Solitary Waves for Nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell Equations

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March 5, 2004

Abstract

In this paper we study the existence of radially symmetric solitary waves for nonlinear Klein-Gordon equations and nonlinear Schrödinger equations coupled with Maxwell equations. The method relies on a variational approach and the solutions are obtained as mountain-pass critical points for the associated energy functional.

1 Introduction

This paper has been motivated by the search of nontrivial solutions for the following nonlinear equations of the Klein-Gordon type:

\[
\frac{\partial^2 \psi}{\partial t^2} - \Delta \psi + m^2 \psi - |\psi|^{p-2} \psi = 0, \quad x \in \mathbb{R}^3,
\]

or of the Schrödinger type:

\[
\mathcal{i} \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi - |\psi|^{p-2} \psi, \quad x \in \mathbb{R}^3,
\]

where \(\hbar > 0, m > 0, p > 2, \psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}\).

In recent years many papers have been devoted to find standing waves of (1.1) or (1.2), i.e. solutions of the form

\[
\psi(x,t) = e^{i\omega t} u(x), \quad \omega \in \mathbb{R}.
\]

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With this Ansatz the nonlinear Klein-Gordon equation, as well as the nonlinear Schrödinger equation, is reduced to a semilinear elliptic equation and existence theorems have been established whether $u$ is radially symmetric and real (see [8], [9]), or $u$ is non-radially symmetric and complex (see [13], [16]). In this paper we want to investigate the existence of nonlinear Klein-Gordon or Schrödinger fields interacting with an electromagnetic field $\mathbf{E} - \mathbf{H}$; such a problem has been extensively pursued in the case of assigned electromagnetic fields (see [3], [4], [12]). Following the ideas already introduced in [5], [6], [7], [10], [11], [14], [15], we do not assume that the electromagnetic field is assigned. Then we have to study a system of equations whose unknowns are the wave function $\psi = \psi(x,t)$ and the gauge potentials $A, \Phi$, 

$$A : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3, \quad \Phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$$

which are related to $\mathbf{E} - \mathbf{H}$ by the Maxwell equations

$$\begin{align*}
\mathbf{E} &= - \left( \nabla \Phi + \frac{\partial A}{\partial t} \right) \\
\mathbf{H} &= \nabla \times \mathbf{A}.
\end{align*}$$

Let us first consider equation (1.1). The Lagrangian density related to (1.1) is given by

$$\mathcal{L}_{KG} = \frac{1}{2} \left[ \left| \frac{\partial \psi}{\partial t} \right|^2 - |\nabla \psi|^2 - m^2 |\psi|^2 \right] + \frac{1}{p} |\psi|^p.$$ 

The interaction of $\psi$ with the electromagnetic field is described by the minimal coupling rule, that is the formal substitution

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + ie\Phi, \quad \nabla \mapsto \nabla - ie\mathbf{A},$$

where $e$ is the electric charge. Then the Lagrangian density becomes:

$$\mathcal{L}_{KGM} = \frac{1}{2} \left[ \left| \frac{\partial \psi}{\partial t} + ie\Phi \right|^2 - |\nabla \psi - ieA\psi|^2 - m^2 |\psi|^2 \right] + \frac{1}{p} |\psi|^p.$$ 

If we set

$$\psi(x,t) = u(x,t)e^{iS(x,t)},$$

where $u, S : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$, the Lagrangian density takes the form

$$\mathcal{L}_{KGM} = \frac{1}{2} \left\{ u_t^2 - |\nabla u|^2 - \left[ |\nabla S - eA|^2 - (S_t + e\Phi)^2 + m^2 \right] u^2 \right\} + \frac{1}{p} |u|^p.$$
Now consider the Lagrangian density of the electromagnetic field $\mathbf{E} - \mathbf{H}$,

$$\mathcal{L}_0 = \frac{1}{2}(|\mathbf{E}|^2 - |\mathbf{H}|^2) = \frac{1}{2}|\mathbf{A}_t + \nabla \Phi|^2 - \frac{1}{2}|
abla \times \mathbf{A}|^2. \quad (1.3)$$

Therefore, the total action is given by

$$S = \int \int \mathcal{L}_{KGM} + \mathcal{L}_0.$$

Making the variation of $S$ with respect to $u$, $S$, $\Phi$ and $\mathbf{A}$ respectively, we get

$$u_{tt} - \Delta u + \left[|\nabla S - e\mathbf{A}|^2 - (S_t + e\Phi)^2 + m^2\right]u - |u|^{p-2}u = 0, \quad (1.4)$$

$$\frac{\partial}{\partial t} [(S_t + e\Phi)u^2] - \text{div}[(\nabla S - e\mathbf{A})u^2] = 0, \quad (1.5)$$

$$\text{div}(\mathbf{A}_t + \nabla \Phi) = e(S_t + e\Phi)u^2, \quad (1.6)$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{\partial}{\partial t}(\mathbf{A}_t + \nabla \Phi) = e(\nabla S - e\mathbf{A})u^2. \quad (1.7)$$

