Integral and Differential Calculus in Riesz spaces and applications

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Abstract

In this paper we outline a new theory about integral and differential calculus for Riesz space-valued mappings defined on suitable Riesz spaces. In our abstract context, we prove some theorems similar to the classical ones, like for example the Fundamental Formula of Calculus and the theorem about exchanging order between limits and derivatives. As applications, we give some results about power series, a fixed point theorem, and some models of differential functional equations.

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0 Introduction

In this paper a new theory is presented, concerning integral and differential calculus for functions defined in a suitable Riesz space and with values in another Riesz space, linked together with a "product" structure. This approach is, in a certain sense, a generalization of the one given in [1]. The concepts of uniform continuity, uniform differentiability, Riemann integrability are introduced and investigated, and some theorems like the corresponding classical ones are proved: among them we quote the Fundamental Formula of Calculus. Moreover a version of the Taylor formula is demonstrated: here we express the "remainder term" by means of our introduced abstract integral. One can find applications for example in the Itô formula, proved in [2], however we shall not deal with it here. Furthermore, a theory about exchanging order between limits and derivatives, power series and analyticity in our abstract context is given: a fixed point theorem, and some examples of differential and functional equations are then deduced. These equations too might have interesting formulations in the Stochastic Calculus, and some of them also in Theory of Fractals, though we chose not to treat them here.
1 Basic definitions and assumptions

A Riesz space $R$ is said to be Dedekind complete if every nonempty subset $A \subset R$, bounded from above, has supremum in $R$.

From now on, we assume that $R$ is a Dedekind complete Riesz space.

Given a bounded sequence $(p_n)_n$ in $R$, we set:

$$\limsup_n p_n = \inf_{n \in \mathbb{N}} \sup_{m \geq n} p_m; \quad \liminf_n p_n = \sup_{n \in \mathbb{N}} \inf_{m \geq n} p_m;$$

and we say that $\lim_n p_n = l \in R$ if $\limsup_n p_n = \liminf_n p_n = l$.

This corresponds to the classical definition of order convergence or $(o)$-convergence (see also [5], [6]).

Assumptions 1.1 Let $R_1$, $R_2$, $R$ be three Dedekind complete Riesz spaces. We say that $(R_1, R_2, R)$ is a product triple if there exists a map $\cdot : R_1 \times R_2 \to R$, which we will call product, such that

1.1.1) $(r_1 + s_1) \cdot r_2 = r_1 \cdot r_2 + s_1 \cdot r_2$, \hspace{1cm} $r_1 \cdot (r_2 + s_2) = r_1 \cdot r_2 + r_1 \cdot s_2$,

1.1.2) $[r_1 \geq s_1, r_2 \geq 0] \Rightarrow [r_1 \cdot r_2 \geq s_1 \cdot r_2], \quad [r_1 \geq 0, r_2 \geq s_2] \Rightarrow [r_1 \cdot r_2 \geq r_1 \cdot s_2]$ for all $r_j, s_j \in R_j, j = 1, 2$;

1.1.3) if $(a_\lambda)_{\lambda \in \Lambda}$ is any family in $R_1$ with $a_\lambda \geq 0 \ \forall \lambda$ and $\inf_\lambda a_\lambda = 0$, and $R_2 \ni b \geq 0$, then $\inf_\lambda (a_\lambda \cdot b) = 0$; if $(b_\lambda)_{\lambda}$ is any family in $R_2$ with $b_\lambda \geq 0 \ \forall \lambda$ and $\inf_\lambda b_\lambda = 0$, and $R_1 \ni a \geq 0$, then $\inf_\lambda (a \cdot b_\lambda) = 0$. 
A Dedekind complete Riesz space $R$ is called an algebra if $(R, R, R)$ is a product triple.

## 2 A Riemann-type integral in Riesz spaces

Let $(R_1, R_2, R)$ be a product triple of Riesz spaces.

