) weakly in \( H^1_0(\Omega) \), strongly in \( L^2(\Omega) \) and a.e. in \( \Omega \). Moreover

\[
(4.9) \quad \frac{|g(x,u_n)|}{\|u_n\|} \leq \frac{a(x)}{\|u_n\|} + b|v_n|,
\]

so that we can assume that there exists \( w \in L^2(\Omega) \) such that

\[
\frac{g(x,u_n)}{\|u_n\|} \to w \quad \text{in } L^2(\Omega).
\]

Since \( \Delta^{-1} : L^2(\Omega) \to H^1_0(\Omega) \) is compact, we get that

\[
\Delta^{-1} \left( \frac{g(x,u_n)}{\|u_n\|} \right) \to \Delta^{-1} w \quad \text{in } H^1_0(\Omega).
\]

Claim: there exists \( m \in L^\infty(\Omega) \) such that \( \underline{a} \leq m \leq \overline{a} \) a.e. in \( \Omega \) and \( w = mu \). Indeed, if \( u(x) > 0 \), then \( u_n(x) = v_n(x)\|u_n\| \to +\infty \) so that

\[
\liminf_{n \to \infty} \frac{g(x,u_n)}{\|u_n\|} = \liminf_{n \to \infty} \frac{g(x,u_n)}{u_n} \frac{u_n}{\|u_n\|} \geq \underline{a}(x)u(x)
\]

and in the same way

\[
\limsup_{n \to \infty} \frac{g(x,u_n)}{\|u_n\|} \leq \overline{a}(x)u(x).
\]

If \( u(x) < 0 \) we get the reversed sign

\[
\liminf_{n \to \infty} \frac{g(x,u_n)}{\|u_n\|} \leq \underline{a}(x)u(x), \quad \limsup_{n \to \infty} \frac{g(x,u_n)}{\|u_n\|} \geq \overline{a}(x)u(x),
\]
while if \( u(x) = 0 \) we get
\[
\left| g(x,u_n) \right| \leq \frac{a(x)}{\|u_n\|} + b|v_n(x)| \to 0.
\]

Now let us observe that if a sequence \((\phi_n)_n\) converges weakly to a function \( \phi \) in \( L^2(\Omega) \) and there exist two measurable functions \( \psi_1, \psi_2 \) such that \( \psi_1 \leq \liminf \phi_n \leq \limsup \phi_n \leq \psi_2 \) a.e. in \( \Omega \), then
\[
(4.10) \quad \psi_1 \leq \phi \leq \psi_2 \quad \text{a.e. in } \Omega.
\]

In this way we get
\[
\alpha(x)u(x) \leq w(x) \leq \alpha(x)u(x) \quad \text{ if } u(x) > 0,
\]
\[
\alpha(x)u(x) \geq w(x) \geq \alpha(x)u(x) \quad \text{ if } u(x) > 0.
\]

In any case, setting
\[
m(x) := \begin{cases} u(x) & \text{ if } u(x) \neq 0, \\ 0 & \text{ if } u(x) = 0,
\end{cases}
\]
we get \( \alpha \leq m \leq \bar{\alpha} \) a.e. in \( \Omega \). Moreover \( m \) is measurable and bounded, since by (4.10) applied to \( v_n \) and (4.9) imply \( |m(x)| \leq b \) a.e. in \( \Omega \). The claim is proved.

Since \( \nabla f(u_n) \to 0 \), from the equality
\[
\frac{\nabla f(u_n)}{\|u_n\|} = v_n + \frac{\Delta^{-1}(g(x,u_n))}{\|u_n\|}
\]
we get that \( v_n \to u \) strongly in \( H^1_0(\Omega) \), so that \( \|u\| = 1 \), and \( 0 = u + \Delta^{-1}(g(x,u)) \), that is \( u \) is a solution of
\[
(P_0) \quad \begin{cases} -\Delta u = mu & \text{ in } \Omega, \\ u = 0 & \text{ on } \partial \Omega.
\end{cases}
\]