We are interested in finding standing (or solitary) waves of (1.4)-(1.7), that is solutions having the form

$$u = u(x), \quad S = \omega t, \quad \mathbf{A} = 0, \quad \Phi = \Phi(x), \quad \omega \in \mathbb{R}.$$

Then the equations (1.5) and (1.7) are identically satisfied, while (1.4) and (1.6) become

$$-\Delta u + [m^2 - (\omega + e\Phi)^2]u - |u|^{p-2}u = 0, \quad (1.8)$$

$$-\Delta \Phi + e^2u^2\Phi = -e\omega u^2. \quad (1.9)$$

In [6] the authors proved the existence of infinitely many symmetric solutions $(u_n, \Phi_n)$ of (1.8)-(1.9) under the assumption $4 < p < 6$, by using an equivariant version of the mountain pass theorem (see [1], [2]).

The object of the first part of this paper is to extend this result as follows.

**Theorem 1.1.** Assume that one of the following two hypotheses hold: either

a) $m > \omega > 0$ and $4 \leq p < 6$,

or

b) $m\sqrt{p-2} > \sqrt{2}\omega > 0$ and $2 < p < 4$. 


Then the system (1.8) − (1.9) has infinitely many radially symmetric solutions \((u_n, \Phi_n)\), \(u_n \neq 0\) and \(\Phi_n \neq 0\), with \(u_n \in H^1(\mathbb{R}^3)\), \(\Phi_n \in L^6(\mathbb{R}^3)\) and \(|\nabla \Phi_n| \in L^2(\mathbb{R}^3)\).

In the second part of the paper we study the Schrödinger equation for a particle in an electromagnetic field. Consider the Lagrangian associated to (1.2):

\[
L_S = \frac{1}{2} \left[ \frac{i}{\hbar} \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} |\nabla \psi|^2 \right] + \frac{1}{p} |\psi|^p.
\]

By using the formal substitution

\[
\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + \frac{i}{\hbar} \Phi, \quad \nabla \mapsto \nabla - \frac{i}{\hbar} A,
\]

we obtain

\[
L_{SM} = \frac{1}{2} \left[ \frac{i}{\hbar} \frac{\partial \psi}{\partial t} - e\Phi |\psi|^2 - \frac{\hbar^2}{2m} |\nabla \psi - \frac{e}{\hbar} A\psi|^2 \right] + \frac{1}{p} |\psi|^p.
\]

Now take

\[
\psi(x, t) = u(x, t) e^{i S(x, t)/\hbar}.
\]

With this Ansatz the Lagrangian \(L_{SM}\) becomes

\[
L_{SM} = \frac{1}{2} \left[ i\hbar \frac{\partial u}{\partial t} - \frac{\hbar^2}{2m} |\nabla u|^2 - \left( S_t + e\Phi + \frac{1}{2m} |\nabla S - eA|^2 \right) u^2 \right] + \frac{1}{p} |\psi|^p.
\]

Proceeding as in [5], we consider the total action \(S = \int \int |L_{SM} + \frac{1}{8\pi} (|E|^2 - |H|^2)|\) of the system “particle-electromagnetic field”. Then the Euler-Lagrange equations associated to the functional \(S = S(u, S, \Phi, A)\) give rise to the following system of equations:

\[
- \frac{\hbar^2}{2m} \Delta u + \left( S_t + e\Phi + \frac{1}{2m} |\nabla S - eA|^2 \right) u - |u|^{p-2} u = 0, \quad (1.10)
\]

\[
\frac{\partial u^2}{\partial t} + \frac{1}{m} \text{div}[(\nabla S - eA)u^2] = 0, \quad (1.11)
\]

\[
eu^2 = -\frac{1}{4\pi} \text{div} \left( \frac{\partial A}{\partial t} + \nabla \Phi \right), \quad (1.12)
\]

\[
\frac{e}{2m} (\nabla S - eA)u^2 = \frac{1}{4\pi} \left[ \frac{\partial}{\partial t} \left( \frac{\partial A}{\partial t} + \nabla \Phi \right) + \nabla \times (\nabla \times A) \right]. \quad (1.13)
\]
If we look for solitary wave solutions in the electrostatic case, i.e. 
\[ u = u(x), \ S = \omega t, \ \Phi = \Phi(x), \ A = 0, \ \omega \in \mathbb{R}, \]
then (1.11) and (1.13) are identically satisfied, while (1.10) and (1.12) become
\[ -\frac{R^2}{2m} \Delta u + e\Phi u - |u|^{p-2} u + \omega u = 0, \quad (1.14) \]
\[ -\Delta \Phi = 4\pi eu^2. \quad (1.15) \]

The existence of solutions of (1.14)-(1.15) was already studied for \( 4 < p < 6 \): in [5] existence of infinitely many radial solutions was proved, while in [13] existence of a non radially symmetric solution was established. In the second part of the paper we prove the following result.

