Stieltjes-Type Integrals for Metric Semigroup-Valued Functions Defined on Unbounded Intervals

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Abstract

We introduce the $GH_k$ integral for functions defined on (possibly) unbounded subintervals of the extended real line and with values in metric semigroups. Basic properties and convergence theorems for this integral are deduced.

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1 Introduction.

Stieltjes-type integrals are widely studied in the literature: for example, meaningful results can be found in [8, 9, 10, 23]. In particular, in [13, 14, 15] and in a more abstract setting in [8, 9], an integral ($GH_k$ integral) for real-valued functions defined in a compact subinterval of the real line has been investigated, which generalizes the integral studied by Š. Schwabik in [24]: the latter includes also the classical Kurzweil-Henstock and Henstock-Stieltjes integrals. Some examples of other particular cases of the $GH_k$ integral are illustrated in [8, 9].

In this paper we extend the $GH_k$ integral to the case of metric semigroup-valued functions, defined on (possibly) unbounded subintervals of the extended real line, and we prove some convergence theorems. Similar results were proved in [5] in the context of the Kurzweil-Henstock integral, for which the $GH_k$ integral is substantially a particular case; moreover, in this paper we prove also an extension Cauchy-type theorem.

For a literature existing on the Kurzweil-Henstock integral in the context of metric semigroups, we refer to [5, 16, 26] and their bibliography, while for Riesz-space valued functions we recall [1, 2, 3, 4, 17, 18, 19, 20, 21, 22]. A particular example of metric semigroup is the set $L(R)$ of fuzzy numbers (see also Section 2 and [5]).

2 Metric semigroups.

Definition 2.1. A metric semigroup is a structure $(X, \rho, +, \cdot)$, where $\rho : X \times X \to \mathbb{R}$, $+: X \times X \to X$, $\cdot : \mathbb{R} \times X \to X$ satisfy the following conditions:

(i) $(X, \rho)$ is a complete metric space;

(ii) $(X, +)$ is a commutative semigroup endowed with a neutral element 0;

(iii) $\rho(w + y, z + t) \leq \rho(w, z) + \rho(y, t)$ for any $w, y, z, t \in X$;

(iv) $\rho(\alpha w, \alpha y) \leq |\alpha| \rho(w, y)$ for all $\alpha \in \mathbb{R}$ and $w, y \in X$;

(v) $\alpha(w + y) = \alpha w + \alpha y$ for each $\alpha \in \mathbb{R}$, $w, y \in X$;

(vi) $(\alpha + \beta)w = \alpha w + \beta w$ for every $\alpha, \beta \in \mathbb{R}_0^+$, $w \in X$, $0 \cdot w = 0$ and $1 \cdot w = w$ for each $w \in X$.

A metric semigroup $(X, \rho, +, \cdot)$ is called invariant, if

$$\rho(w + z, y + z) = \rho(w, y)$$

for any $w, y, z \in X$.

Observe that a consequence of invariance and of the triangular property is the following condition, which will be useful in the sequel:
(vii) $\rho(w + y, z) \leq \rho(w, t) + \rho(y + t, z)$ whenever $x, y, z, t \in X$.

An example of metric semigroup is the set of all fuzzy numbers (see also [5, 26]).

**Definition 2.2.** A fuzzy number is a function $\mu : \mathbb{R} \to [0, 1]$ satisfying the following conditions:

(j) there exists $x_0 \in \mathbb{R}$ such that $\mu(x_0) = 1$;

(jj) the $\alpha$-cut set $\mu_\alpha = \{ x \in \mathbb{R} : \mu(x) \geq \alpha \}$ is convex for $\alpha \in [0, 1]$;

(jjj) $\mu$ is upper semi-continuous, i.e. any $\alpha$-cut $\mu_\alpha$ is a closed subset of $\mathbb{R}$;

(jv) the support $\{ x \in \mathbb{R} : \mu(x) > 0 \}$ of the function $\mu$ is a compact set.

Any real number $u_0$ can be identified with a fuzzy number $\mu_0$ in the following way:

$$\mu_0(x) = \chi_{\{u_0\}}(x),$$

i.e. $\mu_0(u_0) = 1$, and $\mu_0(x) = 0$, if $x \neq u_0$.

The set of all fuzzy numbers is denoted by $L(\mathbb{R})$.

We now endow $L(\mathbb{R})$ with a metric and a linear structure (see also [5, 26]). We define the Hausdorff distance $\mathcal{H}$ on the set of all compact possibly degenerate intervals in $\mathbb{R}$:

$$\mathcal{H}([a, b], [c, d]) = \max(|c - a|, |d - b|).$$

Let $\mu, \nu \in L(\mathbb{R})$. It is easy to check that, for every $\alpha \in (0, 1]$, there exist $a, b, c, d \in \mathbb{R}$ (depending on $\alpha$) such that $\mu_\alpha = [a, b], \nu_\alpha = [c, d]$ So, for $\mu, \nu \in L(\mathbb{R})$, set

$$\rho(\mu, \nu) = \sup\{ \mathcal{H}(\mu_\alpha, \nu_\alpha) : \alpha \in (0, 1] \}.$$ 

Using this definition, $(L(\mathbb{R}), \rho)$ becomes a complete metric space.

To define a linear structure on $L(\mathbb{R})$, recall that every fuzzy number is completely determined by its $\alpha$-cuts. Hence, for any $\mu, \nu \in L(\mathbb{R})$, $\alpha \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}$, set

$$\begin{align*}
(\mu + \nu)_\alpha &= \mu_\alpha + \nu_\alpha, \\
(\lambda \mu)_\alpha &= \lambda \mu_\alpha
\end{align*}$$

(here, $V + Z = \{ v + z : v \in V, z \in Z \}; \lambda V = \{ \lambda v : v \in V \}$).

Finally, we note that $(L(\mathbb{R}), +)$ is not a group, but only a semigroup (see also [5]), in fact let $\mu \in L(\mathbb{R})$ be defined by the formula:

$$\mu(x) = \begin{cases}
    x, & \text{if } x \in [0, 1]; \\
    2 - x, & \text{if } x \in [1, 2]; \\
    0, & \text{otherwise}.
\end{cases}$$
Then $-\mu = (-1) \cdot \mu$ is given by

$$-\mu(x) = \begin{cases} 
-x, & \text{if } x \in [-1, 0]; \\
2 + x, & \text{if } x \in [-2, -1]; \\
0, & \text{otherwise.}
\end{cases}$$

Note that $\mu(x) + (-\mu(x))$ is not the zero element $0 \equiv \chi_{\{0\}}(x)$, but

$$\mu(x) + (-\mu(x)) = \begin{cases} 
1 - \frac{x}{2}, & \text{if } x \in [0, 2]; \\
1 + \frac{x}{2}, & \text{if } x \in [-2, 0]; \\
0, & \text{otherwise.}
\end{cases}$$

On the other hand the subset $R_0 \subset L(\mathbb{R})$ consisting of all functions $\chi_{\{a\}}, a \in \mathbb{R}$, is group isomorphic to the commutative group $(\mathbb{R}, +)$.