If we prove that any solution \( v \) of \( (P_0) \) is the trivial one, then we get a contradiction, since \( \|u\| = 1 \). Thus, let us write \( v = v_1 + v_2 \), where \( v_1 \in H_1 \) and \( v_2 \in H^1_\perp \). Multiplying the equation in \( (P_0) \) by \( v_1 \) and by \( v_2 \) gives
\[
\int_\Omega |Dv_1|^2 \, dx - \int_\Omega mv_1^2 \, dx = \int_\Omega mv_1v_2 \, dx, \quad \int_\Omega |Dv_2|^2 \, dx - \int_\Omega mv_2^2 \, dx = \int_\Omega mv_1v_2 \, dx,
\]
so that
\[
\int_\Omega |Dv_1|^2 \, dx - \int_\Omega mv_1^2 \, dx = \int_\Omega |Dv_2|^2 \, dx - \int_\Omega mv_2^2 \, dx
\]
By (3.2) and (3.3) we get
\[
0 \geq \int_\Omega (\lambda_i - m)v_1^2 \, dx \geq \int_\Omega (\lambda_{i+1} - m)v_2^2 \, dx \geq 0,
\]
so that all integrands are zero (since they don’t change sign). But since \( \lambda_i - m \leq \lambda_i - \alpha < 0 \) and \( \lambda_{i+1} - m \geq \lambda_{i+1} - \bar{\alpha} > 0 \), we get \( v_1 = v_2 = 0 \), as desired.

In this way the Palais–Smale condition is completely proved, Theorem 4.0.5 can be applied and Theorem 4.1.1 is completely proved.
Proposition 4.1.3. Under the assumptions of Theorem 4.1.1, if in addition there exist two measurable functions $\beta, \overline{\beta} : \Omega \to \mathbb{R}$ and $i \geq 1$ such that

$$\lambda_i < \beta(x) \leq \frac{g(x, s_1) - g(x, s_2)}{s_1 - s_2} \leq \overline{\beta}(x) < \lambda_{i+1} \quad \forall s_2 \neq s_1 \text{ and for a.e. } x \in \Omega,$$

then the solution of problem (Q) is unique.

Proof. It follows the line of the end of the proof above. Assume $v_1, v_2$ are two solutions of problem (Q); then $v = v - v_1$ solves

$$\begin{cases} -\Delta v = g(x, v_1) - g(x, v_2) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial \Omega. \end{cases}$$

Multiplying first by $v_1$ and integrating, then by $v_2$ and integrating, we can conclude that $v = 0$. \qed

Corollary 4.1.4. If $\lambda_i < \alpha < \lambda_{i+1}$ for some $i \geq 1$, then for any $h \in L^2(\Omega)$

$$\begin{cases} -\Delta u - \alpha u = h \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial \Omega \end{cases}$$

has a unique solution.
Chapter 5

A new multiplicity abstract theorem

Definition 5.0.5. Let $H$ be a Hilbert space, $f : H \rightarrow \mathbb{R}$ a $\mathcal{C}^1$ function, $X$ a closed subspace of $H$, $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$. We say that condition $(\nabla)(f,X,a,b)$ holds if there exists $\gamma > 0$ such that $\inf \left\{ \| P_X \nabla f(u) \| \mid a \leq f(u) \leq b, \text{dist}(u,X) \leq \gamma \right\} > 0$, where $P_X : H \rightarrow X$ is the orthogonal projection of $H$ onto $X$.

Theorem 5.0.6 ((\nabla)-Theorem, [97]). Let $H$ be a Hilbert space and $H_i$, $i = 1, 2, 3$ three subspaces of $H$ such that $H = H_1 \oplus H_2 \oplus H_3$ and $\dim(H_i) < \infty$ for $i = 1, 2$. Denote by $P_i$ the orthogonal projection of $H$ onto $H_i$. Let $f : H \rightarrow \mathbb{R}$ be a $\mathcal{C}^1$ function. Let $\rho, \rho', \rho'', \rho_1$ be such that $\rho_1 > 0$, $0 \leq \rho' < \rho < \rho''$ and define

$$\Delta = \{ u \in H_1 \oplus H_2 \mid \rho' \leq \| P_2 u \| \leq \rho'', \| P_1 u \| \leq \rho_1 \} \quad \text{and} \quad T = \partial_{H_1 \oplus H_2} \Delta,$$

$$S_{23} = \{ u \in H_2 \oplus H_3 \mid \| u \| = \rho \} \quad \text{and} \quad B_{23} = \{ u \in H_2 \oplus H_3 \mid \| u \| \leq \rho \}.$$

Assume that

$$a' = \sup f(T) < \inf f(S_{23}(\rho)) = a''.$$

Let $a$ and $b$ be such that $a' < a < a''$ and $b > \sup f(\Delta)$. Assume $(\nabla)(f,H_1 \oplus H_3,a,b)$ holds and that $(PS)_c$ holds for every $c$ in $[a,b]$. Then $f$ has at least two critical points in $f^{-1}([a,b])$.

