The contraction principle in Riesz spaces and applications to differential and stochastic equations

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Abstract

Some versions of the contraction principle for Riesz space-valued functions defined in order intervals of suitable Riesz spaces are proved, and some applications to functional, differential and stochastic equations are given.

1 Introduction

In [2] and [3] we introduced the fundamental principles of Integral and Differential Calculus for Riesz space-valued functions, defined in suitable order intervals of Riesz spaces, and pointed out the structural analogies and differences between our approach and the classical setting. In that context we outlined the theory of series and proved some basic theorems of calculus, involving e.g. term-by-term differentiation: this clearly helps in working with elementary functions. Finally, we investigated some types of differential and functional equations, which can be solved by means of our theory, and gave also applications to reconstruction of signals.

In this paper we continue the study of calculus in Riesz spaces, proving some versions of the contraction principle and then stating existence and uniqueness theorems for differential equations. Also some Cauchy-Kovalevskaja-type theorems are proved, for equations of the first order. We then give applications in the solution of several functional, differential and stochastic equations.

2 Preliminaries

A Riesz space R is said to be *Dedekind complete* if every nonempty subset $A \subset R$, bounded from above, has supremum in R. If R is a Riesz space, then R^+ is the set of those elements $r \in R$ such that $r \geq 0$ and $r \neq 0$. Moreover, if $r_1, r_2 \in R$, we say that $r_2 > r_1$ if $r_2 - r_1 \in R^+$.

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Given a Dedekind complete Riesz space R and a bounded sequence $(p_n)_n$ in R, we set:

$$\limsup_n p_n = \inf_{n\in\mathbb{N}} [\sup_{m\geq n} p_m]; \quad \liminf_n p_n = \sup_{n\in\mathbb{N}} [\inf_{m\geq n} p_m];$$

and we say that $(O)\lim_n p_n = l \in R$ if $\limsup_n p_n = \liminf_n p_n = l$. This is the classical definition of order convergence or (O)-convergence (see also [8, 10]).

From now on, we suppose that R is a commutative *algebra,* that is a Dedekind complete Riesz space endowed with a commutative binary operation $\cdot$, called *product,* satisfying distributive laws with respect to the sum and conditions of compatibility with respect to product for scalars, ordering and infima of monotone decreasing nets of positive elements.

Occasionally, we shall also assume that there exists a *multiplicative* unit in R, which will be denoted by the symbol $\underline{1}$ (when this happens, clearly the unit is unique, by commutativity).

From now on, we always suppose that $[a,b] \subset R$ is an order interval of R, that is $[a,b] = \{x \in R : a \leq x \leq b\}$.

Definition 2.1 A map $f : [a,b] \to R$ is said to be *Lipschitz* if there exists $L \in R^+$ such that $|f(v) - f(u)| \leq L|v-u|$ whenever $u, v \in [a,b]$. Such an element L will be called a *Lipschitz constant.* In case the Lipschitz constant L belongs to some subspace $\mathcal{V}$ of R, we shall emphasize this fact by saying that f is *$\mathcal{V}$-Lipschitz.*

Definition 2.2 We say that a function $f : [a,b] \to R$ is *uniformly continuous* in $[a,b]$ if

$$\inf_{r\in R^+} [\sup\{|f(v) - f(u)| : u, v \in [a,b],\ 0 \leq v - u \leq r\}] = 0.$$

Definition 2.3 We say that a function $f : [a,b] \to R$ is *uniformly differentiable* in $[a,b]$ if there exist a bounded function $f' : [a,b] \to R$ and an increasing family $(p_r)_{r\in R^+}$ such that $\inf_{r\in R^+} p_r = 0$ and

$$|f(v) - f(u) - (v-u)\, f'(x)| \leq (v-u)\, p_r \tag{1}$$

for every $r \in R^+$ and whenever $u, v, x \in [a,b]$, $u \leq x \leq v$, $v - x \leq r$ and $x - u \leq r$. In this case we say that f' is a *uniform derivative* of f or, when no confusion can arise, that f' is a *derivative* of f.

We observe that, in general, f' is not unique (see also [2]).

Conversely, in [3] we proved that two differentiable functions f and g which admit the same derivative in an interval $[a,b]$ satisfy necessarily $f - g\ =\ constant$.

We now recall the Riemann-type integral introduced in [2].

Given an order interval $[a,b] \subset R$, a *decomposition* of $[a,b]$ is any finite set $\mathcal{E} = \{x_0, x_1, \ldots, x_{2n}\} \subset [a,b]$, such that $x_0 = a$, $x_{2n} = b$ and $x_i \leq x_{i+1}$ for all $i = 0, \ldots, 2n-1$. The *mesh* of a division $\mathcal{E}$ is the quantity $|\mathcal{E}| := \sup_{i=1}^{2n} (x_i - x_{i-1})$.
If $f : [a,b] \to R$ is a map and $\mathcal{E}$ is a decomposition of $[a,b]$, $\mathcal{E} = \{x_0, x_1, \ldots, x_{2n}\}$, we denote by $S(f, \mathcal{E})$ and call *Riemann sum* associated with $\mathcal{E}$ the element of R given by

$\sum_{j=1}^{n} (x_{2j} - x_{2(j-1)}) \cdot f(x_{2j-1})$. A function $f : [a, b] \to R$ is said to be *Riemann integrable* (shortly, *integrable*) in $[a, b]$ if there exists an element $Y \in R$ such that

$$\inf_{r \in R^+} (\sup\{|S(f, \mathcal{E}) - Y| : |\mathcal{E}| \leq r\}) = 0.$$

In this case we write $\int_a^b f(t)\, dt = Y$.