**Theorem 1.2.** Let \( \omega > 0 \) and \( 4 \leq p < 6 \). Then the system (1.14) – (1.15) has at least a radially symmetric solution \((u, \Phi)\), \( u \neq 0 \) and \( \Phi \neq 0 \), with \( u \in H^1(\mathbb{R}^3) \), \( \Phi \in L^6(\mathbb{R}^3) \) and \( |\nabla \Phi| \in L^2(\mathbb{R}^3) \).

### 2 Nonlinear Klein-Gordon Equations coupled with Maxwell Equations

In this section we will prove Theorem 1.1. For sake of simplicity, assume \( e = 1 \) so that (1.8)-(1.9) give rise to the following system in \( \mathbb{R}^3 \):
\[ -\Delta u + [m^2 - (\omega + \Phi)^2]u - |u|^{p-2} u = 0, \quad (2.16) \]
\[ -\Delta \Phi + u^2 \Phi = -\omega u^2. \quad (2.17) \]

Assume that one of the following hypotheses hold: either

a) \( m > \omega > 0 \), \( 4 \leq p < 6 \),

or

b) \( m\sqrt{p-2} > \sqrt{2}\omega > 0 \), \( 2 < p < 4 \).
We note that \( q = 6 \) is the critical exponent for the Sobolev embedding \( H^1(\mathbb{R}^3) \subset L^q(\mathbb{R}^3) \).

It is clear that (2.16)-(2.17) are the Euler-Lagrange equations of the functional \( F: H^1 \times D^{1,2} \to \mathbb{R} \) defined as

\[
F(u, \Phi) = \frac{1}{2} \int_{\mathbb{R}^3} \left( |\nabla u|^2 - |\nabla \Phi|^2 + |m^2 - (\omega + \Phi)^2|u^2 \right) dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx.
\]

Here \( H^1 \equiv H^1(\mathbb{R}^3) \) denotes the usual Sobolev space endowed with the norm

\[
\|u\|_{H^1} \equiv \left( \int_{\mathbb{R}^3} \left( |\nabla u|^2 + |u|^2 \right) dx \right)^{1/2}
\]

and \( D^{1,2} \equiv D^{1,2}(\mathbb{R}^3) \) is the completion of \( C_0^\infty(\mathbb{R}^3, \mathbb{R}) \) with respect to the norm

\[
\|u\|_{D^{1,2}} \equiv \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{1/2}.
\]

The following two propositions hold.

**Proposition 2.1.** The functional \( F \) belongs to \( C^1(H^1 \times D^{1,2}, \mathbb{R}) \) and its critical points are the solutions of (2.16) – (2.17).

(For the proof we refer to [6]).

**Proposition 2.2.** For every \( u \in H^1 \), there exists a unique \( \Phi = \Phi[u] \in D^{1,2} \) which solves (2.17). Furthermore

(i) \( \Phi[u] \leq 0 \);

(ii) \( \Phi[u] \geq -\omega \) in the set \( \{ x \mid u(x) \neq 0 \} \);

(iii) if \( u \) is radially symmetric, then \( \Phi[u] \) is radial, too.

**Proof.** Fixed \( u \in H^1 \), consider the following bilinear form on \( D^{1,2} \):

\[
a(\phi, \psi) = \int_{\mathbb{R}^3} (\nabla \psi \nabla \phi + u^2 \phi \psi) \ dx.
\]

Obviously \( a(\phi, \phi) \geq \| \phi \|^2_{D^{1,2}} \). Observe that, since \( H^1(\mathbb{R}^3) \subset L^3(\mathbb{R}^3) \), then \( u^2 \in L^{3/2}(\mathbb{R}^3) \). On the other hand \( D^{1,2} \) is continuously embedded in \( L^6(\mathbb{R}^3) \), hence, by Hölder’s inequality,

\[
a(\phi, \psi) \leq \| \phi \|_{D^{1,2}} \| \psi \|_{D^{1,2}} + \| u^2 \|_{L^{3/2}} \| \phi \|_{L^6} \| \psi \|_{L^6} \leq (1 + C \| u \|^2_{L^3}) \| \phi \|_{D^{1,2}} \| \psi \|_{D^{1,2}}
\]
for some positive constant \( C \), given by Sobolev inequality (see [20]). Therefore \( a \) defines an inner product, equivalent to the standard inner product in \( D^{1,2} \).

Moreover \( H^1(\mathbb{R}^3) \subset L^{12/5}(\mathbb{R}^3) \), and then

\[
\left| \int_{\mathbb{R}^3} u^2 \psi \, dx \right| \leq \| u^2 \|_{L^{12/5}} \| \psi \|_{L^6} \leq C \| u \|_{L^{12/5}}^2 \| \psi \|_{D^{1,2}}^2.
\] (2.20)