Given two elements $a, b \in R_1$, with $a \leq b$, we denote by $[a, b]$ and call order interval (or in short interval) the set of all elements $r \in R_1$, such that $a \leq r \leq b$. Given an order interval $[a, b] \subset R_1$, a division of $[a, b]$ is any finite set $T = \{x_0, x_1, \ldots, x_n\} \subset [a, b]$, such that $x_0 = a$, $x_n = b$ and $x_i \leq x_{i+1}$, $x_i \neq x_{i+1}$ for all $i = 0, \ldots, n - 1$. The mesh of a division $T$ is the quantity $\eta(T) = \sup_{i=1}^n (x_i - x_{i-1})$.

A decomposition of $[a, b]$ is a set $E = \{([x_{i-1}, x_i], \xi_i) : i = 1, \ldots, n\}$, where $\{x_0, x_1, \ldots, x_n\}$ is a division $T$ of $[a, b]$ and $\xi_i \in [x_{i-1}, x_i] \ \forall \ i = 1, \ldots, n$. For such a decomposition $E$, we shall put $|E| = \eta(T)$.

We now introduce a Riemann-type integral in our setting, which will be useful in the sequel in order to prove our version of the Taylor formula. If $f : [a, b] \rightarrow R_2$ is a map and $E$ is a decomposition of $[a, b]$, $E = \{([x_{i-1}, x_i], \xi_i) : i = 1, \ldots, n\}$, we denote by $S(f, E)$ and call Riemann sum associated with $E$ the element of $R$ given by $\sum_{i=1}^n (x_i - x_{i-1}) \cdot f(\xi_i)$.

A function $f : [a, b] \rightarrow R_2$ is said to be Riemann integrable (in short,
integrable) in \([a, b]\) if there exists an element \(Y \in R\) such that

\[
\inf_{r \in R_1^+} (\sup\{|S(f, E) - Y| : |E| \leq r\}) = 0,
\]

where \(R_1^+\) is the set of all elements \(r \in R_1\) such that \(r \geq 0\) and \(r \neq 0\). In this case we write \(\int_a^b f(t) \, dt = Y\).

It is easy to see that such an element \(Y\) is uniquely determined.

The following results are easy to prove and will be useful in the sequel.

**Proposition 2.1** If \(f_1\) and \(f_2\) are integrable in \([a, b]\) and \(\alpha_1, \alpha_2 \in R\), then \(\alpha_1 f_1 + \alpha_2 f_2\) is integrable in \([a, b]\) too, and in this case we have

\[
\int_a^b (\alpha_1 f_1 + \alpha_2 f_2)(t) \, dt = \alpha_1 \int_a^b f_1(t) \, dt + \alpha_2 \int_a^b f_2(t) \, dt.
\]

If \(f_1\) and \(f_2\) are integrable in \([a, b]\) and \(f_1 \leq f_2\), then

\[
\int_a^b f_1(t) \, dt \leq \int_a^b f_2(t) \, dt.
\]

If \(f\) is integrable in an order interval \([a, b]\), then \(f\) is also integrable in any order interval \(I \subset [a, b]\).

Thus it follows that for any integrable function \(f : [a, b] \to R_2\) the indefinite integral is defined as an additive \(R\)-valued interval function on the family of all intervals in \([a, b]\). We shall denote it by \(F(I) = \int_I f\), with the convention to define \(F([b, a]) = -F([a, b])\) \((\forall a, b \in R_1, \ a \leq b)\), so that \(F([a, a]) = 0\ \forall a \in R_1\), and we shall call integral function associated
with \( f \) the map defined (with abuse of notation) as follows:

\[
F(x) \equiv F([a, x]), \quad x \in [a, b].
\]

Like in the classical case, uniform continuity implies integrability.

**Definition 2.2** We say that a function \( f : [a, b] \to \mathbb{R}^2 \) is uniformly continuous in \([a, b]\) if

\[
\inf_{r \in \mathbb{R}^+} \{ \sup \{|f(v) - f(u)| : u, v \in [a, b], u \leq v, v - u \leq r\} \} = 0.
\]

**Proposition 2.3** Every uniformly continuous function \( f : [a, b] \to \mathbb{R}^2 \) is integrable in \([a, b]\) too.