We now recall a version of the Taylor formula in our context (see [2]).

Definition 2.4 If a function $f : [a, b] \to R$ is uniformly differentiable and if its derivative f' is uniformly differentiable with derivative f'', we will say that f'' is a *uniform second derivative* or, when no confusion can arise, *second derivative* of f. By induction it is possible to introduce the (uniform) derivatives of order n for every $n \in \mathbb{N}$.

A map $f : [a, b] \to R$ is *of class* C^n if it has uniformly continuous derivatives up to the order n in $[a, b]$; *of class* C^∞, if it is of class C^n for all n.

The following theorem has been proved in [2].

Theorem 2.5 *Let $f : [a, b] \to R$ have derivatives up to the order $n+1$: f', f'', $\ldots$, $f^{(n)}$, $f^{(n+1)}$. Fix arbitrarily $x_0 \in [a, b]$ and $h \in R$, such that $x_0 + h \in [a, b]$. Then we have:*

$$f(x_0 + h) = f(x_0) + \frac{h\, f'(x_0)}{1!} + \ldots + \frac{h^n\, f^{(n)}(x_0)}{n!} + B(x_0, h),$$

where $|B(x_0, h)| \leq \frac{|h|^{n+1}}{n!} \sup_{x \in [a,b]} |f^{(n+1)}(x)|$.

Definition 2.6 Let $a \in R^+$ be fixed. We say that $f : [-a, a] \to R$ is *analytic* if it has derivatives of order n for every $n \in \mathbb{N}$ and if its Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

converges to f in $[-a, a]$. This definition might be easily extended to more general cases.

Some special power series have been studied in [2]. We recall that a *power series* in R is an expression of the type

$$\sum_{n=0}^{\infty} a_n x^n, \tag{2}$$

where $(a_n)_n$ is a fixed sequence in R, and $x \in R$ is the variable, with the convention that $a_0 x^0 := a_0$.

We now recall the following notations, introduced in [2]. For every positive real number k, and for all $r \in R^+$, we denote by $S_k(r)$ the following subset of R: $S_k(r) = \{x \in R : |x| \leq k \cdot r\}$; moreover, for each positive real number t, we set $U_t(r) = \bigcup_{0<k<t} S_k(r)$, $R_r = \bigcup_{t>0} U_t(r)$. In case a unit $\underline{1}$ exists, we shall simply write S_k and U_t rather than $S_k(\underline{1})$ and $U_t(\underline{1})$.

We now simply list the main results on power series obtained in [2].

Theorem 2.7 *If the power series (2) converges at some element $x \neq 0$, then it converges uniformly and absolutely in every set of the type $S_k(x)$, $0 < k < 1$. In these sets, the sum of the series is differentiable, and the usual formula of term-by-term differentiation holds.*

If a unit $\underline{1}$ exists, then

1) the geometric *series* $\sum_{n=0}^{\infty} x^n$ *absolutely converges in the set U_1; moreover, for every element $x \in U_1$ there exists the* inverse *of $\underline{1} - x$ in the algebra R: such inverse is the sum of the geometric series above;*

2) the exponential *series* $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ *converges everywhere in the set $R_{\underline{1}}$. The sum of the exponential series is denoted by* $\exp(x)$*, whenever it exists. Moreover, usual techniques show that this function coincides with its derivative (see also Theorem 6.7 of [2]) and obeys the usual algebraic rules of the exponential function, therefore* $\exp(x)$ *has always an inverse element, i.e.* $\exp(-x)$*. Furthermore, it is easy to check that* $\exp(0) = \underline{1}$ *and* $\exp(x) > \underline{1}$ *whenever $x > 0$.*

3 Other analytic functions

In this section, we shall assume that a unit $\underline{1}$ exists in R.

Besides the exponential function exp, other functions can be introduced in a similar way as in the classical case. One of them is the *logarithm*, whose basic properties are the usual ones: in the previous paper [3], the set

$$V = \bigcup_{0<\varepsilon<1} \left[\varepsilon \underline{1}, \frac{1}{\varepsilon} \underline{1}\right] \tag{3}$$

has been introduced, and the following result was proved:

Theorem 3.1 *For each $\zeta \in V$ there exists a (unique) element $v \in R_{\underline{1}}$, such that* $\exp(v) = \zeta$.

Moreover, if $x \in U_1$, then

$$\log(\underline{1} + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n.$$

From now on, we call *logarithm of ζ* the element v as in Theorem 3.1, and we write shortly $\log \zeta = v$.

We also notice that, by virtue of Theorem 3.1, every element $z \in V$ has an *inverse* element.

Other important functions are the *trigonometric* ones: for example, we define:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{(2n+1)!}, \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n}}{(2n)!} :$$

these mappings are defined in $R_{\underline{1}}$ and enjoy the usual properties of the real sin and cos functions. In particular, one can prove the following

Proposition 3.2 *In $R_{\underline{1}}$, we have*

$$\sin' x = \cos x, \quad \cos' x = -\sin x, \quad \sin^2 x + \cos^2 x = \underline{1}.$$

Proof: The formulas concerning derivatives are immediate consequences of the term-by-term differentiation of the power series defining the functions.

As to the last formula, we observe that the functions $x \mapsto \sin^2 x$ and $x \mapsto -\cos^2 x$ have the same derivative (i.e. $2 \sin x \cos x$), hence their difference is constant. Finally, the value of the constant can be easily evaluated by setting $x = 0$. $\square$

4 The contraction principle

Before investigating the equations we present in the next sections, we recall a version of the Banach-Caccioppoli contraction principle, stated in [2]. For this formulation we don't require that a unit exists, but we need a piece of notation.