Therefore the linear map \( \mathbb{R} \to \int_{\mathbb{R}^3} u^2 \psi \, dx \) is continuous. By Lax-Milgram Lemma we get the existence of a unique \( \Phi \in D^{1,2} \) such that

\[
\int_{\mathbb{R}^3} (\nabla \Phi \nabla \psi + u^2 \Phi \psi) \, dx = -\omega \int_{\mathbb{R}^3} u^2 \psi \, dx \quad \forall \psi \in D^{1,2},
\]

i.e. \( \Phi \) is the unique solution of (2.17). Furthermore \( \Phi \) achieves the minimum

\[
\inf_{\phi \in D^{1,2}} \int_{\mathbb{R}^3} \left( \frac{1}{2} \left( |\nabla \phi|^2 + u^2 |\phi|^2 \right) + \omega u^2 \phi \right) \, dx
= \int_{\mathbb{R}^3} \left( \frac{1}{2} \left( |\nabla \phi|^2 + u^2 |\phi|^2 \right) + \omega u^2 \phi \right) \, dx.
\]

Note that also \( -|\Phi| \) achieves such a minimum; then, by uniqueness, \( \Phi = -|\Phi| \leq 0 \). Now let \( O(3) \) denote the group of rotations in \( \mathbb{R}^3 \). Then for every \( g \in O(3) \) and \( f : \mathbb{R}^3 \to \mathbb{R} \), set \( T_g(f)(x) = f(gx) \). Note that \( T_g \) does not change the norms in \( H^1, D^{1,2} \) and \( L^p \). In Lemma 4.2 of [6] it was proved that \( T_g \Phi[u] = \Phi[T_g u] \). In this way, if \( u \) is radial, we get \( T_g \Phi[u] = \Phi[u] \).

Finally, following the same idea of [17], fixed \( u \in H^1 \), if we multiply (2.17) by \((\omega + \Phi[u])^+ \equiv -\min\{\omega + \Phi[u], 0\} \), which is an admissible test function, since \( \omega > 0 \), we get

\[
-\int_{\Phi[u] < -\omega} |D \Phi[u]|^2 \, dx - \int_{\Phi[u] < -\omega} (\omega + \Phi[u])^2 u^2 \, dx = 0,
\]

so that \( \Phi[u] \geq -\omega \) where \( u \neq 0 \).

**Remark 2.3.** The result (ii) of Proposition 2.2 can be strengthened in some cases. Indeed, take \( \overline{u} \) in \( H^1(\mathbb{R}^3) \cap C^\infty \) radially symmetric such that

\[
\overline{u} > 0 \text{ in } B(0,R), \overline{u} \equiv 0 \text{ in } \mathbb{R}^3 \setminus B(0,R)
\]

for some \( R > 0 \). Then there results

\[
-\omega \leq \Phi[\overline{u}](x) \leq 0 \quad \forall x \in \mathbb{R}^3.
\]
In fact, since $\Phi[\Phi]$ solves (2.17), by standard regularity results for elliptic equations, $u \in C^\infty$ implies $\Phi[\Phi] \in C^\infty$. By Proposition 2.2, $\Phi[\Phi]$ is radial; moreover $\Phi[\Phi]$ is harmonic outside $B(0, R)$. Since $\Phi[\Phi] \in D^{1,2}$, then

$$\Phi[\Phi](x) = -\frac{c}{|x|}, \quad |x| \geq R,$$

for some $c > 0$. Setting $\Phi(r) = \Phi[\Phi](x)$ for $|x| = r$, it results $\Phi'(R) > 0$ and $\Phi(r) > \Phi(R)$ for every $r > R$. Therefore the minimum of $\Phi[\Phi]$ is achieved in $B(0, R)$. Let $\pi$ be a minimum point for $\Phi[\Phi]$. Then (2.17) implies

$$\Phi[\Phi](\pi) = \frac{-\omega \pi^2(\pi) + \Delta \Phi[\Phi](\pi)}{\pi^2(\pi)} \geq -\omega.$$

In view of proposition 2.2, we can define the map

$$\Phi : H^1 \rightarrow D^{1,2}$$

which maps each $u \in H^1$ in the unique solution of (2.17). From standard arguments it results $\Phi \in C^1(H^1, D^{1,2})$ and from the very definition of $\Phi$ we get

$$F'_\Phi(u, \Phi[u]) = 0 \quad \forall u \in H^1. \quad (2.21)$$

Now let us consider the functional

$$J : H^1 \rightarrow \mathbb{R}, \quad J(u) := F(u, \Phi[u]).$$

By proposition 2.1, $J \in C^1(H^1, \mathbb{R})$ and, by (2.21),

$$J'(u) = F'_u(u, \Phi[u]).$$

By definition of $F$, we obtain

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left( |\nabla u|^2 - |\nabla \Phi[u]|^2 + |m^2 - \omega^2|u^2 - u^2 \Phi[u]^2 \right) dx$$

$$-\omega \int_{\mathbb{R}^3} u^2 \Phi[u] - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx.$$

Multiplying both members of (2.17) by $\Phi[u]$ and integrating by parts, we obtain

$$\int_{\mathbb{R}^3} |\nabla \Phi[u]|^2 dx + \int_{\mathbb{R}^3} |u|^2 |\Phi[u]|^2 dx = -\omega \int_{\mathbb{R}^3} |u|^2 |\Phi[u]| dx. \quad (2.22)$$
Using (2.22), the functional $J$ may be written as

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + |m^2 - \omega^2|u^2 - \omega u^2 \Phi[u]\right) \, dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \, dx.$$  \quad (2.23)

The next lemma states a relationship between the critical points of the functionals $F$ and $J$ (the proof can be found in [6]).