Moreover, if $a_1 < \inf f(B_{23}) > -\infty$ and $(PS)_c$ holds at any $c$ in $[a_1,b]$, then $f$ has another critical level in $[a_1,a']$.

See [97] for the proof.

5.1 An application

Let us consider again the problem

$$(P) \quad \begin{cases} -\Delta u - \lambda u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

where $\Omega$ is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, $\lambda \in \mathbb{R}$ and $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is superlinear and subcritical in the usual sense (see Chapter 3) with $g(x,0) = 0$, so that $u \equiv 0$ is a solution of $(P)$.

In this section, contrary to [6], we need to assume $(g_4)$ globally and with a special value of $\mu$, precisely

$$0 < pG(x,s) \leq g(x,s)s \quad \forall s \neq 0 \text{ and for a.e. } x \in \Omega.$$
Remark 5.1.1. It is worth noting that in the general case in $(g_4)$ one asks the existence of $\mu > 2$ such that the inequality holds. But by $(g_2)$ and Remark 2.1.3, one immediately gets $\mu \leq p$ (and so $p > 2$ automatically). So we require a stronger assumption on $\mu$, which is however satisfied whenever $g(x, s)$ “behaves” like $|s|^{p-2}s$.

There are many multiplicity results, in particular for the autonomous case, that is $g(x, u) = g(u)$, or when $\lambda = 0$ ([41], [129]). The literature concerning such results is quite extensive (especially considering the peculiar features that $g$ has in different cases) and we just refer to [34], [17], [39], [53], [54], the other papers cited here and the references quoted therein.

A common way to face this problem is to look for solutions having one sign in their domain ([6]) and then possibly sign-changing solutions ([40]). We will follow a different approach: with the help of Theorem 5.0.6, we will get the existence of two nontrivial solutions, and then we will get the third one by an additional linking structure.

Concerning the nonautonomous case, we refer, as inspiring results and papers to which compare our result, to [2], [3], [6], [77], [114], [122] and [123], although the nonlinearity $g$ has, in some cases, a different behaviour.

In particular in [6], in a substantially similar situation, the existence of a positive and a negative solution is proved for any $\lambda < \lambda_1$, while in [122] a larger number of solution is provided, but for a different nonlinearity: in fact, while the equation in problem $(P)$ generalizes the equation $-\Delta u - \lambda u - |u|^{s-2}u = 0$, the equation studied in [122] (as well as in the one in [3]
and [38]) generalizes the equation $-\Delta u - \lambda u + |u|^{n-2}u = 0$ and a minimization approach can be used. On the other hand in [123] a problem which is a perturbation of a symmetric one is considered, providing the existence of infinitely many solutions and infinitely many solutions are also provided when some symmetries on the related functional are assumed ([2], [6], [77]). It is also worth noting that our results is related to the one found in [4], although under different assumptions on $g$.

More precisely, the result presented here is in the spirit of the result of [114], where the author studies the problem
\[
\begin{cases}
-\Delta u - \lambda u + t((u + \alpha e_1)^-) p = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where $t, \alpha > 0$ and $e_1$ is the first (positive) eigenfunction of $-\Delta$ on $H^1_0(\Omega)$ and she proves the analogous of our Theorem.

We can now state the main result of this chapter.

**Theorem 5.1.2.** Assume $(g_1) - (g_4)$ with condition (5.1). Then $\forall i \geq 2$ there exists $\delta_i > 0$ such that $\forall \lambda \in (\lambda_i - \delta_i, \lambda_i)$, problem $(P)$ has at least three nontrivial solutions.

Let us note that in [129] the existence of three nontrivial solutions for problem $(P)$ is proved with Morse theory techniques when $\lambda = 0$ and $g = g(u)$ is of class $C^1$, but actually that proof works also for $\lambda < \lambda_1$.

Since we are interested in values of $\lambda$'s which are in a left neighborhood of $\lambda_i$, $i \geq 2$, we can assume that $\lambda_{i-1} < \lambda < \lambda_i = \ldots = \lambda_j < \lambda_{j+1}$ for some $i$, $j \geq 2$, since $\lambda_i$ may have multiplicity greater than one. We will prove that the topological situation described in Theorem 5.0.6 holds with $X_1 = H_{i-1}, X_2 = \text{Span}(e_i, \ldots, e_j)$ and $X_3 = H_j^\perp$ and that $(\nabla)(f, H_{i-1} \oplus H_j^\perp, a, b)$ holds for suitable $a$ and $b$.