Definition 4.1 We say that an element $\kappa \in R$ is a *geometric element* if the geometric series converges at $|\kappa|$. In the sequel, the set of all geometric elements will be denoted by G.

Theorem 4.2 *Let $f : [a, b] \to [a, b]$ be any mapping, satisfying*

$$|f(x_2) - f(x_1)| \leq L\ |x_2 - x_1|$$

for a suitable positive element $L \in G$, and all $x_1, x_2 \in [a, b]$. Then there exists a unique fixed point s for f. Moreover, s is the limit of every sequence $(s_n)_n$ defined by choosing s_0 arbitrarily in $[a, b]$ and requiring

$$s_{n+1} = f(s_n) \quad \forall\, n \in \mathbb{N} \cup \{0\}.$$

We remark here that the *contraction* condition expressed in the previous theorem is satisfied as soon as f is uniformly differentiable in $[a, b]$, and $|f'(x)| \leq L$ for all $x \in [a, b]$, with $L \in G$: this is a consequence of the Taylor formula stated in Theorem 2.5, with $n = 0$.

We now give a version of the contraction principle in the context of function spaces. Let $I \subset R$ be an order interval, and $B(I)$ be the space of all bounded functions defined in I and with values in R. Of course, also $B(I)$ is a Riesz algebra. Now, if Y is any bounded element of $B(I)$, the *norm* $\|Y\|$ is an R-valued map defined by

$$\|Y\| = \sup\{|Y(u)| : u \in I\}.$$

We first deduce a completeness property of $B(I)$ with respect to this norm.

Proposition 4.3 *Let $(Y_n)_n$ be any sequence in $B(I)$, such that the set $\{\|Y_n\| : n \in \mathbb{N}\}$ is bounded, and assume that $(p_n)_n$ is an (O)-sequence, where*

$$p_n := \sup\{\|Y_m - Y_n\| : m > n\}.$$

Then there exists an element $Y \in B(I)$ such that $(Y_n)_n$ (O)-converges to Y uniformly.

Proof. Let us fix any element $u \in I$: then the sequence $(Y_n(u))_n$ is clearly Cauchy, and hence has a limit $Y(u)$, with respect to the order convergence. Thanks to the boundedness condition on the norms $\|Y_n\|$, the function Y is in $B(I)$. Now we have, for each $u \in I$ and every $n \in \mathbb{N}$:

$$|Y_n(u) - Y(u)| = (O) \lim_m |Y_n(u) - Y_m(u)| \leq \sup_{m>n} |Y_n(u) - Y_m(u)| \leq p_n.$$

Therefore, $\|Y_n - Y\| \leq p_n$ for all n, which clearly means uniform convergence. □

A contraction principle will now be stated in connection with this norm, as follows:

Theorem 4.4 *Let $T : B(I) \to B(I)$ be any mapping, satisfying the condition*

$$\|T(Y_1) - T(Y_2)\| \leq K\|Y_1 - Y_2\|$$

for some positive geometric element $K \in R$. Then T admits a unique fixed point, which is obtained as the limit of the sequence $(X_n)_n$ in $B(I)$ defined in the usual way: X_0 arbitrary, and $X_n = T(X_{n-1})$ for $n \geq 1$.

Proof. Let us fix X_0 arbitrarily in $B(I)$, and define $(X_n)_n$ by iterating T. We have for each $m \in \mathbb{N}$:

$$\|X_m - X_0\| \leq \sum_{j=0}^{m-1} K^j \|X_1 - X_0\|$$

and also, for every $m, n \in \mathbb{N}$:

$$\|X_{n+m} - X_n\| \leq \sum_{j=0}^{m-1} K^{n+j} \|X_1 - X_0\|.$$

Convergence of the geometric series at K and Proposition 4.3 give convergence of the sequence $(X_n)_n$. Then a routine argument concludes the proof. □

We remark that, in the previous theorem, the space $B(I)$ can be replaced by any subspace C, stable with respect to uniform convergence.

5 Existence and uniqueness theorems for differential equations

We shall consider first order differential equations in R, with functional argument. More precisely, we begin with studying Cauchy problems of the kind:

$$Y'(x) = f(Y(\psi(x))), \quad Y(x_0) = y_0, \tag{4}$$

where $x \in I := [a, b]$, $b - a \in span(G)$, Y is the unknown (uniformly differentiable) function, $f : R \to R$ is $span(G)$-Lipschitz, $\psi : I \to I$ is such that $\psi(x_0) = x_0$ and $0 \leq \psi(v) - \psi(u) \leq v - u$ whenever $u, v \in I$, $u \leq v$. We recall that the $span(G)$-Lipschitz property means that there exists a positive element $K \in span(G)$ such that $|f(u) - f(v)| \leq K|u - v|$ for all $u, v \in R$. Without loss of generality, we can also assume that $K = \xi\,\kappa$ for some positive real number ξ and positive geometric element κ. First of all, we prove that the map

$$\Psi(x) := f(Y(\psi(x)), \qquad x \in I$$

is uniformly continuous: this is a consequence of uniform continuity of Y and our assumptions on f and ψ. Indeed, for every $r \in R^+$, we have:

$$\begin{aligned} 0 &\leq \sup\{|Y(\psi(v)) - Y(\psi(u))| : u, v \in I, 0 \leq v - u \leq r\} \\ &\leq \sup\{|Y(w) - Y(t)| : t, w \in I, 0 \leq w - t \leq r\}, \end{aligned}$$

hence, thanks to uniform continuity of Y, we get

$$\inf_{r\in R^+} [\sup\{|Y(\psi(v)) - Y(\psi(u))| : u, v \in I,\ 0 \leq v - u \leq r\}] = 0. \tag{5}$$

From Lipschitzianity of f and (5), finally we obtain

$$\inf_{r\in R^+} [\sup\{|f(Y(\psi(v))) - f(Y(\psi(u)))| : u, v \in I,\ 0 \leq v - u \leq r\}] = 0,$$

that is uniform continuity of Ψ.