### 3 The construction of the integral.

From now on we denote by capital letters the elements of the extended real line and by small letters the real numbers. Let $[A, B]$ be a (possibly unbounded) interval of the extended real line, and $\mathcal{F}$ be the family of all closed convex subsets. By partition (or $k$-partition ) of a set $W \in \mathcal{F}$ we denote a finite collection

$$\Pi = \{(\xi_1; F_{1,1}, \ldots, F_{1,k}), \ldots, (\xi_q; F_{q,1}, \ldots, F_{q,k})\} = \{(\xi_1; E_1), \ldots, (\xi_q; E_q)\} \quad (1)$$

such that

(i) $F_{i,j} \in \mathcal{F}$ for all $i = 1, \ldots, q$ and $j = 1, \ldots, k$;

(ii) $\bigcup_{j=1}^{k} F_{i,j} = E_i$ for all $i = 1, \ldots, q$;

(iii) $\bigcup_{i=1}^{q} E_i = W$;

(iv) $\xi_i \in E_i (i = 1, \ldots, q)$;

(v) the $F_{i,j}$’s are pairwise non-overlapping;

(vi) $\sup F_{i,j} = \inf F_{i,j+1}$ whenever $i = 1, \ldots, q$ and $j = 1, \ldots, k - 1$.

A finite collection $\Pi$ as in (1), satisfying conditions (i), (ii), (iv), (v) and (vi), but not necessarily (iii), is said to be a decomposition (or $k$-decomposition ) of $W$. 

Definitions 3.1. • A gauge is a map \( \gamma \) defined in \([A, B]\) and taking values in the set of all open intervals in \( \tilde{\mathbb{R}} \), such that \( \xi \in \gamma(\xi) \) for every \( \xi \in [A, B] \) and \( \gamma(\xi) \) is a bounded open interval (with respect to the topology of \([A, B]\)) for every \( \xi \in \mathbb{R} \cap [A, B] \).

• Given a gauge \( \gamma \), a \( k \)-decomposition of \([A, B]\) of the type
  \[
  \Pi = \{ (\xi_i; E_i), i = 1, \ldots, q \}
  \]
  is said to be \( \gamma \)-fine if \( \xi_i \in E_i \subset \gamma(\xi_i) \) for all \( i = 1, \ldots, q \). Observe that for any gauge \( \gamma \) there always exists a \( \gamma \)-fine \( k \)-partition (see also \([8, 11]\)).

• Given \([a, b] \subset \mathbb{R} \) and a map \( \delta : [a, b] \to \mathbb{R}^+ \), a partition \( \Pi \) of \([a, b]\) as in (2) is said to be \( \delta \)-fine if \( \xi_i \in E_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)) \) for all \( i = 1, \ldots, q \). In any case we note that, if \( E_i \) is an unbounded interval, then the element \( \xi_i \) associated with \( E_i \) is necessarily \( +\infty \) or \( -\infty \): otherwise \( \gamma(\xi_i) \) should be a bounded interval and contain an unbounded interval, a contradiction.

From now on, we assume that \( X \) is an invariant metric semigroup. Given any \( k \)-decomposition \( \Pi \) as in (1) and a function \( U : [A, B] \times \mathcal{F}_k \to X \), we call \textit{Riemann sum} of \( U \) (and we write \( \sum \Pi U \)) the expression
  \[
  \sum_{i=1}^{q} U(\xi_i; F_{i,1}, \ldots, F_{i,k}).
  \]

We now introduce the \( GH_k \)-integral for \( X \)-valued functions defined on \([A, B]\times \mathcal{F}_k\). We will show that this concept can be formulated equivalently both with gauges and with positive maps \( \delta \).

**Definition 3.2.** We say that a function \( U : [A, B] \times \mathcal{F}_k \to X \) is \( GH_k \)-integrable on \([A, B]\) if there exists \( I \in X \) such that for all \( \varepsilon > 0 \) there correspond a function \( \delta : [A, B] \to \mathbb{R}^+ \) and a positive real number \( P \) such that
  \[
  \rho \left( \frac{1}{I}, \frac{\sum \Pi U}{\sum \Pi I} \right) \leq \varepsilon
  \]
whenever \( \Pi \) is a \( \delta \)-fine \( k \)-partition of any bounded interval \([a, b]\) with \([a, b] \supset [A, B] \cap [-P, P]\). In this case we say that \( I \) is the \( GH_k \)-integral of \( U \), and we denote the element \( I \) by the symbol \((GH_k) \int_A^B U\), writing usually \( U \in GH_k[A, B]\).

Analogously it is possible to define the integral \((GH_k) \int_c^d U\) for each subinterval \([c, d] \subset [A, B]\).

**Remark 3.3.** We note that the \( GH_k \)-integral is well-defined, that is there exists at most one element \( I \), satisfying condition (4) (see also \([5]\)).
We now give the following characterization of $GH_k$ integrability.

**Theorem 3.4.** A function $U : [A, B] \times F^k \to X$ is $GH_k$ integrable if and only if there is $J \in X$ such that for all $\varepsilon > 0$ there exists a gauge $\gamma$ such that
\[
\rho \left( J, \sum_{\Pi} U \right) \leq \varepsilon
\] (5)
whenever $\Pi$ is a $\gamma$-fine partition of $[A, B]$, and in this case we have $\int_{A}^{B} f = J$.

**Proof:** See also [3], Theorem 3.3., and [5]. □

### 4 Elementary properties of the $GH_k$ integral

The proof of the following proposition is similar to the corresponding one in [5].