To do that we start from the following notations: if $j < k$ in $\mathbb{N}$ we set:
\[
S^+_k(\rho) = \left \{ u \in H^+_k \mid ||u|| = \rho \right \},
\]
\[
T_{j,k}(R) = \left \{ u \in H_k \mid ||u|| = R \right \} \bigcup \left \{ u \in H_j \mid ||u|| \leq R \right \}.
\]

**Lemma 5.1.3.** Assume $\lambda_{i-1} < \lambda_i = \ldots = \lambda_j < \lambda_{j+1}$ and $\lambda \in (\lambda_{i-1}, \lambda_j)$. Then there exist $R$ and $\rho > 0$, $R > \rho > 0$, such that
\[
\sup_{\lambda} f_{\lambda}\left(T_{i-1,j}(R)\right) < \inf_{\lambda} f_{\lambda}\left(S^+_{i-1}(\rho)\right).
\]

**Proof.** By $(g_2)$ and $(g_3)$ we get in a standard way that, given $\epsilon > 0$, there exists $\delta > 0$ such that $G(x, u) \leq \epsilon |u|^2 + C(\delta) |u|^{n+1}$. Moreover, by the fact that $\int |Dz|^2 \geq \lambda_i \int z^2 \forall z \in H^+_i$, we get the existence of $\rho > 0$ such that
\[
\inf_{\lambda} f_{\lambda}\left(S^+_{i-1}(\rho)\right) > 0.
\]

Moreover $f_{\lambda}(H_{i-1}) \leq 0$. To conclude the proof it is enough to show that
\[
\lim_{\|u\| \to \infty, u \in H_i} f_{\lambda}(u) = -\infty.
\]
Such a result easily follows from Remark 2.1.3. In fact if $u \in H_i$, then
\[
f_{\lambda}(u) \leq \frac{1}{2} \|u\|^2 - \frac{\lambda}{2} \int_{\Omega} u^2 dx - c_1 \int_{\Omega} |u|^p dx,
\]
and since all norms in $H_i$ are equivalent, the thesis follows. \qed
Now take \( a \in \left( \sup f_{\lambda}(T_{i-1,j}(R)), \inf f_{\lambda}(S_{i-1}^+(\rho)) \right) \) and \( b > \sup f_{\lambda}(B_j(R)) \), where \( B_j(R) \) is the ball in \( H_j \) with radius \( R \). If one shows that \( (\nabla)(f_{\lambda}, H_{i-1} \oplus H_j^+), a, b \) holds, then Theorem 5.0.6 can be applied and Theorem 5.1.8 can be proved.

Let us start with two Lemmas which will be useful to prove the \((\nabla)\)-condition.

**Lemma 5.1.4.** Assume \( \lambda_{i-1} < \lambda_i = \ldots = \lambda_j < \lambda_{j+1} \) for some \( i \leq j \) in \( \mathbb{N} \). Then \( \forall \delta > 0 \ \exists \varepsilon_0 > 0 \) such that \( \forall \lambda \) in \( [\lambda_{i-1} + \delta, \lambda_{j+1} - \delta] \), the unique critical point \( u \) of \( f_{\lambda} \) constrained on \( H_{i-1} \oplus H_j^+ \) such that \( f_{\lambda}(u) \in [-\varepsilon_0, \varepsilon_0] \), is the trivial one.

**Proof.** Assume by contradiction that there exist \( \delta > 0 \), \( \lambda_n \in [\lambda_{i-1} + \delta, \lambda_{j+1} - \delta] \) and \( u_n \) in \( H_{i-1} \oplus H_j^+ \setminus \{0\} \) such that

\[
f_{\lambda_n}(u_n) = \frac{1}{2} \int_{\Omega} |Du_n|^2 \, dx - \frac{\lambda_n}{2} \int_{\Omega} u_n^2 \, dx - \int_{\Omega} G(x, u_n) \, dx \to 0
\]

and such that \( \forall z \) in \( H_{i-1} \oplus H_j^+ \)

\[
(5.2) \quad \int_{\Omega} D u_n \cdot D z \, dx - \lambda_n \int_{\Omega} u_n z \, dx - \int_{\Omega} g(x, u_n) z \, dx = 0.
\]