We now prove the following result.

Theorem 5.1 *Let f, ψ satisfy the hypotheses as above. Then the Cauchy problem (4) admits a unique solution on the whole of I.*

Proof: We can choose $x_0 = a$, without loss of generality. Let $L \in span(G)$ be a Lipschitz constant for f, and pick $b' := a+\vartheta(b-a)$, with $\vartheta > 0$, in such a way that $0 \leq L\,(b'-a) \leq \kappa$, with $0 < \kappa \in G$.

By Proposition 4.3, it is easy to see that $C^0([a, b'])$ has the same completeness property as $B(I)$ with respect to the norm defined there. Moreover, for each $Y \in C^0([a, b'])$, set

$$T(Y)(x) = y_0 + \int_a^x f(Y(\psi(\tau)))\, d\tau, \qquad x \in [a, b']. \tag{6}$$

As $\psi(\tau) \in [a, b']$ whenever $\tau \in [a, b']$, and since every uniformly continuous function is integrable (see [2]), then the integral in (6) is well-defined. By the Torricelli-Barrow theorem (see [2]), $T(Y)$ is C^1 in $[a, b']$. Furthermore, for every Y_1, $Y_2 \in C^0([a, b'])$ and $x \in [a, b']$ we get:

$$\begin{aligned} \|T(Y_1)(x) - T(Y_2)(x)\| &\leq \int_a^x \|(f(\tau, Y_1(\tau)) - f(\tau, Y_2(\tau))\|\, d\tau \leq L \int_a^x \|Y_1 - Y_2\|\, d\tau \\ &\leq L(b' - a)\|Y_1 - Y_2\| \leq \kappa\|Y_1 - Y_2\|. \end{aligned}$$

So, by the remark following Theorem 4.4, it is easy to see that the map T has a unique fixed point $Y \in C^0([a, b'])$: thus, Y will be the solution of our Cauchy problem in $[a, b']$. The usual continuation method allows to extend the found solution to the whole of $[a, b]$ after a finite number of steps.

This concludes the proof of the theorem. □

Remark 5.2 We remark here that, in case a unit $\underline{1}$ exists, a similar theorem like 5.1 can be formulated and proved also if I is unbounded, e.g. $I = \bigcup_{n=1}^{\infty}[-n\underline{1}, A]$, for some positive element $A \in R_{\underline{1}}$. This will be useful later.

In the following examples, we shall always suppose that a unit $\underline{1}$ exists in R.

Examples 5.3 **1.** Our first example deals with the equation

$$Y'(x) = Y(sX - \nu),$$

which has a solution, as soon as $s \in U_1,\ s \geq 0$, and $\nu \geq 0$ is any fixed element of $R_{\underline{1}}$. In particular, in [2] and [3] we found a concrete expression of the solutions when $s = 1$ and a series expansion of the solutions for the case $\nu = 0$ (and s arbitrary).

2. A slightly different equation is

$$Y'(x) = Y(s - x):$$

in the set $\cup_{n\in\mathbb{N}}[-n\underline{1}, s]$ all the conditions in the previous theorems are satisfied. In order to find solutions of class C^2, we simply differentiate again the equation, thus obtaining

$$Y''(x) = -Y(x):$$

now it's clear that the solutions must be of the type

$$Y(x) = c_1 \sin x + c_2 \cos x$$

where the constants c_1 and c_2 must be related (to each other and to s) by a suitable condition: $c_1(\underline{1} - \sin s) = c_2 \cos s$.

3. Also *partial* differential equations can be considered, though a suitable theory is still lacking. Let us fix any positive element $s \in R_{\underline{1}},\ s \neq 0$, and consider the *shifted* heat equation

$$\frac{\partial Z}{\partial t}(t, x) = \gamma \frac{\partial^2 Z}{\partial x^2}(t, x + s), \tag{7}$$

where γ is a fixed real constant, $\gamma \neq 0$, t runs in some real interval $[\alpha, \beta]$ and x in $R_{\underline{1}}, x \geq 0$. We shall not give precise definitions of the partial derivatives involved: we simply look for solutions in the form

$$Z(t, x) = g_1(t)\, g_2(x),$$

so the meaning of our derivatives is clear. Simple computations give the following class of solutions:

$$g_1(t) = re^{\gamma t},\ g_2(x) = r \exp(\lambda x),$$

where $r \in R$ is any constant, and λ is the element of $R_{\underline{1}} \cap R^+$ satisfying

$$\lambda = \exp(-\lambda s/2):$$

the existence of such constant was proved in [3] by means of the contraction principle. So, the functions

$$Z(t,x) = r\, e^{\gamma t} \exp(\lambda x)$$

satisfy the equation (7), for all $r \in R$.

If in (7) $s = 0$, then a class of solutions is given by the functions

$$Z(t,x) = r\, e^{\gamma t} \exp(x), \quad r \in R. \tag{8}$$

6 A Cauchy-Kovalevskaja Theorem

The Cauchy-Kovalevskaja Theorem is well-known: roughly speaking, it shows that, in case a differential equation has analytic data, then also its solution is analytic. A result of this kind holds also in our setting, and will be used later. A first formulation is the following.