**Lemma 2.4.** The following statements are equivalent:

i) $(u, \Phi) \in H^1 \times D^{1,2}$ is a critical point of $F$;

ii) $u$ is a critical point of $J$ and $\Phi = \Phi[u]$.

Then, in order to get solutions of (2.16)-(2.17), we look for critical points of $J$.

**Theorem 2.5.** Assume hypotheses a) and b). Then the functional $J$ has infinitely many critical points $u_n \in H^1$ having a radial symmetry.

**Proof.** Our aim is to apply the equivariant version of the Mountain-Pass Theorem (see [1], Theorem 2.13, or [18], Theorem 9.12). Since $J$ is invariant under the group of translations, there is clearly a lack of compactness. In order to overcome this difficulty, we consider radially symmetric functions. More precisely we introduce the subspace

$$H^1_r = \left\{ u \in H^1 \mid u(x) = u(|x|) \right\}.$$  

We divide the remaining part of the proof in three steps.

**Step 1.** Any critical point $u \in H^1_r$ of $J_{|H^1_r}$ is also a critical point of $J$.

The proof can be found in [6].

**Step 2.** The functional $J_{|H^1_r}$ satisfies the Palais-Smale condition, i.e.

any sequence $\{u_n\}_n \subset H^1_r$ such that $J(u_n)$ is bounded and $J'_{|H^1_r}(u_n) \to 0$ contains a convergent subsequence.

For the sake of simplicity, from now on we set $\Omega = m^2 - \omega^2 > 0$. Let $\{u_n\}_n \subset H^1_r$ be such that

$$|J(u_n)| \leq M, \quad J'_{|H^1_r}(u_n) \to 0.$$
for some constant $M > 0$. Then, using the form of $J$ given in (2.23),

$$pJ(u_n) - J'(u_n)u_n = \left( \frac{p}{2} - 1 \right) \int_{\mathbb{R}^3} \left( |\nabla u_n|^2 + \Omega |u_n|^2 \right) dx$$

$$- \omega \left( \frac{p}{2} - 2 \right) \int_{\mathbb{R}^3} u_n^2 \Phi[u_n] dx + \int_{\mathbb{R}^3} u_n^2 (\Phi[u_n])^2 dx$$

$$\geq \left( \frac{p}{2} - 1 \right) \int_{\mathbb{R}^3} \left( |\nabla u_n|^2 + \Omega |u_n|^2 \right) dx - \omega \left( \frac{p}{2} - 2 \right) \int_{\mathbb{R}^3} u_n^2 \Phi[u_n] dx. \quad (2.24)$$

We distinguish two cases: either $p \geq 4$ or $2 < p < 4$.

If $p \geq 4$, by (2.24), using Proposition 2.2, we immediately deduce

$$pJ(u_n) - J'(u_n)u_n \geq \left( \frac{p}{2} - 1 \right) \int_{\mathbb{R}^3} \left( |\nabla u_n|^2 + \Omega |u_n|^2 \right) dx. \quad (2.25)$$

Moreover, by hypothesis a)

$$\left( \frac{p}{2} - 1 \right) \int_{\mathbb{R}^3} \left( |\nabla u_n|^2 + \Omega |u_n|^2 \right) dx \geq c_1 \|u_n\|^2, \quad (2.26)$$

and by assumption

$$pJ(u_n) - J'(u_n)u_n \leq pM + c_2 \|u_n\| \quad (2.27)$$

for some positive constants $c_1$ and $c_2$.

Combining (2.25), (2.26), (2.27), we deduce that \{u_n\}_n is bounded in $H^1$.

If $2 < p < 4$, by Proposition 2.2 and (2.24) we get

$$pJ(u_n) - J'(u_n)u_n \geq \left( \frac{p}{2} - 1 \right) \int_{\mathbb{R}^3} \left( |\nabla u_n|^2 + \Omega |u_n|^2 \right) dx - \omega^2 \left( 2 - \frac{p}{2} \right) \int_{\mathbb{R}^3} u_n^2 dx$$

$$= \left( \frac{p}{2} - 1 \right) \int_{\mathbb{R}^3} |\nabla u_n|^2 + \left( \frac{m^2(p-2) - 2\omega^2}{2} \right) \int_{\mathbb{R}^3} |u_n|^2 dx.$$ 

By hypothesis b) $m^2(p-2) - 2\omega^2 > 0$, then we repeat the same argument as for $p \geq 4$ and obtain the boundedness of \{u_n\}_n in $H^1$.