Of course, up to a subsequence, we can assume \( \lambda_n \to \lambda \) in \( [\lambda_{i-1} + \delta, \lambda_{j+1} - \delta] \). Choose \( z = u_n \) in (5.2). Then

\[
0 = \int_{\Omega} |Du_n|^2 \, dx - \lambda_n \int_{\Omega} u_n^2 \, dx - \int_{\Omega} g(x, u_n) u_n \, dx = 2f_{\lambda_n}(u_n) + \int_{\Omega} [2G(x, u_n) - g(x, u_n) u_n] \, dx \leq 2f_{\lambda_n}(u_n) + (2 - p) \int_{\Omega} G(x, u_n) \, dx
\]

by \((g_1)\). In particular

\[
(5.3) \quad \lim_{n \to \infty} \int_{\Omega} G(x, u_n) \, dx = 0.
\]

Now take \( v_n \) in \( H_{i-1} \) and \( w_n \) in \( H_j^+ \) such that \( u_n = v_n + w_n \ \forall n \) and choose \( z = v_n - w_n \) in (5.2). Then

\[
\int_{\Omega} |Dv_n|^2 \, dx - \lambda_n \int_{\Omega} v_n^2 \, dx - \left( \int_{\Omega} |Dw_n|^2 \, dx - \lambda_n \int_{\Omega} w_n^2 \, dx \right) = \int_{\Omega} g(x, u_n)(v_n - w_n) \, dx.
\]

Since \( \int |Dz|^2 \leq \lambda_{i-1} \int z^2 \ \forall z \) in \( H_{i-1} \) and \( \int |Dz|^2 \geq \lambda_{j+1} \int z^2 \ \forall z \) in \( H_j^+ \), we get that there exists \( c > 0 \) independent of \( n \) (\( c = \delta/\lambda_{j+1} \)) such that

\[
(5.4) \quad c \|u_n\|^2 \leq \int_{\Omega} g(x, u_n)(w_n - v_n) \, dx \quad \forall n.
\]

Here we used the fact that \( \|v_n\|^2 + \|w_n\|^2 = \|u_n\|^2 \), since \( v_n \) and \( w_n \) are orthogonal.

Moreover

\[
\left| \int_{\Omega} g(x, u_n)(w_n - v_n) \, dx \right| \leq \left( \int_{\Omega} |g(x, u_n)|^p \, dx \right)^{1-1/p} \left( \int_{\Omega} |w_n - v_n|^p \, dx \right)^{1/p}.
\]
By the Sobolev’s embedding theorem there exists a universal constant \( c_0 > 0 \) such that \( \| v_n - w_n \| \leq c_0 \| v_n - w_n \| = \gamma_p \| u_n \|. \) In this way, since \( u_n \neq 0 \), (5.4) implies that there exists \( c' > 0 \) such that

\[
(5.5) \quad \| u_n \| \leq c' \left( \int_{\Omega} |g(x, u_n)|^{p'} \, dx \right)^{1-1/p} \forall n.
\]

Up to subsequences, there are two cases: \( \| u_n \| \to \infty \) or \( \| u_n \| \) is bounded.

**First case:** \( \| u_n \| \) is unbounded. Then we can suppose that there exists \( u \) in \( H_{i-1} \oplus H_i^\perp \) such that \( u_n / \| u_n \| \to u \).

By (\( g_2 \)), (\( g_4 \)) and Remark 2.1.3 we get

\[
\int_{\Omega} |g(x, u_n)|^{p'} \, dx \leq a_1' + a_2' \int_{\Omega} G(x, u_n) \, dx.
\]

But the last quantity is bounded (by (5.3)) and (5.5) leads to a contradiction.

**Second case:** \( \| u_n \| \) is bounded. We can suppose there exists \( u \) in \( H_{i-1} \oplus H_i^\perp \) such that \( u_n \to u \).

Moreover (5.3) implies \( u = 0 \).

If \( u_n \to 0 \), then (5.5) would give

\[
1 \leq \lim_{n \to \infty} c' \left( \frac{\int_{\Omega} |g(x, u_n)|^{p'} \, dx}{\| u_n \|} \right)^{1-1/p} = 0,
\]

the limit being 0 by (\( g_2 \)) and (\( g_3 \)).

So there should exist \( \sigma > 0 \) such that \( \| u_n \| \geq \sigma \forall n \). But also in this case (\( g_3 \)) and (5.5) would give

\[
\sigma \leq \lim_{n \to \infty} c' \left( \frac{\int_{\Omega} |g(x, u_n)|^{p'} \, dx}{\| u_n \|} \right)^{1-1/p} = 0.
\]

The Lemma is thus completely proved. \( \square \)

Now we denote by \( P : H_0^1(\Omega) \to \text{Span}(e_i, \ldots, e_j) \) and \( Q : H_0^1(\Omega) \to H_{i-1} \oplus H_i^\perp \) the orthogonal projections.