Theorem 6.1 *Consider the following problem:*

$$Y'(x) = f(Y(x)), \quad Y(0) = 0, \tag{9}$$

where $f, Y : R \to R$. *If* f *is analytic in an order interval of the type* $[-\theta^*, \theta^*]$, *with* $\theta^* \in G$, $\theta^* > 0$, *and if there exists a constant element* $F \in span(G)$, $F \geq 0$, *such that* $|f^{(n)}(x)| \leq F$ *for all* $n \in \mathbb{N}$ *and all* $x \in [-\theta^*, \theta^*]$, *then there exists an order interval of the type* $[-\theta', \theta']$, *with* $\theta' \in G$, *such that the problem (9) admits there a unique solution* Y, *which is analytic.*

Proof: First of all we observe that, by Theorem 5.1, the problem (9) has a unique solution Y, defined at least in an order interval of the type $[-\theta, \theta]$, with $\theta \in G$, and Y admits there derivatives of any order, because f is analytic in $[-\theta^*, \theta^*]$ and, by virtue of Theorem 6.6. of [2], all the derivatives $f^{(n)}$, $n \in \mathbb{N}$, are there analytic too: indeed, we can see that

$$Y' = f(Y), Y'' = f'(Y)f(Y),$$

and so on, by applying the equation and the chain rule, which holds at least whenever the *outer* function is analytic (see [3], Proposition 3.2). Our next step is to prove that

$$|Y^{(n)}(x)| \leq (n-1)! F^n \tag{10}$$

for all $n \in \mathbb{N}$ and $x \in [-\theta, \theta]$. Of course, the domination is true, when $n = 1$ and $n = 2$. By differentiating Y'', the product $f'(Y)f(Y)$ gives rise to two summands, each dominated by F^3:

$$Y^{(3)} = f''(Y)f(Y)f(Y) + f'(Y)f'(Y)f(Y).$$

Now assume by recursion that $Y^{(n)}$ is the sum of $(n-1)!$ summands, each resulting as the product of n factors of the type $f(Y)$ or some derivative $f^{(k)}(Y)$: then differentiating again each summand gives rise to n new summands, each dominated by F^{n+1}, and clearly the total number of these new summands is $(n-1)!n = n!$; therefore $|Y^{(n+1)}| \leq n! F^{n+1}$, thus proving (10) by induction.

Now we recall from 2.5 that Y satisfies a Taylor formula of the type

$$Y(h) = Y(0) + \frac{h\,Y'(0)}{1!} + \ldots + \frac{h^n\,Y^{(n)}(0)}{n!} + \frac{h^{n+1}}{n!}\,B(h),$$

for all $n \in \mathbb{N}$ and $h \in [-\theta, \theta]$, where

$$\inf_{x\in[-\theta,\theta]} Y^{(n+1)}(x) \le B(h) \le \sup_{x\in[-\theta,\theta]} Y^{(n+1)}(x).$$

Thus we can see that, for each $n \in \mathbb{N}$ and $h \in [-\theta, \theta]$,

$$\left| Y(h) - \sum_{j=0}^{n} \frac{Y^{(j)}(0)}{j!} h^j \right| \le \frac{|h|^{n+1}}{n!} n! F^{n+1} = (|h|F)^{n+1}. \tag{11}$$

Clearly then, as F is in *span*(G), it is possible to find a positive element $\theta' \in G$ such that $|hF| \in G$ as soon as $|h| \le \theta'$: for these values of h, the last member of (11) tends to 0 as $n \to +\infty$, which shows analyticity of Y in $[-\theta', \theta']$. This concludes the proof. □

It is possible to drop the condition about uniform boundedness of all the derivatives of f in Theorem 6.1, but then a particular kind of analytic function is needed, which can be defined only when a unit $\underline{1}$ exists in R.

Definition 6.2 Suppose that a unit $\underline{1}$ exists in R. Let $\delta > 0$ be any fixed real number, and assume that the following *real* power series converges:

$$\sum_{n=0}^{\infty} a_n \delta^{*n}$$

for some $\delta^* > \delta$. Then the function

$$f : [-\delta\underline{1}, \delta\underline{1}] \to R, \quad f(u) = a_0\underline{1} + \sum_{n=1}^{\infty} a_n u^n$$

exists and is analytic: such function will be called *real-analytic*. To avoid confusion, we shall denote by f_ρ the real-valued function defined as the sum of the power series $\sum_{n=0}^{\infty} a_n \xi^n$, converging in the real interval $[-\delta, \delta]$. Clearly we have

$$f(\xi\underline{1}) = [f_\rho(\xi)] \cdot \underline{1}$$

as soon as $\xi \in [-\delta, \delta]$.

We shall state a Cauchy-Kovalevskaja-type theorem, for real-analytic data. But first we need some preliminary results, concerning products and compositions of real-analytic functions.

Proposition 6.3 *Let f and g be two real-analytic functions in $[-\delta\underline{1}, \delta\underline{1}]$. Then their product fg is real-analytic in $[-\delta\underline{1}, \delta\underline{1}]$ too, and the expansion of fg coincides with the one of $f_\rho g_\rho$.*

Proof: Suppose that $f(u) = \sum_{n=0}^{\infty} a_n u^n$, $g(u) = \sum_{n=0}^{\infty} b_n u^n$, for $|u| \leq \delta\underline{1}$, are real-analytic.