**Proposition 4.1.** If $U_1, U_2 \in GH_k[A, B]$ and $c_1, c_2 \in \mathbb{R}$, then $c_1 U_1 + c_2 U_2 \in GH_k[A, B]$, and
\[
(GH_k) \int_{A}^{B} (c_1 U_1 + c_2 U_2) = c_1 (GH_k) \int_{A}^{B} U_1 + c_2 (GH_k) \int_{A}^{B} U_2.
\]
(Here we intend by $-U$ the entity $( -1 ) \cdot U$)

**Theorem 4.2.** A map $U : [A, B] \times F^k \to X$ is $GH_k$ integrable if and only if for all $\varepsilon > 0$ there exists a gauge $\gamma = \gamma(\varepsilon)$ on $[A, B]$ such that
\[
\rho \left( \sum_{\Pi} U, \sum_{\Pi'} U \right) \leq \varepsilon
\] (6)
whenever $\Pi, \Pi'$ are $\gamma$-fine $k$-partitions of $[A, B]$.

**Proof:** We follow the lines of the proof of Proposition 3.5 of [5].

The necessary part is straightforward.

We now turn to the sufficient part. Let $U$ satisfy (6), and set $\varepsilon = 1/n$, with $n \in \mathbb{N}$. Then for all $n$ there exists a gauge $\gamma_n$ on $[A, B]$ such that
\[
\rho \left( \sum_{\Pi_1} U, \sum_{\Pi_2} U \right) \leq \frac{1}{n}
\]
whenever $\Pi_1, \Pi_2$ are $\gamma_n$-fine partitions of $[A, B]$. Put $\eta_n = \gamma_1 \cap \gamma_2 \cap \ldots \cap \gamma_n$ for all $n \in \mathbb{N}$, and set
\[
A_n = \{ x \in X : \exists \eta_n \text{-fine partition } \Pi_1 : x = \sum_{\Pi_1} U \}, \quad n \in \mathbb{N}.
\]
If $x, y \in A_n$, then $\rho(x, y) \leq 1/n$, and hence
\[
\text{diam } \overline{A_n} = \text{diam } A_n \leq \frac{1}{n}.
\]
Since $\eta_{n+1} \subset \eta_n$, we obtain $A_{n+1} \subset \overline{A_n}$. Since $X$ is complete, there exists exactly one element $I \in \cap_{n=1}^{\infty} \overline{A_n}$.

Pick arbitrarily $\varepsilon > 0$, and choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. If $\Pi$ is any $\eta_n$-fine partition, then
\[
\sum_{\Pi} U \in A_n.
\]
Since $I \in \overline{A_n}$, we obtain
\[
\rho(I, \sum_{\Pi} U) \leq \frac{1}{n} < \varepsilon.
\]
Therefore $U$ is $GH_k$ integrable on $[A, B]$ and $I = \int_A^B U$. □

We now investigate $GH_k$ integrability on subintervals, by proceeding similarly as in [8].

**Proposition 4.3.** If $U \in GH_k[A, B]$, then $U \in GH_k[c, d]$ for each $[c, d] \subset [A, B]$, and
\[
(GH_k) \int_A^B U = (GH_k) \int_A^c U + (GH_k) \int_c^B U
\]
whenever $A < c < B$.

**Proof:** We begin with the first statement. Without loss of generality, we can assume that $[c, d] = [A, d]$, with $A < d < B$. Let $\gamma$ be any gauge on $[A, B]$, pick any two $\gamma$-fine $k$-partitions $\Pi_1, \Pi_2$ of $[A, d]$, and let $\Pi'$ be a $\gamma$-fine $k$-partition of $[d, B]$. Such a partition does exist, by virtue of the Cousin lemma. Then, for $j = 1, 2$, $\Pi_j' := \Pi' \cup \Pi_j$ is a $\gamma$-fine partition of $[A, B]$. Since
\[
\rho \left( \sum_{\Pi_1} U, \sum_{\Pi_2} U \right) = \rho \left( \sum_{\Pi_1'} U, \sum_{\Pi_2'} U \right),
\]
then the assertion follows from the Cauchy criterion

We now turn to the last part. For every $\varepsilon > 0$ there exists a gauge $\gamma$ such that for each $\gamma$-fine $k$-partition $\Pi_1$ of $[A, c]$ and $\Pi_2$ of $[c, B]$ we get
\[
\rho \left( \sum_{\Pi_1} U, (GH_k) \int_A^c U \right) \leq \varepsilon, \quad \rho \left( \sum_{\Pi_2} U, (GH_k) \int_c^B U \right) \leq \varepsilon.
\]
Hence, if $\Pi = \Pi_1 \cup \Pi_2$, we have also
\[
\rho \left( \sum_{\Pi} U, (GH_k) \int_A^B U \right) \leq \varepsilon.
\]
We obtain:

\[
0 \leq \rho \left( (GH_k) \int_A^c U + (GH_k) \int_c^B U \right)
\]

\[
\leq \rho \left( \sum_{I_1} U, (GH_k) \int_A^c U \right) + \rho \left( \sum_{I_2} U, (GH_k) \int_c^B U \right) + \rho \left( \sum U, (GH_k) \int_A^B U \right)
\]

\[
\leq 3 \varepsilon.
\]

By arbitrariness of \( \varepsilon \in \mathbb{R}^+ \) we get that

\[
(GH_k) \int_A^B U = (GH_k) \int_c^c U + (GH_k) \int_c^B U.
\]

This completes the proof. \( \square \)

In order to establish a converse of the previous result, we now introduce the following property.

**Definition 4.4.** Let \( U : [A, B] \times \mathcal{F}_k \to X \) and fix a point \( x_0 \in [A, B] \). We say that \( U \) satisfies condition \( \text{[H1] at } x_0 \) if for all \( \varepsilon > 0 \) there exists a positive real number \( \eta = \eta(\varepsilon; x_0) \) such that

\[
\rho \left( U(x_0; [w_0^{(0)}, w_1^{(0)}], \ldots, [w_k^{(0)}, w_k^{(0)}]); U(x_0; [w_0^{(1)}, w_1^{(1)}], \ldots, [w_k^{(1)}, w_1^{(1)}]) \right) + \rho \left( U(x_0; [w_0^{(2)}, w_1^{(2)}], \ldots, [w_k^{(2)}, w_k^{(2)}]) \right) \leq \varepsilon
\]

whenever

\[
\bigcup_{l=0}^2 \left( \bigcup_{i=1}^k \{w_i^{(l)}\} \right) \subset [x_0 - \eta, x_0 + \eta] \text{ and } w_0^{(0)} = w_0^{(1)}, w_k^{(0)} = w_k^{(2)}.
\]

Note that \( \text{[H1]} \) is a kind of "quasi-additivity" of the set function \( U \). In many cases, when \( X = \mathbb{R} \), \( U \) is defined by means of suitable "differences" (for example, \( U(t; [u, v]) = V(t; v) - V(t; u) \) when \( k = 1 \) or

\[
U(t; [w_0, w_1], \ldots, [w_{k-1}, w_k]) = V(t; w_1, \ldots, w_k) - V(t; w_0, \ldots, w_{k-1})
\]

for \( k \geq 2 \); then, if \( k = 1 \), property \( \text{[H1]} \) is automatically satisfied (see also [24], Theorem 1.11, pp. 10-12); while for \( k \geq 2 \) it is implied by the condition of "existence of the iterated limit \( J \)" used by A. G. Das and S. Kundu (see [8], Definition 2.9., p. 69).