**Lemma 5.1.5.** Suppose \( \lambda_{i-1} < \lambda_i = \cdots = \lambda_j < \lambda_{j+1} \), \( \lambda \in \mathbb{R} \) and \( (u_n)_n \) in \( H_0^1(\Omega) \) is such that \( (f_\lambda(u_n))_n \) is bounded, \( Pu_n \to 0 \) and \( Q\nabla f_\lambda(u_n) \to 0 \). Then \( (u_n)_n \) is bounded.

**Proof.** Assume by contradiction that \( (u_n)_n \) is unbounded. Then we can suppose that there exists \( u \) in \( H_0^1(\Omega) \) such that \( u_n / \| u_n \| \to u \).

Note that \( u_n = Pu_n + Qu_n, Pu_n \to 0 \) and \( Q\nabla f_\lambda(u_n) \to 0 \), where \( \nabla f_\lambda(u_n) = u_n + \Delta^{-1}(\lambda u_n + g(x, u_n)) \). In particular

\[
\langle Q\nabla f_\lambda(u_n), u_n \rangle = \langle \nabla f_\lambda(u_n), u_n \rangle - \langle P\nabla f_\lambda(u_n), u_n \rangle = \| u_n \|^2 - \lambda \int_{\Omega} u_n^2 \, dx
\]

\[
- \int_{\Omega} g(x, u_n)u_n \, dx - \int_{\Omega} D \left( Pu_n + \Delta^{-1}(\lambda u_n + g(x, u_n)) \right) \cdot Du_n \, dx.
\]

But for every \( z \) in \( H_0^1(\Omega) \), \( Pz \) is a smooth function and \( Pz \perp Qz \), so that the last integral in the previous equation is equal to

\[
\int_{\Omega} |DPu_n|^2 \, dx - \lambda \int_{\Omega} |Pu_n|^2 \, dx - \int_{\Omega} g(x, u_n)Pu_n \, dx.
\]
In this way
\[
\langle Q \nabla f_\lambda(u_n), u_n \rangle = 2f_\lambda(u_n) + 2 \int_{\Omega} G(x, u_n) \, dx
\]
(5.6)
\[- \int_{\Omega} g(x, u_n) u_n \, dx - \|Pu_n\|^2 + \lambda \int |Pu_n|^2 \, dx + \int_{\Omega} g(x, u_n) Pu_n \, dx.
\]
Now observe that \((g_2)\) implies that
\[
\lim_{n \to \infty} \int_{\Omega} |g(x, u_n)Pu_n| \, dx = 0,
\]
since \(|g(x, u_n)Pu_n| \leq \|Pu_n\|_\infty (a_1 + a_2|u_n|^{p-1})\) and \(\|Pu_n\|_\infty \to 0\). In this way, starting from (5.6), using \((g_4)\) and dividing by \(\|u_n\|^{p-1}\), we get \((p > 2)\)
\[
\lim_{n \to \infty} \int_{\Omega} G(x, u_n) \, dx = 0.
\]
Thus Remark 2.1.3 implies
\[
\lim_{n \to \infty} \int_{\Omega} |u_n|^p \, dx = 0,
\]
and so \(u \equiv 0\). In this way, dividing \(2f_\lambda(u_n)\) by \(\|u_n\|^2\), we get
\[
\lim_{n \to \infty} \int_{\Omega} G(x, u_n) \, dx = 1,
\]
and so there exists a constant \(C > 0\) such that
\[
\int \|u_n\|^\mu \, dx \leq C\|u_n\|^2.
\]
Now let us show that
\[
\lim_{n \to \infty} \int_{\Omega} g(x, u_n) Pu_n \, dx = 0,
\]
which is obvious if \(s \leq 2\) (this is the case if \(N \geq 6\)). In fact
\[
\lim_{n \to \infty} \int_{\Omega} \frac{|g(x, u_n)Pu_n|}{\|u_n\|^2} \, dx \leq \frac{\|Pu_n\|_\infty}{\|u_n\|^2} \left( a_1 + a_2 \int_{\Omega} |u_n|^{p-1} \, dx \right)
\]
\[
\leq \|Pu_n\|_\infty \left[ \frac{a_1}{\|u_n\|^2} + \frac{a_2'}{\|u_n\|^{2/p}} \left( \frac{\int |u_n|^p \, dx}{\|u_n\|^2} \right)^{1/p'} \right],
\]
and the thesis follows from (5.8).