For every $k \in \mathbb{N}$, set

$$q_k := \sum_{i=0}^{k} a_i b_{k-i}.$$

The product series

$$\sum_{k=0}^{\infty} q_k \xi^k$$

is a real power series, convergent in $[-\delta, \delta]$ to $f_\rho(\xi) g_\rho(\xi)$, and hence the series

$$\sum_{k=0}^{\infty} q_k u^k$$

converges to a function φ, real-analytic in $[-\delta\underline{1}, \delta\underline{1}]$. We now prove that $\varphi(u) = f(u)g(u)$ for each $u \in [-\delta\underline{1}, \delta\underline{1}]$.

For every $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$\sum_{n > N_0} (|a_n| + |b_n|)\delta^n \leq \varepsilon.$$

Hence we get

$$\left| f(u) - \sum_{n \leq N} a_n u^n \right| + \left| g(u) - \sum_{n \leq N} b_n u^n \right| \leq \varepsilon\underline{1}$$

for all $N > N_0$.

Now, fixed $N > N_0$, let us compare the quantities

$$\left(\sum_{n=0}^{N} a_n u^n \right) \left(\sum_{n=0}^{N} b_n u^n \right), \quad \text{and} \quad \sum_{k=0}^{2N} q_k u^k :$$

the second summation involves all the terms in the first one; extra summands are involved too, but their sum is dominated by the product $\left(\sum_{n>N} |a_n||u|^n\right) \left(\sum_{n>N} |b_n||u|^n\right)$, which in turn is dominated by $\varepsilon^2\underline{1}$.

From this we deduce that, for every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\left| \sum_{k=0}^{2N} q_k u^k - \left(\sum_{n=0}^{N} a_n u^n \right) \left(\sum_{n=0}^{N} b_n u^n \right) \right| \leq \varepsilon\underline{1}$$

whenever $N > N_0$. As N tends to $+\infty$, taking into account of boundedness of f and g, we obtain that the series $\sum_{k=1}^{\infty} q_k u^k$ converges to $f(u)g(u)$ in $[-\delta\underline{1}, \delta\underline{1}]$. $\square$

As a consequence, we deduce that, as soon as f is real-analytic in $[-\delta\underline{1}, \delta\underline{1}]$, any function $p(f)$ is real-analytic in the same interval, provided p is a real *polynomial.*

A second tool is given in the following Proposition.

Proposition 6.4 *Let f and g be two real-analytic functions, defined in a set of the type $[-\delta\underline{1}, \delta\underline{1}]$, and such that $g(0) = 0$. Then there exists a real number $\sigma > 0$ such that the function $f(g)$ is real-analytic in U_σ, and the expansion of $f(g)$ is the same as for $f_\rho(g_\rho)$.*

Proof: Let us set

$$f(u) = \sum_{j=0}^{\infty} b_j u^j, \;\; g(u) = \sum_{n=1}^{\infty} a_n u^n :$$

by assumption the real power series $f_\rho(\xi) := \sum_{j=0}^{\infty} b_j \xi^j$ and $g_\rho(\xi) := \sum_{n=1}^{\infty} a_n \xi^n$ both converge in the interval $[-\delta, \delta]$. (In this proof, we shall use the symbol ξ to denote *real* numbers, and u to denote elements of R). Moreover, there exists $\sigma > 0$, $\sigma \leq \delta$, such that $\sum_{n=1}^{\infty} |a_n| \, |\xi|^n \leq \delta$ as soon as $|\xi| \leq \sigma$. This clearly implies that

$$\sum_{n=1}^{\infty} |a_n| \, |u|^n \in [-\delta\underline{1}, \delta\underline{1}]$$

as soon as $u \in [-\sigma\underline{1}, \sigma\underline{1}]$, and therefore $f(g)$ is defined in $[-\sigma\underline{1}, \sigma\underline{1}]$. Now, according with the previous remark, every power of g is real-analytic; hence, for every integer j, let us set

$$g^j(u) = \sum_{n=j}^{\infty} p_n^{(j)} u^n,$$

$|u| \leq \sigma\underline{1}$.

As the power series defining g_ρ is dominated by $\sum_{n=1}^{\infty} |a_n|\sigma^n$, then also the absolute series $\sum_{n=1}^{\infty} |p_n^{(j)}||\xi|^n$ is dominated by the expansion of $(\sum_{n=1}^{\infty} |a_n|\sigma^n)^j$, for every j.

We get then

$$f(g(u)) = b_0 + \sum_{j=1}^{\infty} b_j \left(\sum_{n=j}^{\infty} p_n^{(j)} u^n \right),$$

for $|u| \leq \sigma\underline{1}$. We shall show that

$$f(g(u)) = b_0 + \sum_{n=1}^{\infty} \left(\sum_{j=1}^{n} b_j p_n^{(j)} \right) u^n \tag{12}$$

for $|u| \leq \sigma\underline{1}$. To this aim, fix $\varepsilon > 0$: then there exists an integer J such that

$$\sum_{j>J} |b_j||\delta|^j \leq \varepsilon.$$

Next, for each integer $1 \leq j \leq J$ there exists $N(j) \in \mathbb{N}$ such that

$$\sum_{n>N(j)} |b_j| \, |p_n^{(j)}| \, |\sigma|^j \leq \varepsilon / J :$$

without loss of generality, we can assume $J < N(1) < N(2) < \ldots < N(J)$.