We now prove the following result on additivity.

**Theorem 4.5.** Let \( U : [A, B] \times \mathcal{F}_k \to X \) satisfy condition \( \text{[H1]} \) at \( c \in ]A, B[ \). If \( U \in GH_k[A, c] \) and \( U \in GH_k[c, B] \), then \( U \in GH_k[A, B] \) and

\[
(GH_k) \int_A^B U = (GH_k) \int_A^c U + (GH_k) \int_c^B U.
\]
**Proof:** By hypothesis, for every \( \varepsilon > 0 \) there exist a function \( \delta^*: [a, b] \to \mathbb{R}^+ \) and a positive real number \( P \) (without loss of generality, greater than \(|c|\)) with the following property: for all \( \delta^* \)-fine \( k \)-partitions \( \Pi_1 \) of any bounded interval \([a_1, b_1] \subset [a, c] \), \([a_1, b_1] \supset [a, c] \cap [-P, P] \) and \( \Pi_2 \) of every bounded interval \([a_2, b_2] \subset [c, B] \), \([a_2, b_2] \supset [c, B] \cap [-P, P] \) we get

\[
\rho \left( \sum_{\Pi_1} U, (GH_k) \int_A^c U \right) \leq \varepsilon, \quad \rho \left( \sum_{\Pi_2} U, (GH_k) \int_c^B U \right) \leq \varepsilon.
\]

Let \( \eta = \eta(\varepsilon;c) \) be related to condition \( \text{H1} \) at \( c \), and set \( \overline{\delta}(x) = \min\{\delta^*(x), |x - c|\} \) if \( x \in [A, B] \setminus \{c\} \), \( \delta(c) = \min\{\delta^*(c), \eta\} \). Pick now any bounded interval \([a, b] \subset [A, B] \), \([a, b] \supset [A, B] \cap [-P, P] \), and any \( \delta^* \)-fine \( k \)-partition

\[
\Pi = \{(\xi_i; F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, q \}
\]

of \([a, b] \). There exists \( m \) with \( 1 \leq m \leq q \), such that \( c = \xi_m \) and \( \bigcup_{j=1}^k F_{i,j} \) contains \( c \) if and only if \( i = m \) (see also [8, 24]). We get:

\[
\sum_{\Pi} U = \sum_{i=1}^{m-1} U(\xi_i; F_{i,1}, \ldots, F_{i,k}) + U(c; F_{m,1}, \ldots, F_{m,k}) + \sum_{i=m+1}^q U(\xi_i; F_{i,1}, \ldots, F_{i,k}).
\]

Consider now the points

\[
c - \delta(c) < x_{m-1,k} = y_{m,0} < \ldots < y_{m,k} = c = \hat{z}_{m,0} < \ldots < \hat{z}_{m,k} = x_{m+1,0} < c + \delta(c).
\]

The parts of the partition \( \Pi \) for \( i = 1, \ldots, m-1 \) (\( i = m+1, \ldots, q \)) and the single family \( \{[c; y_{m,0}, y_{m,1}], \ldots, [y_{m,k-1}, y_{m,k}]\} \cap \{[c; \hat{z}_{m,0}, \hat{z}_{m,1}], \ldots, [\hat{z}_{m,k-1}, \hat{z}_{m,k}]\}) \) form a \( \delta^* \)-fine \( k \)-partition \( \Pi_1 \) \( \Pi_2 \) of \([a, c] \) \([c, b] \). So, we have:

\[
\rho \left( \sum_{\Pi} U, (GH_k) \int_A^c U + (GH_k) \int_c^B U \right) \leq \rho \left( \sum_{\Pi_1} U, (GH_k) \int_A^c U \right) + \rho \left( \sum_{\Pi_2} U, (GH_k) \int_c^B U \right) + \rho \left( \sum_{\Pi_1} U, \sum_{\Pi_1} U + \sum_{\Pi_2} U \right)
\]

\[
\leq 2\varepsilon + \rho(U; F_{m,1}, \ldots, F_{m,k}) + U(c; [y_{m,0}, y_{m,1}], \ldots, [y_{m,k-1}, y_{m,k}]) + U(c; [\hat{z}_{m,0}, \hat{z}_{m,1}], \ldots, [\hat{z}_{m,k-1}, \hat{z}_{m,k}]) \leq 3\varepsilon.
\]

From this it follows that \( U \in GH_k [A, B] \) and

\[
(GH_k) \int_A^c U = (GH_k) \int_A^c U + (GH_k) \int_c^B U.
\]

This concludes the proof. \( \square \)
5 Convergence theorems

We begin with a version of the Saks-Henstock lemma (see also [5], Proposition 4.1). Here, the symbol $| \cdot |$ denotes the Lebesgue measure.

**Lemma 5.1.** Let $U : [A, B] \times \mathcal{F}^k \to X$ be $GH_k$ integrable on $[A, B]$. Then for every $\varepsilon > 0$ there exists a gauge $\gamma$ on $[A, B]$ such that, for every $\gamma$-fine $k$-decomposition of $[A, B]$

$$\Pi = \{(t_i; F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, m\} = \{(t_i; E_i), i = 1, \ldots, m\}, \quad (7)$$

where $\bigcup_{j=1}^{k} F_{i,j} = E_i$, $i = 1, \ldots, m$; we have

$$\rho \left( \sum_{i=1, \ldots, m, |F_i| < +\infty} U(t_i; F_{i,1}, \ldots, F_{i,k}), \sum_{i=1}^{m} (GH_k) \int_{E_i} U \right) \leq \varepsilon.$$