On the other hand, using equation (2.17), and proceeding as in (2.20), we get

$$\int_{\mathbb{R}^3} \nabla \Phi[u_n] \cdot dx \leq \int_{\mathbb{R}^3} |
\nabla \Phi[u_n]|^2 dx + \int_{\mathbb{R}^3} |u_n|^2 |\Phi[u_n]|^2 dx = -\omega \int_{\mathbb{R}^3} u_n^2 \Phi[u_n] dx$$

$$\leq C\omega \|u_n\|_{L^{12/5}}^2 \|\Phi[u_n]\|_{L^{12}}.$$
which implies that \( \{ \Phi[u_n] \}_n \) is bounded in \( D^{1,2} \).
Then, up to a subsequence,
\[
    u_n \rightharpoonup u \quad \text{in } H^1_r
\]
\[
    \Phi[u_n] \rightharpoonup \phi \quad \text{in } D^{1,2}.
\]
If \( L : H^1_r \to (H^1_r)' \) is defined as
\[
    L(u) = -\Delta u + \Omega u,
\]
then
\[
    L(u_n) = \omega u_n \Phi[u_n] + |u_n|^{p-2} u_n + \varepsilon_n,
\]
where \( \varepsilon_n \to 0 \) in \( (H^1_r)' \), that is
\[
    u_n = L^{-1}(\omega u_n \Phi[u_n]) + L^{-1}(|u_n|^{p-2} u_n) + L^{-1}(\varepsilon_n).
\]
Now note that \( \{ u_n \Phi[u_n] \} \) is bounded in \( L^{3/2}_r \); in fact, by Hölder’s inequality,
\[
    \|u_n \Phi[u_n]\|_{L^{3/2}_r} \leq \|u_n\|_{L^2_r} \|\Phi[u_n]\|_{L^{6/5}_r} \leq c \|u_n\|_{L^2_r} \|\Phi[u_n]\|_{D^{1,2}}.
\]
Moreover \( \{ |u_n|^{p-2} u_n \} \) is bounded in \( L^p_r \) (where \( 1/p + 1/p' = 1 \)). The immersions \( H^1_r \to L^3_r \) and \( H^1_r \to L^p_r \) are compact (see [8] or [19]) and thus, by duality, \( L^{3/2}_r \) and \( L^p_r \) are compactly embedded in \( (H^1_r)' \). Then by standard arguments \( L^{-1}(\omega u_n \Phi[u_n]) \) and \( L^{-1}(|u_n|^{p-2} u_n) \) strongly converge in \( H^1_r \). Then we conclude
\[
    u_n \rightharpoonup u \quad \text{in } H^1_r.
\]

**Step 3.** The functional \( J |_{H^1_r} \) satisfies the geometrical hypothesis of the equivariant version of the Mountain Pass Theorem.

First of all we observe that \( J(0) = 0 \). Moreover, by Proposition 2.2 and (2.23),
\[
    J(u) \geq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{\Omega}{2} \int_{\mathbb{R}^3} |u|^2 \, dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \, dx.
\]
The hypothesis \( 2 < p < 6 \) and the continuous embedding \( H^1 \subset L^p \) imply that there exists \( \rho > 0 \) small enough such that
\[
    \inf_{\|u\|_{H^1} = \rho} J(u) > 0.
\]
Since $J$ is even, the thesis of step 3 will follow if we prove that for every finite dimensional subset $V$ of $H^1_r$ it results

$$\lim_{\|u\|_{H^1_r} \to +\infty} J(u) = -\infty. \quad (2.28)$$

Let $V$ be an $m$-dimensional subspace of $H^1_r$ and let $u \in V$. By Proposition 2.2 $\Phi[u] \geq -\omega$ where $u \neq 0$, so that

$$J(u) \leq \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + \Omega |u|^2 + \omega^2 |u|^2) \, dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \leq c\|u\|_{H^1_r}^2 - \frac{1}{p}\|u\|_{L^p}^p,$$

and (2.28) follows, since all norms in $V$ are equivalent.

**Proof of Theorem 1.1.** Lemma 2.4 + Theorem 2.5.

**Remark 2.6.** In view of Remark 2.3 the existence of one nontrivial critical point for the functional $J$ follows from the classical mountain pass theorem: more precisely, taken $\pi \in H^1_r \cap C^\infty$ as in Remark 2.3, since $\|\Phi[\pi]\|_{\infty} \leq \omega$, there results

$$J(t\pi) \leq \frac{t^2}{2} \int_{\mathbb{R}^3} (|\nabla \pi|^2 + \Omega |\pi|^2 + \omega^2 |\pi|^2) \, dx - \frac{t^p}{p} \int_{\mathbb{R}^3} |\pi|^p \to -\infty \text{ as } t \to +\infty.$$

### 3 Nonlinear Schrödinger Equations coupled with Maxwell Equations

For sake of simplicity assume $\hbar = m = e = 1$ in (1.14)-(1.15). Then we are reduced to study the following system in $\mathbb{R}^3$:

$$-\frac{1}{2} \Delta u + \Phi u + \omega u - |u|^{p-2} u = 0, \quad (3.29)$$

$$-\Delta \Phi = 4\pi u^2. \quad (3.30)$$

We will assume

a') $\omega > 0$

b') $4 \leq p < 6$. 