From this it follows that

$$\left| f(g(u)) - b_0 - \sum_{j=1}^{J} \left(\sum_{n=j}^{N(j)} b_j p_n^{(j)} u^n \right) \right| \leq 2\varepsilon \underline{1}.$$

Now, let us compare the quantities

$$\sum_{j=1}^{J} \left(\sum_{n=j}^{N(j)} b_j p_n^{(j)} u^n \right), \quad \sum_{n=1}^{N(J)} \left(\sum_{j=1}^{n} b_j p_n^{(j)} \right) u^n :$$

the second summation contains all the terms in the first one, plus:
1) some summands of the series $\sum_{n=1}^{\infty} b_1 \, p_n^{(1)} u^n$, with $n > N(1)$;
2) some summands of the series $\sum_{n=2}^{\infty} b_2 \, p_n^{(2)} u^n$, with $n > N(2)$;
... and so on, up to the series $\sum_{n=J}^{\infty} b_J \, p_n^{(J)} u^n$;
plus some terms whose sum is dominated by $\sum_{j>J} |b_j||\delta|^j \underline{1} \leq \varepsilon \underline{1}$. It follows that for $N > N(J)$ we have

$$\left| f(g(u)) - b_0 - \sum_{n=1}^{N} \left(\sum_{j=1}^{n} b_j p_n^{(j)} \right) u^n \right| \leq 4\varepsilon \underline{1},$$

thus showing (12). □

We are finally ready to prove the Cauchy-Kovalevskaja theorem for real-analytic maps.

Theorem 6.5 *Assume that a unit $\underline{1}$ exists in R, and consider the following problem:*

$$Y'(x) = f(Y(x)), \quad Y(0) = 0, \tag{13}$$

where $f, Y : R_{\underline{1}} \to R_{\underline{1}}$. If f is real-analytic in an order interval of the type $[-\delta\underline{1}, \delta\underline{1}]$, with $\delta \in \mathbb{R}$, $\delta > 0$, then there exists an order interval of the type $[-\sigma\underline{1}, \sigma\underline{1}]$, with $\sigma > 0$, such that the problem (13) *admits there a unique solution Y, which is analytic.*

Proof: Clearly, according with the previous results, there exist a positive real number δ' and a C^∞ function $w : [-\delta'\underline{1}, \delta'\underline{1}] \to R$ satisfying the equation and the initial data, and such solution is unique. Now, as f is real-analytic, let us set

$$f(u) = b_0\underline{1} + \sum_{n=1}^{\infty} b_n u^n,$$

where $b_n \in \mathbb{R} \; \forall n \in \mathbb{N} \cup \{0\}$ and $u \in [-\delta'\underline{1}, \delta'\underline{1}]$, and consider the auxiliary problem

$$Q'_\rho(\xi) = f_\rho(Q_\rho(\xi)), \; Q_\rho(0) = 0,$$

in the real line, where as usual

$$f_\rho(\xi) = b_0 + \sum_{n=1}^{\infty} b_n \xi^n.$$

Thanks to the classical Cauchy-Kovalevskaja theorem (see also [6]), this auxiliary problem admits a unique analytic solution Q_ρ, defined at least in an interval $[-\delta_1, \delta_1]$, with $\delta_1 \le \delta'$. Let us set

$$Q_\rho(\xi) = \sum_{n=1}^{\infty} \alpha_n \xi^n :$$

clearly $\alpha_0 = 0$ thanks to the initial condition.

Now, we claim that the solution w is real-analytic, and that $w_\rho = Q_\rho$ in some set of the type $[-\sigma\underline{1}, \sigma\underline{1}]$.

To prove this, let us denote by $Q : [-\delta_1\underline{1}, \delta_1\underline{1}] \to R$ the following real-analytic map:

$$Q(u) = \sum_{n=1}^{\infty} \alpha_n u^n.$$

We shall show that Q is a solution of the initial problem, at least in a smaller interval $[-\sigma\underline{1}, \sigma\underline{1}]$, from which it will follow by uniqueness that $w = Q$ in $[-\sigma\underline{1}, \sigma\underline{1}]$. First, there exists a positive real number $\sigma \le \delta_1$, such that $|Q(u)| \le \delta\underline{1}$ for all $u \in [-\sigma\underline{1}, \sigma\underline{1}]$. Thanks to the previous Proposition, $f(Q)$ is analytic in $[-\sigma\underline{1}, \sigma\underline{1}]$. So, it will suffice to show that Q' and $f(Q)$ have the same expansion in $[-\sigma\underline{1}, \sigma\underline{1}]$. But we already know that Q and Q_ρ have the same expansion, so the same holds for Q' and Q'_ρ; moreover, also $f(Q)$ and $f_\rho(Q_\rho)$ have the same expansion (this was proved in the previous Proposition); since Q_ρ is a real solution of $Q'_\rho = f_\rho(Q_\rho)$, all expansions coincide, and the proof is finished. □

7 Applications to Stochastic Equations

In this section we shall use the previous results in the solution of some kinds of Stochastic Differential equations. We give also examples, in which explicit formulations of the solutions can be found.

First, we fix an algebra R_0 of real-valued functions, e.g. the space $L_0(X, \mathcal{A}, \mu)$ (where $(X, \mathcal{A}, \mu)$ is a probability space). We then assume that our algebra R is a sub-algebra of $R_0^{[\alpha,\beta]}$, where $[\alpha, \beta]$ is some fixed real interval. Hence, R can be viewed as a space of Stochastic Processes.

Let us assume that $\Phi : R_{\underline{1}} \to R_{\underline{1}}$ is a bounded and Lipschitz map on $R_{\underline{1}}$, and introduce the following Stochastic Differential equation (*sde* from now on):

$$dY_t = \Phi(Y)U(t)dt + X(t)dW_t, \quad Y(\alpha) = 0, \tag{14}$$

where $U, X : [\alpha, \beta] \to \mathbb{R}$ are two fixed C^1 functions.