**Proof:** (see also [5]) Choose arbitrarily $\varepsilon > 0$, and let $\gamma$ be a gauge on $[A, B]$ existing in correspondence with $\varepsilon$, according to Theorem 3.4. Fix arbitrarily any $\gamma$-fine $k$-decomposition $\Pi$ of $[A, B]$ as in (7), and let $\text{int } E_i$ be the interior of $E_i$, $i = 1, \ldots, m$. Since the $E_i$’s are non-overlapping, the set $[A, B] \setminus \bigcup_{i=1}^{m} (\text{int } E_i)$ is empty or is the union of non-overlapping (possibly bounded or not) intervals $B_1, \ldots, B_p$. Let $\eta > 0$. Since $U$ is $GH_k$ integrable on each $B_j$, for each $j = 1, \ldots, p$ there exists a gauge $\gamma_j$ on $B_j$ such that $\gamma_j(x) \subset \gamma(x)$ for all $x \in B_j$ and

$$\rho \left( \sum_{\Pi_j} U, (GH_k) \int_{B_j} U \right) < \frac{\eta}{p + 1}$$

for every $\gamma_j$-fine partition $\Pi_j$ of $B_j$. Let now $\Pi_j$ be such a partition. We observe that

$$\Pi := \{(t_i; F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, m\} \cup \bigcup_{j=1}^{p} \Pi_j$$
is a \( \gamma \)-fine partition of \([A, B]\). Then we have:

\[
\rho \left( \sum_{i=1}^{m} U(t_i; F_{i,1}, \ldots, F_{i,k}), \sum_{i=1}^{m} (GH_k) \int_{E_i} U \right)
\]

\[
= \rho \left( \sum_{i=1}^{m} U(t_i; F_{i,1}, \ldots, F_{i,k}), \sum_{j=1}^{p} \sum_{i=1}^{m} U, \sum_{i=1}^{m} (GH_k) \int_{E_i} U + \sum_{j=1}^{p} \sum_{i=1}^{m} U \right)
\]

\[
\leq \rho \left( \sum_{i=1}^{m} U, (GH_k) \int_{A} U \right)
\]

\[
+ \rho \left( \sum_{i=1}^{m} (GH_k) \int_{E_i} U + \sum_{j=1}^{p} (GH_k) \int_{B_j} U, \sum_{i=1}^{m} (GH_k) \int_{E_i} U + \sum_{j=1}^{p} \sum_{i=1}^{m} U \right)
\]

\[
\leq \varepsilon + \rho \left( \sum_{j=1}^{p} (GH_k) \int_{B_j} U, \sum_{i=1}^{m} \sum_{j=1}^{p} U \right)
\]

\[
\leq \varepsilon + \sum_{j=1}^{p} \rho \left( (GH_k) \int_{B_j} U, \sum_{i=1}^{m} U \right) < \varepsilon + \sum_{j=1}^{p} \eta < \varepsilon + \eta.
\]

Since the inequality

\[
\rho \left( \sum_{i=1}^{m} U(t_i; F_{i,1}, \ldots, F_{i,k}), \sum_{i=1}^{m} (GH_k) \int_{E_i} U \right) < \varepsilon + \eta
\]

holds for any \( \eta > 0 \), then the assertion follows by arbitrariness of \( \eta \). \( \square \)

We now prove a version of a Hake's type theorem, which is an extension of the Cauchy theorem. To do this, let \( U : [A, B] \times \mathcal{F}^k \to X \) belong to \( GH_k[A, c] \) for all \( c \in [A, B] \), fix \( I \in X \) and let us introduce the following condition:

- **H2)** for every \( \varepsilon > 0 \) there exists a left neighborhood \( U \) of \( B \) such that

\[
\rho \left( I, (GH_k) \int_{A} U + U(B; F_1, \ldots, F_k) \right) \leq \varepsilon
\]

whenever \( F_1, \ldots, F_k \in \mathcal{F} \) are pairwise non-overlapping and such that \( U \ni c \) and \( F_j = \inf F_{j+1}, j = 1, \ldots, k-1, \) and \( \sup F_k = B \).

In the literature several situations are considered, when, in the Riemann sums, only the terms where the involved intervals are bounded are taken: this can be done simply by postulating it or by requiring the condition

\[
U(\pm \infty; \Lambda_1, \ldots, \Lambda_k) = 0
\]

(8)

for every choice of \( \Lambda_j \in \mathcal{F}, j = 1, \ldots, k. \)
Observe that, when $B = +\infty$ and we require (8), $H2)$ can be automatically replaced by the simpler condition of existence in $X$ of the limit

$$
\lim_{c \to B^-} (GH_k) \int_{A}^{c} U.
$$

Finally, we note that, when $X = \mathbb{R}$, property $H2)$ is implied by the two conditions of existence in $\mathbb{R}$ of the limit as in (9) and of "existence of the iterated limit (from the left) $J^-$" used by A. G. Das and S. Kundu (see [8]) when $k \geq 2$.

For $k = 1$, $H2)$ is equivalent to the existence in $\mathbb{R}$ of the limit in [24], formula (1.11), p. 15.

**Theorem 5.2.** Let $A \in \mathbb{R}^+$, $U : [A, B] \times J^k \to X$ be such that $U \in GH_k[A, c]$ for every $c \in [A, B]$, and suppose that there is an element $I \in X$ such that $H2)$ holds.

Then $U \in GH_k[A, B]$ and $(GH_k) \int_{A}^{B} U = I$.

Moreover, if $U \in GH_k[A, B]$, then

$$
\lim_{c \to B^-} (GH_k) \int_{A}^{c} U = (GH_k) \int_{A}^{B} U
$$

(last result is independent on $H2)$).

**Proof:** Let $(c_p)_p$ be a strictly increasing sequence in $[A, b]$ with $c_p \uparrow B$ and $c_0 = A$. For every $p \in \mathbb{N}$ and $\varepsilon > 0$ there exists a gauge $\gamma_p : [A, c_p] \to \mathbb{R}^+$, such that

$$
\rho \left( \sum_{I_p} U, (GH_k) \int_{A}^{c_p} U \right) \leq \frac{\varepsilon}{2^p}
$$

whenever $I_p$ is any $\gamma_p$-fine $k$-partition of $[A, c_p]$.