Of course, (3.29)-(3.30) are the Euler-Lagrange equations of the functional $F : H^1 \times D^{1,2} \to \mathbb{R}$ defined as

$$F(u, \Phi) = \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx - \frac{1}{16\pi} \int_{\mathbb{R}^3} |\nabla \Phi|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} (\Phi u^2 + \omega u^2) \, dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \, dx,$$

where $H^1$ and $D^{1,2}$ are defined as in the previous section.

It is easy to prove the analogous of Proposition 2.1, i.e. that $F \in C^1(H^1 \times D^{1,2}, \mathbb{R})$ and that its critical points are solutions of (3.29)-(3.30).

Moreover we have the following proposition.

**Proposition 3.1.** For every $u \in H^1$ there exists a unique solution $\Phi = \Phi[u] \in D^{1,2}$ of (3.30), such that

- $\Phi[u] \geq 0;$
- $\Phi[tu] = t^2 \Phi[u]$ for every $u \in H^1$ and $t \in \mathbb{R}$.

**Proof.** Let us consider the linear map $\phi \in D^{1,2} \mapsto \int_{\mathbb{R}^3} u^2 \phi \, dx$, which is continuous by (2.20). By Lax-Milgram’s Lemma we get the existence of a unique $\Phi \in D^{1,2}$ such that

$$\int_{\mathbb{R}^3} \nabla \Phi \nabla \phi \, dx = 4\pi \int_{\mathbb{R}^3} u^2 \phi \, dx \quad \forall \phi \in D^{1,2},$$

i.e. $\Phi$ is the unique solution of (3.30). Furthermore $\Phi$ achieves the minimum

$$\inf_{\Phi \in D^{1,2}} \left\{ \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi|^2 - 4\pi \int_{\mathbb{R}^3} u^2 \phi \, dx \right\} = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \Phi|^2 \, dx - 4\pi \int_{\mathbb{R}^3} u^2 \Phi \, dx.$$

Note that also $|\Phi|$ achieves such minimum; then, by uniqueness, $\Phi = |\Phi| \geq 0$.

Finally,

$$-\Delta \Phi[tu] = 4\pi t^2 u^2 = -t^2 \Delta \Phi[u] = -\Delta(t^2 \Phi[u]),$$

thus, by uniqueness, $\Phi[tu] = t^2 \Phi[u]$.

Proceeding as in the previous section we can define the map

$$\Phi : H^1 \to D^{1,2},$$

which maps each $u \in H^1$ in the unique solution of (3.30). As before, $\Phi \in C^1(H^1, D^{1,2})$ and

$$F_\Phi(u, \Phi[u]) = 0 \quad \forall u \in H^1.$$
Now consider the functional $J : H^1 \rightarrow \mathbb{R}$ defined by

$$J(u) = F(u, \Phi[u]).$$

$J$ belongs to $C^1(H^1, \mathbb{R})$ and satisfies $J'(u) = F_u(u, \Phi[u])$. Using the definition of $F$ and equation (3.30), we obtain

$$J(u) = \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{\omega}{2} \int_{\mathbb{R}^3} |u|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} |u|^2 \Phi[u] \, dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \, dx.$$

As before, one can prove the following lemma.

**Lemma 3.2.** The following statements are equivalent:

1. $(u, \Phi) \in H^1 \times D^{1,2}$ is a critical point of $F$,
2. $u$ is a critical point of $J$ and $\Phi = \Phi[u]$.

Now we are ready to prove the existence result for equations (3.29)-(3.30).

**Theorem 3.3.** Assume hypotheses a') and b'). Then the functional $J$ has a nontrivial critical point $u \in H^1$ having a radial symmetry.

**Proof.** Let $H^1_r$ be defined as in theorem 2.5.

**Step 1.** Any critical point $u \in H^1_1$ of $J|_{H^1_r}$ is also a critical point of $J$.

The proof is as in theorem 2.5.

**Step 2.** The functional $J|_{H^1_r}$ satisfies the Palais-Smale condition.

Let $\{u_n\}_n \subset H^1_1$ be such that

$$|J(u_n)| \leq M, \quad J'(u_n) \rightharpoonup 0$$

for some constant $M > 0$. Then

$$pJ(u_n) - J'(u_n)u_n$$

$$= \left( \frac{p}{4} - \frac{1}{2} \right) \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \left( \frac{p}{4} - 1 \right) \int_{\mathbb{R}^3} \Phi[u_n] |u_n|^2 dx + \left( \frac{p}{2} - 1 \right) \omega \int_{\mathbb{R}^3} |u_n|^2 dx$$

$$\geq \left( \frac{p}{4} - \frac{1}{2} \right) \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \left( \frac{p}{2} - 1 \right) \omega \int_{\mathbb{R}^3} |u_n|^2 dx$$
by Proposition 3.1, since $p \geq 4$. Moreover

$$
\left( \frac{p}{4} - \frac{1}{2} \right) \int_{\mathbb{R}^3} (|\nabla u|^2 + \omega |u|^2) \, dx \geq c_1 \|u_n\|^2
$$

and by assumption

$$p \mathcal{J}(u_n) - \mathcal{J}'(u_n)u_n \leq pM + c_2 \|u_n\|_{H^1}
$$

for some positive constants $c_1$ and $c_2$.