In this way (5.6), \((g_4)\) and (5.9) imply
\[
\lim_{n \to \infty} \int_{\Omega} \frac{G(x, u_n) \, dx}{\|u_n\|^2} = 0,
\]
which contradicts (5.7).
Now, by Lemma 5.1.4 and Lemma 5.1.5 we can prove the following fundamental result.

**Proposition 5.1.6.** Assume $\lambda_{i-1} < \lambda_i = \ldots = \lambda_j < \lambda_{j+1}$ for some $i \leq j$ in $\mathbb{N}$. Then $\forall \delta > 0 \ \exists \varepsilon_0 > 0$ such that $\forall \lambda \in [\lambda_{i-1} + \delta, \lambda_j + 1 - \delta]$ and $\forall \varepsilon', \varepsilon'' \in (0, \varepsilon_0)$, $\varepsilon' < \varepsilon''$, the condition $(\nabla)(f, H_{i-1} \oplus H_j, \varepsilon', \varepsilon'')$ holds.

**Proof.** Assume by contradiction that there exists $\delta > 0$ such that $\forall \varepsilon_0 > 0$ there exist $\lambda$ in $[\lambda_{i-1} + \delta, \lambda_j + 1 - \delta]$ and $\varepsilon', \varepsilon''$ in $(0, \varepsilon_0)$, such that $(\nabla)(f, H_{i-1} \oplus H_j, \varepsilon', \varepsilon'')$ does not hold.

Take $\varepsilon_0 > 0$ as given by Lemma 5.1.4. Then there exists $(u_n)_n$ in $H^1_0(\Omega)$ such that $d(u_n, H_{i-1} \oplus H_j) \to 0$, $f_\lambda(u_n) \in [\varepsilon', \varepsilon'']$ and $Q\nabla f_\lambda(u_n) \to 0$. Then by Lemma 5.1.5 $(u_n)_n$ is bounded. Assume $u_n \rightharpoonup u$.

Note that $Q\nabla f_\lambda(u_n) = u_n - Pu_n + \Delta^{-1}(\lambda u_n + g(x, u_n))$, where $g(x, u_n) \to g(x, u)$ in $L^{1+1/s}(\Omega)$ by $(g_2)$ and $\Delta^{-1}: H^{-1} \to H^1_0(\Omega)$ is a compact operator. Then $u_n \to u$ and $u$ is a critical point of $f_\lambda$ constrained on $H_{i-1} \oplus H_j$.

By Lemma 5.1.4 $u = 0$, while $0 < \varepsilon' \leq f_\lambda(u_n) \forall n$. The continuity of $f_\lambda$ gives rise to a contradiction.

In order to apply Theorem 5.0.6 we only have to show that $\sup f_\lambda(\overline{B_j(R)})$ is small enough. In order to do that, let us show the following Lemma.

**Lemma 5.1.7.**

$$\lim_{\lambda \to \lambda_j} \sup \lambda f_\lambda(H_j) = 0.$$  

**Proof.** Assume by contradiction that there exist $\lambda_n \to \lambda_j$, $(u_n)_n$ in $H_j$ and $\varepsilon > 0$ such that

$$\sup \lambda f_\lambda(H_j) = f_\lambda(u_n) \geq \varepsilon \quad \forall n.$$  

Note that $f_\lambda$ attains a maximum in $H_j$ by Remark 2.1.3.

If $(u_n)_n$ is bounded, we can assume that $u_n \to u$ in $H_j$. In this way

$$\varepsilon \leq \frac{1}{2} \int_\Omega |Du|^2 \, dx - \frac{\lambda_j}{2} \int_\Omega u^2 \, dx - \int_\Omega G(x, u) \, dx \leq 0.$$  

So we can assume that $||u_n|| \to \infty$. In this case $(g_4)$ implies

$$0 < \varepsilon \leq f_\lambda(u_n) \leq \frac{1}{2} ||u_n||^2 - \frac{\lambda_n}{2} \int_\Omega u_n^2 \, dx - c_1 \int_\Omega |u_n|^p \, dx,$$

and since all norms are equivalent in $H_j$ the right hand side of the last inequality would tend to $-\infty$.

We can now prove the following preliminary result.