For every $\xi \in [A, B]$ there exists exactly one $p = p(\xi) \in \mathbb{N}$ such that $\xi \in [c_{p(\xi)} - 1, c_{p(\xi)}]$. Given $\xi \in [A, B]$, choose $\hat{\gamma}(\xi)$ such that $\hat{\gamma}(\xi) \subset \gamma_{p(\xi)}(\xi)$ and $\hat{\gamma}(\xi) \cap [A, c_{p(\xi)}(\xi))]$. Let $c \in [A, B]$ and

$$
\hat{I} := \{(\xi_i; F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, n\} = \{(\xi_i; E_i), i = 1, \ldots, n\},
$$

with $\bigcup_{j=1}^{k} F_{i,j} = E_i$, $i = 1, \ldots, n$, be a $\hat{\gamma}$-fine $k$-partition of $[A, c]$. For every $i = 1, \ldots, n$ we get:

$$
E_i \subset \hat{\gamma}(\xi_i) \subset [A, c_{p(\xi_i)}(\xi_i)].
$$

Furthermore, $E_i \subset \gamma_{p(\xi_i)}(\xi_i)$. For every $p \in \mathbb{N}$, let us indicate by

$$
\sum_{i=1}^{n, p(\xi_i) = p} \rho \left( U(\xi_i; F_{i,1}, \ldots, F_{i,k}), (GH_k) \int_{E_i} U \right)
$$

the sum of those terms of

$$
\sum_{i=1}^{n} \rho \left( U(\xi_i; F_{i,1}, \ldots, F_{i,k}), (GH_k) \int_{E_i} U \right)
$$
for which \( \xi_i \in [c_{p-1}, c_p] \). By Lemma 5.1 we obtain
\[
\rho \left( \sum_{i=1}^{n} U(\xi_i; F_{i,1}, \ldots, F_{i,k}), (GH_k) \int_{E_i} U \right) \leq \frac{\varepsilon}{2^p}
\]
for all \( p \in \mathbb{N} \). Since \( U \in GH_k[A, c] \) for every \( c \in [A, B] \), then by Proposition 4.3 we have
\[
(GH_k) \int_{A}^{c} U = \sum_{i=1}^{n} (GH_k) \int_{E_i} U.
\]
So we get:
\[
\rho \left( \sum_{i=1}^{n} U(\xi_i; F_{i,1}, \ldots, F_{i,k}), (GH_k) \int_{A}^{c} U \right) = \rho \left( \sum_{i=1}^{n} U(\xi_i; F_{i,1}, \ldots, F_{i,k}), \sum_{i=1}^{n} (GH_k) \int_{E_i} U \right) \leq \sum_{i=1}^{n} \rho \left( U(\xi_i; F_{i,1}, \ldots, F_{i,k}), \sum_{i=1}^{n} (GH_k) \int_{E_i} U \right) \leq \sum_{i=1}^{n} \frac{\varepsilon}{2^p} = \varepsilon.
\]

Let \( U \) be related with condition \( H2 \), and pick a gauge \( \gamma \) on \([A, B]\) such that \( \gamma(\xi) \subset \tilde{\gamma}(\xi) \) if \( \xi \in [A, B] \), and \( \gamma(B) \subset U \). Let
\[
\Pi := \{(\xi_i; F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, n\} = \{(\xi_i; E_i), i = 1, \ldots, n\}
\]
be any arbitrary \( \gamma \)-fine \( k \)-partition of \([A, B]\), where \( \bigcup_{j=1}^{k} F_{i,j} = E_i \) and \( E_i = [x_{i-1,k}, x_{i,k}], i = 1, \ldots, n; \) we get \( x_{n,k} = B \) and hence \( \xi_n = B \) (if not, then \( E_n \subset \tilde{\gamma}(\xi_n) \subset [A, c_{p(\xi_n)}] \) and thus \( x_{n,k} < B \), a contradiction). We have, thanks to the condition formulated in the hypothesis and using property \( (vii) \) of the function \( \rho \),
\[
\rho \left( \bigcup_{i=1}^{n} U \right) \leq \rho \left( \sum_{i=1}^{n} U(\xi_i; F_{i,1}, \ldots, F_{i,k}) + U(B; F_{n,1}, \ldots, F_{n,k}) \right) \leq \rho \left( \sum_{i=1}^{n} U(\xi_i; F_{i,1}, \ldots, F_{i,k}), (GH_k) \int_{A}^{x_{n-1,k}} U \right) + \rho \left( I, (GH_k) \int_{A}^{x_{n-1,k}} U + U(B; F_{n,1}, \ldots, F_{n,k}) \right) \leq \rho \left( \sum_{i=1}^{n} U(\xi_i; F_{i,1}, \ldots, F_{i,k}), (GH_k) \int_{A}^{x_{n-1,k}} U \right) + \varepsilon.
\]
As \(x_{n-1,k} < B\) and \(\{(\xi_i; F_i,1,\ldots, F_i,k), i = 1,\ldots, n-1\}\) is a \(\hat{\gamma}\)-fine \(k\)-partition of \([A, x_{n-1,k}]\), we get

\[
\rho \left( \sum_{i=1}^{n-1} U(\xi_i; F_i,1,\ldots, F_i,k), (GH_k) \int_A^{x_{n-1,k}} U \right) \leq \varepsilon,
\]

and hence

\[
\rho \left( I, \sum_{\Pi} U \right) \leq 2\varepsilon.
\]

From this the assertion of the first part of the theorem follows.

We now turn to the last part. Since, by hypothesis, \(U : [A, B] \times \mathcal{F}^k \to X\) is \(GH_k\) integrable on \([A, B]\), then \(U\) is \(GH_k\) integrable on \([A, c]\) for every \(A < c \leq B\). So for all \(\varepsilon > 0\) and \(c \in [A, B]\) there exists \(\delta_1 : [A, c] \to \mathbb{R}^+\) such that for every \(\delta_1\)-fine \(k\)-partition \(\Pi'\) of \([A, c]\) we get:

\[
\rho \left( \sum_{\Pi'} U, (GH_k) \int_A^c U \right) \leq \varepsilon.
\]

Moreover, by \(GH_k\) integrability on \([A, B]\) (see also Definition 3.2), for any \(\varepsilon > 0\) there exist \(\delta : [A, B] \to \mathbb{R}^+\) and \(P \in]A, B]\) such that for every bounded interval \([d_1, d_2] \subset [A, B]\) with \([d_1, d_2] \supset [-P, P]\) and for each \(\delta\)-fine \(k\)-partition \(\Pi\) of \([d_1, d_2]\) we have

\[
\rho \left( \sum_{\Pi} U, (GH_k) \int_A^B U \right) \leq \varepsilon,
\]

Let now \(\varepsilon > 0\), \(c > P\), \(\delta_2(x) := \min\{\delta(x), \delta_1(x)\}, x \in [A, c]\), and \(\Pi\) be any \(\delta_2\)-fine \(k\)-partition of \([A, c]\). Then we get:

\[
\rho \left( (GH_k) \int_A^c U, (GH_k) \int_A^B U \right) \leq \rho \left( \sum_{\Pi} U, (GH_k) \int_A^c U \right) + \rho \left( \sum_{\Pi} U, (GH_k) \int_A^B U \right) \leq 2\varepsilon.
\]

Thus the theorem is completely proved. \(\square\)

**Remark 5.3.** An analogous version of Theorem 5.2 holds, if we consider, in our ”limit operations” and calculus, the point \(A\) from the right instead of the point \(B\) from the left.