We have thus proved that $\{u_n\}_n$ is bounded in $H^1_r$.

On the other hand, $\|\Phi[u_n]\|_{D^{1,2}}^2 = 4\pi \int_{\mathbb{R}^3} u^2 \Phi[u_n] \, dx$, and then, using inequality (2.20), we easily deduce that $\{\Phi[u_n]\}_n$ is bounded in $D^{1,2}$.

The remaining part of the proof follows as in Step 2 of Theorem 2.5, after replacing $L$ with $\mathcal{L} : H^1_r \to (H^1_r)'$ defined as $\mathcal{L}(u) = -\frac{1}{2} \Delta u + \omega u$.

**Step 3.** The functional $\mathcal{J}_{|H^1_r}$ satisfies the three geometrical hypothesis of the mountain pass theorem.

By Proposition 3.1 it results

$$
\mathcal{J}(u) \geq \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{\omega}{2} \int_{\mathbb{R}^3} |u|^2 \, dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \, dx.
$$

Then, using the continuous embedding $H^1 \subset L^p$, we deduce that $\mathcal{J}$ has a strict local minimum in 0.

We introduce the following notation: if $u : \mathbb{R}^3 \to \mathbb{R}$, we set

$$u_{\lambda,\alpha,\beta}(x) = \lambda^\beta u(\lambda^\alpha x), \quad \lambda > 0, \alpha, \beta \in \mathbb{R}.
$$

Now fix $u \in H^1_r$. We want to show that

$$
\Phi[u_{\lambda,\alpha,\beta}] = (\Phi[u])|_{\lambda,\alpha,2(\beta-\alpha)}.
$$

In fact

$$
-\Delta \Phi[u_{\lambda,\alpha,\beta}](x) = 4\pi u_{\lambda,\alpha,\beta}^2(\lambda^\alpha x) = 4\pi \lambda^{2\beta} u^2(\lambda^\alpha x)
$$

$$
= -\lambda^{2\beta} (\Delta \Phi[u])(\lambda^\alpha x) = -\Delta ((\Phi[u])|_{\lambda,\alpha,2(\beta-\alpha)})(x).
$$

By uniqueness (see Proposition 3.1), (3.31) follows.

Now take $u \neq 0$ in $H^1_r$ and evaluate

$$
\mathcal{J}(u_{\lambda,\alpha,\beta}) = \frac{\lambda^{2\beta-\alpha}}{4} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{\omega}{2} \lambda^{2\beta-2\alpha} \int_{\mathbb{R}^3} u^2 \, dx
$$
Solitary waves

\[ + \frac{\lambda^{\beta-5\alpha}}{4} \int_{\mathbb{R}^3} u^2 \Phi[u] \, dx - \frac{\lambda^{\beta p-3\alpha}}{p} \int_{\mathbb{R}^3} |u|^p \, dx. \]

We want to prove that \( J(u_{\lambda,\alpha,\beta}) < J(0) \) for some suitable choice of \( \lambda, \alpha \) and \( \beta \).

For example assume

\[
\begin{align*}
\beta p - 3\alpha &< 0, \\
\beta p - 3\alpha &< 2\beta - \alpha, \\
\beta p - 3\alpha &< 2\beta - 3\alpha, \\
\beta p - 3\alpha &< 4\beta - 5\alpha,
\end{align*}
\] (3.32)

then it is clear that \( J(u_{\lambda,\alpha,\beta}) \to -\infty \) as \( \lambda \to 0 \).

So we look for a couple \((\alpha, \beta)\) which satisfies (3.32). From the third inequality we get \( \beta < 0 \). Combining the second and the fourth ones, we derive

\[
4 - p < \frac{2\alpha}{\beta} < p - 2.
\] (3.33)

Such an inequality is satisfied by taking \( \beta = 2\alpha \), which also satisfies the first inequality in (3.32).

In a similar way one can prove that if

\[
\begin{align*}
\beta p - 3\alpha &> 0, \\
\beta p - 3\alpha &> 2\beta - \alpha, \\
\beta p - 3\alpha &> 2\beta - 3\alpha, \\
\beta p - 3\alpha &> 4\beta - 5\alpha,
\end{align*}
\] (3.34)

then \( J(u_{\lambda,\alpha,\beta}) \to -\infty \) as \( \lambda \to +\infty \) with the same choice \( \beta = 2\alpha \).

**Remark 3.4.** Notice that the systems (3.32) or (3.34) have a solution for every \( p > 3 \). More precisely for every \( p > 3 \) there is a couple \((\alpha, \beta)\) which satisfies the inequality (3.33) and, consequently, \( J(u_{\lambda,\alpha,\beta}) \to -\infty \). The restriction \( p \geq 4 \) appears in proving the Palais-Smale condition.

**Proof of Theorem 1.2 Lemma 3.2 + Theorem 3.3.**

**References**