The equation (14) is equivalent to the integral equation

$$Y_t = \int_\alpha^t \Phi(Y)(\tau)U(\tau)d\tau + \int_\alpha^t X(\tau)dW_\tau, \tag{15}$$

where the first integral is a Riemann one, while the second is of Itô type (here X is assumed to be integrable w.r.t. W_t, and W_t denotes the usual Brownian Motion process, though more general cases can be considered: we refer to [4] for details).

Let us define now the transformation $T : R_{\underline{1}} \to R_{\underline{1}}$ as follows:

$$T(Y)(t) := \int_\alpha^t \Phi(Y)(\tau)U(\tau)d\tau + \int_\alpha^t X(\tau)dW_\tau :$$

we have

$$|T(Y_1) - T(Y_2)|(t) \leq L \int_\alpha^t \|Y_1 - Y_2\| d\tau \leq L(t-\alpha)\|Y_1 - Y_2\|,$$

where L is a Lipschitz constant for Φ, including also a domination for U. Thus, as soon as $t - \alpha$ is small enough, we get a unique solution of the equation (15) and therefore of (14), in $[\alpha, t]$. Then the usual method of continuation yields a solution Y in $[\alpha, \beta]$. Such solution is also unique.

The meaning of (15) can be described as follows: fixed Φ, every choice of the function X gives rise to a different *sde*, so that X can be interpreted as a kind of *control* of the system, and therefore it is unknown too. In order to explain this situation, we state a particular case of (14):

$$dY_t = \Phi(Y)d^2H_t + dH_t, \tag{16}$$

where $dH_t = X(t)dW_t$ unifies X and W, in a unique *control* process. The quantity d^2H_t here simply means $X(t)^2dt$, as we are assuming W_t to be the Brownian Motion. The equation (16) is quite similar to the *Langevin* one, but the driving process here (i.e. H_t) is not necessarily W_t.

Let us suppose now that Φ is real-analytic, and denote by Φ_1 any *primitive* of Φ . We look for solutions of (16) in the form $Y_t = E(t, W_t)$, for suitable *controls* X and functions E.

In particular, we shall find E in the form:

$$E(t, x) = \eta(Z_1(t)Z_2(x))$$

with $\eta,\ Z_1,\ Z_2$ suitable real-analytic functions of *one* variable.

Indeed, if we assume that $Y_t = E(t, W_t)$ (see [4] for details), we obtain

$$dY_t = \left(\frac{\partial E}{\partial t} + \frac{1}{2}\frac{\partial^2 E}{\partial x^2}\right)dt + \frac{\partial E}{\partial x}dW_t$$

(the meaning of the *partial* derivatives here is clear); thus (16) is satisfied as soon as $X(t) = \dfrac{\partial E}{\partial x}(t, W_t)$ and

$$\frac{\partial E}{\partial t} + \frac{1}{2}\frac{\partial^2 E}{\partial x^2} = \Phi(E)\left(\frac{\partial E}{\partial x}\right)^2. \tag{17}$$

The latter is a well-known equation, called *Burger's* equation, and can be solved in our setting more or less like in the classic case: let $t \mapsto Z_1(t)$ and $x \mapsto Z_2(x)$ be chosen in such a way that the product $Z_1(t)Z_2(x)$ satisfies the heat equation (these solutions are described in (8), choosing $\gamma = -\dfrac{1}{2}$), and set

$$E(t, x) = \eta(Z_1(t)Z_2(x)),$$

where the function $\eta : R_{\underline{1}} \to R_{\underline{1}}$ satisfies the following differential equation:

$$\eta''(y) = 2\Phi(\eta(y))\eta'(y)^2, \tag{18}$$

which in turn reduces to

$$\eta'(y) = \exp\left[2\Phi_1(\eta(y))\right], \tag{19}$$

where $\Phi_1' = \Phi$. (Analyticity of this solution has been proved in 6.5).

Once we have found E, we deduce also X and Y, by setting

$$Y_t = E(t, W_t), \quad X(t) = \frac{\partial E}{\partial x}(t, W_t),$$

and the problem (16) is solved.

References

[1] A. BOCCUTO, *Differential and Integral Calculus in Riesz Spaces,* Tatra Mountains Math. Publ. **14** (1998), 293-323.

[2] A. BOCCUTO & D. CANDELORO, *Integral and Differential Calculus in Riesz spaces and applications,* (2005), to appear on J. Appl. Funct. Anal.

[3] A. BOCCUTO & D. CANDELORO, *Several types of equations in Riesz spaces and applications,* Acta Math. (Nitra), **8** (2005), 7-21.

[4] A. BOCCUTO, D. CANDELORO & E. KUBIŃSKA, *Kondurar Theorem and Itô formula in Riesz spaces,* J. Concr. Appl. Math. **4** (2006), 67-90.

[5] D. CANDELORO, *Riemann-Stieltjes Integration in Riesz Spaces,* Rend. Mat. (Roma) **16** (1996), 563-585.

[6] R. COURANT & D. HILBERT, *Methods of Mathematical Physics, Vol. II, Partial Differential Equations,* (1962), John Wiley & Sons.

[7] M. DUCHOŇ & B. RIEČAN, *On the Kurzweil-Stieltjes integral in ordered spaces,* Tatra Mountains Math. Publ. **8** (1996), 133-142.

[8] W. A. J. LUXEMBURG & A. C. ZAANEN, *Riesz Spaces, I,* (1971), North-Holland Publ. Co.

[9] Z. SCHUSS, *Theory and Applications of Stochastic Differential Equations,* (1980), John Wiley & Sons.

[10] B. Z. VULIKH, *Introduction to the theory of partially ordered spaces,* (1967), Wolters - Noordhoff Sci. Publ.

[11] A. C. ZAANEN, *Riesz spaces, II,* (1983), North-Holland Publ. Co.