This concept will be useful in the sequel.

**Definition 5.4.** A sequence of integrable functions \((U_h : [A, B] \times \mathcal{F}^k \to X)_h\) is said to be *equiintegrable* if for any \(\varepsilon > 0\) there exists a gauge \(\gamma\) on \([A, B]\) such that

\[
\rho \left( \sum_{\Pi} U_h, (GH_k) \int_A^B U_h \right) \leq \varepsilon
\]

for any \(\gamma\)-fine partition \(\Pi\) and every \(h \in \mathbb{N}\).
We now prove the following convergence theorems for the \( \text{GH}_k \) integral in the context of metric semigroups.

**Theorem 5.5.** Let \((U_h)_h\) be an equiintegrable sequence and let
\[
\lim_{h \to +\infty} \rho(U_h(t; \Lambda_1, \ldots, \Lambda_k), U(t; \Lambda_1, \ldots, \Lambda_k)) = 0
\]
for any \( t \in [A, B] \) and uniformly with respect to \( \Lambda_1, \ldots, \Lambda_k \in \mathcal{F} \). Then \( U \) is \( \text{GH}_k \) integrable on \([A, B]\), and
\[
\lim_{h \to +\infty} \rho \left( \left( \text{GH}_k \right)^{\int_A^B U_h}, \left( \text{GH}_k \right)^{\int_A^B U} \right) = 0.
\]

**Proof:** First of all, we observe that for each \( \varepsilon > 0 \), there exist: a non-negative function \( E : [A, B] \times F^k \to \mathbb{R} \), strictly positive on \(([A, B] \cap \mathbb{R}) \times F^k\), \( \text{GH}_k \) integrable in \([A, B]\), with
\[
(\text{GH}_k)^{\int_A^B E} \leq \frac{\varepsilon}{2}
\]
(for example,
\[E(t; \Lambda_1, \ldots, \Lambda_k) = \sum_{j=1}^k \frac{|\Lambda_j| \varepsilon}{2\pi(1 + t^2)}, \quad t \in [A, B],\]
with the convention \( E(\pm \infty; \Lambda_1, \ldots, \Lambda_k) = 0 \) for every choice of \( \Lambda_j \in F, j = 1, \ldots, k \); a gauge \( \gamma_0 \) on \([A, B]\), such that
\[
\sum_{i=1, \ldots, n, |I_i| < +\infty} E(t_i; F_{i,1}, \ldots, F_{i,k}) \leq \varepsilon \tag{11}
\]
for each \( \gamma_0 \)-fine partition \( \Pi \) of \([A, B]\),
\[
\Pi := \{(t_i; F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, n\} = \{(t_i; I_i), i = 1, \ldots, n\},
\]
with \( \bigcup_{j=1}^k F_{i,j} = I_i, i = 1, \ldots, n. \)

Let now \( \varepsilon > 0, \gamma \) be as in Definition 5.4, \( \hat{\gamma} = \gamma \cap \gamma_0 \), and
\[
\Pi := \{(t_i; F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, n\} = \{(t_i; I_i), i = 1, \ldots, n\},
\]
be any \( \hat{\gamma} \)-fine \( k \)-partition of \([A, B]\), where \( \bigcup_{j=1}^k F_{i,j} = I_i, i = 1, \ldots, n. \) Then for each \( i = 1, \ldots, n \) there exists a positive integer \( h_i \) such that
\[
\rho(U_{h_i}(t_i; F_{i,1}, \ldots, F_{i,k}), U(t_i; F_{i,1}, \ldots, F_{i,k})) \leq E(t_i; F_{i,1}, \ldots, F_{i,k}) \tag{12}
\]
whenever \( h \geq h_i \). Pick now \( h \geq \max_{i=1,\ldots,n} h_i \). From (11) and (12) we have:

\[
\rho \left( \sum_{\Pi} U_h, \sum_{\Pi} U \right)
= \rho \left( \sum_{i=1,\ldots,n, |I_i| < +\infty} U_h(t_i; F_{i,1}, \ldots, F_{i,k}), \sum_{i=1,\ldots,n, |I_i| < +\infty} U(t_i; F_{i,1}, \ldots, F_{i,k}) \right)
\leq \sum_{i=1,\ldots,n, |I_i| < +\infty} \rho(U_h(t_i; F_{i,1}, \ldots, F_{i,k}), U(t_i; F_{i,1}, \ldots, F_{i,k}))
\leq \sum_{i=1,\ldots,n, |I_i| < +\infty} \varepsilon(t_i; F_{i,1}, \ldots, F_{i,k}) \leq \varepsilon.
\]

It follows that

\[
\lim_{h \rightarrow +\infty} \rho \left( \sum_{\Pi} U_h, \sum_{\Pi} U \right) = 0.
\]

Now we get:

\[
\rho \left( \sum_{\Pi} U, (GH_k) \int_A^B U_h \right) \leq \rho \left( \sum_{\Pi} U, \sum_{\Pi} U_h \right) + \rho \left( \sum_{\Pi} U_h, (GH_k) \int_A^B U_h \right) \leq 2\varepsilon.
\]

Choose now arbitrarily two \( \mathfrak{F} \)-fine partitions \( \Pi \) and \( \Pi' \) of \([A, B] \), and let \( h^* = \max\{\max_i h_i, \max_j h'_j\} \), where the integers \( h_i, h'_j \) associated to \( \Pi \) and \( \Pi' \) respectively have the same role as the \( h'_i \)'s in (12). We get:

\[
\rho \left( \sum_{\Pi} U, \sum_{\Pi'} U \right) \leq \rho \left( \sum_{\Pi} U, (GH_k) \int_A^B U_{h^*} \right) \leq 4\varepsilon.
\]

Integrability of \( U \) on \([A, B] \) follows from (13) and the Cauchy criterion 4.2.