**Proof.** The proof is similar to the one in [4]. \( \square \)

## 3 An abstract derivative in Riesz spaces

Throughout this section, we always assume that \( R_1 \) and \( R_2 \) are two Dedekind complete Riesz spaces and that \((R_1, R_2, R_2)\) is a product triple; let now \([a, b] \subset R_1\) be an interval. We begin with the following:

**Definition 3.1** We say that a function \( f : [a, b] \to \mathbb{R}^2 \) is uniformly differentiable in \([a, b]\) if there exist a bounded function \( f' : [a, b] \to \mathbb{R}^2 \) and an increasing family \((p_r)_{r \in \mathbb{R}^+}\) such that \( \inf_{r \in \mathbb{R}^+} p_r = 0 \) and

\[
|f(v) - f(u) - (v - u) f'(x)| \leq (v - u) p_r \tag{1}
\]
for every $r \in R_1^+$ and whenever $u, v, x \in [a, b]$, $u \leq x \leq v$, $v - x \leq r$ and $x - u \leq r$. In this case we say that $f'$ is a uniform derivative of $f$ or, when no confusion can arise, that $f'$ is a derivative of $f$.

We observe that, in general, $f'$ is not unique. Indeed, let $R_1$ and $R_2$ be the spaces of all bounded measurable real-valued functions, defined on $[0, 1]$, vanishing on $[0, 1/2]$ and $[1/2, 1]$ respectively. For every $\psi_1 \in R_1$ and $\psi_2 \in R_2$, $\psi_1 \cdot \psi_2$ is identically zero (here, \cdot is the usual product between functions): thus, it is not difficult to see that $(R_1, R_2, \{0\})$ is a product triple with respect to this product. Let $[a, b]$ be any arbitrary order interval of $R_1$, and $f : [a, b] \to R_2$ be any constant function: then clearly every function $f_1 : [a, b] \to R_2$ is a derivative of $f$.

This fact will not affect our results, and it will be clear from the context in which sense we deal with derivatives.

For instance, it is quite clear that every function $f : [a, b] \to R_2$, uniformly differentiable in $[a, b]$, is uniformly continuous in $[a, b]$.

Usual differentiation rules hold in our setting, for example:

**Proposition 3.2** Let $(R_1, R_2, R_2)$, $(R_1, S_2, S_2)$, $(R_2, S_2, T_1)$, $(R_1, T_1, T_1)$ be four product triples, and $[a, b] \subset R_1$ be an interval. If $f : [a, b] \to R_2$, $g : [a, b] \to S_2$ are two uniformly differentiable functions with derivatives $f'$, $g'$ respectively, then the map $h = f \cdot g : [a, b] \to T_1$ is uniformly
differentiable too, with derivative $h'$ given by $h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$, $x \in [a, b]$.

Therefore every "polynomial" function (in a commutative algebra $R$) is uniformly differentiable, and the usual differentiation rule is valid.

The following results are fundamental theorems of Integral Calculus, as in the classical case. We begin with the following version of the Torricelli-Barrow theorem: the proof is easy.

**Theorem 3.3** Let $(R_1, R_2, R_2)$ be a product triple, and $f : [a, b] \to R_2$ be a uniformly continuous function (in $[a, b]$). Then its integral function $F$ is uniformly differentiable in $[a, b]$ and $F'(x) = f(x) \forall x \in [a, b]$.

We now turn to a version of the Fundamental Formula of Integral Calculus in an abstract setting.

**Theorem 3.4** Let $(R_1, R_2, R_2)$ be a product triple, $[a, b] \subset R_1$ be an interval and $f : [a, b] \to R_2$ be a uniformly differentiable function, with derivative $f'$. Then, $f'$ is integrable, and

$$\int_a^b f'(t) \, dt = f(b) - f(a).$$

**Proof.** Choose arbitrarily $r \in R_1^+$ and take any decomposition $E = \{([x_{i-1}, x_i], \xi_i) : i = 1, \ldots, n\}$ of $[a, b]$, with $|E| \leq r$. Let $(p_r)_{r \in R_1^+}$ be a
family as in the definition of uniform differentiability. We get:

\[
0 \leq \left| \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot f'(\xi_i) - [f(b) - f(a)] \right| \\
\leq \sum_{i=1}^{n} |f(x_i) - f(x_{i-1}) - (x_i - x_{i-1}) \cdot f'(\xi_i)| \\
\leq \left( \sum_{i=1}^{n} (x_i - x_{i-1}) \right) \cdot p = (b - a) p_r.
\]