**Theorem 5.1.8.** Assume $\lambda_{i-1} < \lambda_i = \ldots = \lambda_j < \lambda_{j+1}$ for some $i \leq j$ in $\mathbb{N}$. Then there exists $\delta_j > 0$ such that, $\forall \lambda$ in $(\lambda_j - \delta_j, \lambda_j)$, problem $(P)$ has at least two nontrivial solution.

**Proof.** Take $\delta' > 0$ and find $\varepsilon_0$ as in Proposition 5.1.6. Fix $\varepsilon' < \varepsilon'' < \varepsilon_0$. By Lemma 5.1.7 there exists $\delta' \leq \delta'$ such that, if $\lambda \in (\lambda_j - \delta_j, \lambda_j)$, then $\sup \lambda f_\lambda(H_j) < \varepsilon''$ and by Proposition 5.1.6, $(\nabla)(f, H_{i-1} \oplus H_j, \varepsilon', \varepsilon'')$ holds. Moreover, since $\lambda < \lambda_j$, the topological situation of Lemma 5.1.3 is satisfied. By Theorem 5.0.6 there exist two critical point $u_1, u_2$ of $f_\lambda$ such that $f_\lambda(u_i) \in [\varepsilon', \varepsilon'']$, $i = 1, 2$. In particular $u_1$ and $u_2$ are nontrivial solutions of problem $(P)$.  

\[ \Box \]
In order to prove the existence of a third nontrivial solution, let us prove the following Lemma.

**Lemma 5.1.9.** Suppose $\lambda_{i-1} < \lambda_i = \ldots = \lambda_j < \lambda_{j+1}$. Then there exist $\delta_i > 0$, $\rho_1 > 0$ and $R_1 > \rho_1$ such that $\forall \lambda$ in $(\lambda_i - \delta_i, \lambda_i)$

$$\inf f_\lambda \left( S_j^+ (\rho_1) \right) > \sup f_\lambda \left( T_{j,j+1} (R_1) \right).$$

In particular there exists a critical point $u$ of $f_\lambda$ such that $f_\lambda(u) \geq \inf f_\lambda \left( S_j^+ (\rho_1) \right)$.

**Proof.** Take $\lambda$ in $[\lambda_{i-1}, \lambda_i)$. If $v \in H_j^+$, then $\int |Dv|^2 \geq \lambda_{j+1} \int u^2$, so that $(g_2)$ and $(g_3)$ imply that $\forall \tau > 0$ there exists $\rho_1 > 0$ such that, if $v \in H_j^+$ and $\|v\| = \rho_1$, then

$$f_\lambda(v) \geq \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda_{j+1}} - \tau \right) \|v\|^2.$$

Of course one take $\tau$ so small that the right hand side of the last inequality is greater than $C\rho_1^2$, where $C$ is independent on $\lambda$ and $C > 0$ (for example $C = 1 - \lambda_j/\lambda_{j+1} - \tau$).

By Remark 2.1.3 we get

$$f_\lambda(u) \leq \frac{1}{2} \|u\|^2 - \frac{\lambda}{2} \|u\|_{L^2(\Omega)}^2 - c_1 \|u\|_{L^p(\Omega)},$$

and since all norms are equivalent in $H_{j+1}$, we get that $f_\lambda(u) \to -\infty$ if $u \in H_{j+1}$ and $\|u\| \to \infty$.

By Lemma 5.1.7 there exists $\delta_i > 0$ such that $\forall \lambda$ in $(\lambda_i - \delta_i, \lambda_i)$ it results

$$\sup f_\lambda(H_j) < C\rho_1^2.$$

Of course we can always assume that $\delta_i \leq \delta_1$, $\delta_1$ being the one given in Theorem 5.1.8.

In this way, the classical Linking Theorem (see [116]) shows the existence of a critical point $u$ of $f_\lambda$ such that $f_\lambda(u) \geq C\rho_1^2$. \qed

Note that, although the topological structure found in Lemma 5.1.9 is equal to the one of Lemma 5.1.3, it is not possible to apply Theorem 5.0.6 again, since it is not clear if $(\nabla)(f_\lambda, H_j \oplus H_{j+1}^+, C\rho_1^2, \sup f_\lambda(B_{j+1}(R_1)))$ holds.

**Proof of Theorem 5.1.2.** Take $\delta_i$ as given in Lemma 5.1.9. Then the critical point $u$ found there is different from the critical points $u_i$ found in Theorem 5.1.8, since

$$f_\lambda(u_i) \leq \sup f_\lambda(H_j) < C\rho_1^2 \leq f_\lambda(u).$$
Bibliography