Finally, to every \( \varepsilon > 0 \) there corresponds a gauge \( \mathfrak{G} \) on \([A, B] \) such that for any \( \mathfrak{G} \)-fine \( k \)-partition \( \Pi \) there exists \( h \in \mathbb{N} \) with

\[
\rho \left( (GH_k) \int_A^B U_h, (GH_k) \int_A^B U \right) \leq \rho \left( (GH_k) \int_A^B U_h, \sum_{\Pi} U_h \right)
+ \rho \left( \sum_{\Pi} U_h, \sum_{\Pi} U \right) + \rho \left( \sum_{\Pi} U, (GH_k) \int_A^B U \right) \leq 3\varepsilon
\]

for all \( h \geq \overline{h} \). This implies that

\[
\lim_{h \rightarrow +\infty} \rho \left( (GH_k) \int_A^B U_h, (GH_k) \int_A^B U \right) = 0. \quad \square
\]
The next step is to prove a version of the convergence theorem with respect to the "uniform convergence". To this aim we introduce the following concept.

**Definition 5.6.** Given a sequence of functions \( (U_n : [A, B] \times F^k \to X)_{n \in \mathbb{N} \cup \{0\}} \), we say that the \( U_n \)'s, \( n \geq 1 \), variationally uniformly converge to \( U_0 \) if to every \( \varepsilon > 0 \) an integer \( n_0 \) can be found, such that

\[
\rho \left( \sum_{i=1}^{q} U_n(t_i; F_{i,1}, \ldots, F_{i,k}), \sum_{i=1}^{q} U_0(t_i; F_{i,1}, \ldots, F_{i,k}) \right) \leq \varepsilon
\]

for every \( n \geq n_0 \) and any \( k \)-partition \( \Pi = \{(t_i, I_i), i = 1, \ldots, q\} \) of \([A, B]\), where \( \bigcup_{j=1}^{k} F_{i,j} = I_i, i = 1, \ldots, q \).

Observe that, if \( k = 1 \) and \( U_n(t; [u, v]) = [g(v) - g(u)] \cdot f_n(t), \quad n \in \mathbb{N} \cup \{0\} \),

where \( g : [A, B] \to \mathbb{R} \) is of bounded variation and the sequence \( (f_n : [A, B] \to X)_{n} \) is uniformly convergent to \( f_0 \) on \([A, B]\), then the \( U_n \)'s variationally uniformly converge to \( U_0 \). In this case, under the hypothesis of uniform convergence of \((f_n)_{n}\) to \( f_0 \), if the \( f_n \)'s, \( n \geq 1 \), are Henstock-Stieltjes integrable with respect to \( g \), then \( f_0 \) is too, and we get the exchange of limits under the sign of integral.

An example in which this happens is when we take \( X = L(\mathbb{R}) \) (i. e. the set of all fuzzy numbers), and define \( f_n : [0, 1] \to X \) by setting \( f_n(x) = \chi_{[0,1]} \cap [x-1/n, x+1/n], \quad n \in \mathbb{N} \), then the sequence \((f_n)_{n}\) is uniformly convergent to the "identity" function (in the sense that the generic element \( x \in [0, 1] \) is identified with the element \( \chi_{\{x\}} \)).

**Theorem 5.7.** Let \( (U_n : [A, B] \times F^k \to X)_{n} \) be a sequence of functions, \( GH_k \)-integrable on \([A, B]\) and variationally uniformly convergent to a map \( U \).

Then \( U \) is \( GH_k \)-integrable on \([A, B]\) and

\[
\lim_{n \to +\infty} \rho \left( \left( GH_k \right) \int_{A}^{B} U_n, \left( GH_k \right) \int_{A}^{B} U \right) = 0.
\]

**Proof:** Let \( \varepsilon > 0 \), and take \( n_0 = n_0(\varepsilon) \) according to variationally uniform convergence. Then

\[
\rho \left( \sum_{\Pi_1} U, \sum_{\Pi_2} U \right) \leq \rho \left( \sum_{\Pi_1} U, \sum_{\Pi_1} U_{n_0} \right) + \rho \left( \sum_{\Pi_1} U_{n_0}, \sum_{\Pi_2} U \right) \leq 2\varepsilon + \rho \left( \sum_{\Pi_1} U_{n_0}, \sum_{\Pi_2} U_{n_0} \right)
\]
for any two partitions $\Pi_1, \Pi_2$ of $[A, B]$. Since $U_{n_0}$ is $GH_k$ integrable on $[A, B]$, then there is a map $\delta = \delta_{n_0} : [A, B] \to \mathbb{R}^+$, such that, for any two $\delta$-fine $k$-partitions $\Pi_1, \Pi_2$ of $[A, B]$,

$$\rho \left( \sum_{\Pi_1} U_{n_0}, \sum_{\Pi_2} U_{n_0} \right) \leq \varepsilon,$$

and hence

$$\rho \left( \sum_{\Pi_1} U, \sum_{\Pi_2} U \right) \leq 3\varepsilon.$$

Thus $U$ is $GH_k$ integrable on $[A, B]$, by virtue of the Cauchy criterion 4.2 So there exists a map $\delta' : [A, B] \to \mathbb{R}^+$ such that

$$\rho \left( \sum_{\Pi} U, \int_{A}^{B} (GH_k) U \right) \leq \varepsilon$$

for each $\delta'$-fine partition $\Pi$ of $[A, B]$. Fix $n \geq n_0$ and choose $\kappa_n : [A, B] \to \mathbb{R}^+$ such that

$$\rho \left( \sum_{\Pi} U_n, \int_{A}^{B} (GH_k) U_n \right) \leq \varepsilon$$

whenever $\Pi$ is a $\kappa_n$-fine partition of $[A, B]$. Put $\delta_n = \min\{\delta', \kappa_n\}$: for any $\delta_n$-fine $k$-partition $\Pi$ of $[A, B]$ we obtain

$$\rho \left( \int_{A}^{B} U_n, (GH_k) \int_{A}^{B} U_n \right) \leq \rho \left( \int_{A}^{B} U_n, \sum_{\Pi} U \right) + \rho \left( \int_{A}^{B} U_n, \sum_{\Pi} U_n \right) + \rho \left( \sum_{\Pi} U_n, \int_{A}^{B} (GH_k) U_n \right) \leq 3\varepsilon,$$

and thus the last part of the assertion. □

References