Thus the assertion follows. \(\square\)

**Remark 3.5** We can observe that Theorem 3.4 is true also if the endpoints \(a\) and \(b\) are not comparable, provided they are contained in a larger interval \([A, B]\) where \(f\) is uniformly differentiable, and \(f'\) is its derivative. In fact, in case \(A \leq a, b \leq B\), we can set \(h = b - a\), and define

\[
\int_{a}^{b} f'(t)dt = \int_{a}^{a+h} f'(t)dt = \int_{a}^{a+h^+} f'(t)dt - \int_{a+h}^{a+h^+} f'(t)dt.
\]

Indeed, as \(B - a \geq 0\), from \(h = b - a \leq B - a\) it follows \(h^+ \leq B - a\), and hence \([a, a + h^+] \subset [A, B]\); moreover, it follows also \([a + h, a + h^+] = [b, a + h^+] \subset [b, B]\). Thus, applying 3.4 to the last member of (2), it follows easily

\[
\int_{a}^{b} f'(t)dt = f(a + h^+) - f(a) + f(a + h) - f(a + h^+) = f(b) - f(a).
\]
4 The Taylor formula

We shall prove a version of the Taylor formula in our context. Besides the obvious applications in approximating functions, this formula has applications in stochastic integration (see [2]).

**Definition 4.1** If a function $f : [a, b] \to \mathbb{R}$ is uniformly differentiable and if its derivative $f'$ is uniformly differentiable with derivative $f''$, we will say that $f''$ is a uniform second derivative or, when no confusion can arise, second derivative of $f$. By induction it is possible to introduce the (uniform) derivatives of order $n$ for every $n \in \mathbb{N}$. If $f : [a, b] \to \mathbb{R}$ is uniformly differentiable up to the order $n$, and if its $n$-th derivative $f^{(n)}$ is uniformly continuous, we say that $f$ is of class $C^n([a, b])$. Furthermore, if $S \subset \mathbb{R}_1$ contains at least an order interval, we say that $f : S \to \mathbb{R}$ is of class $C^n(S)$ if it is of class $C^n([a, b])$ for every order interval $[a, b] \subset S$, and that $f : S \to \mathbb{R}$ is of class $C^\infty(S)$ if it is of class $C^n(S) \forall n \in \mathbb{N}$.

**Theorem 4.2** Let $R$ be an algebra, $[a, b] \subset R$ be an interval, and $f : [a, b] \to R$ have derivatives up to the order $n + 1$: $f'$, $f''$, $\ldots$, $f^{(n)}$, $f^{(n+1)}$.

Fix arbitrarily $x_0 \in [a, b]$ and $h \in R$, such that $x_0 + h \in [a, b]$. Then we have:

$$f(x_0 + h) = f(x_0) + \frac{h f'(x_0)}{1!} + \ldots + \frac{h^n f^{(n)}(x_0)}{n!} + B(x_0, h),$$
where $|B(x_0, h)| \leq \frac{|h|^{n+1}}{n!} \sup_{x \in [a, b]} |f^{(n+1)}(x)|$.

**Proof.** Fix $x_0$ and $h$ as in the hypotheses, and define an auxiliary function $F : [a, b] \to \mathbb{R}$ as follows:

$$F(t) = f(x_0 + h) - f(t) - \frac{(x_0 + h - t) f'(t)}{1!} - \cdots - \frac{(x_0 + h - t)^n f^{(n)}(t)}{n!}.$$  

By hypothesis, $F$ is uniformly differentiable and we have, $\forall t \in [a, b]$: 

$$F'(t) = -\frac{(x_0 + h - t)^n f^{(n+1)}(t)}{n!},$$ 

and $F'$ is bounded. Put $M = \sup_{x \in [a, b]} |f^{(n+1)}(x)|$. By Theorem 3.4 and Remark 3.5 we get:

$$F(x_0) = -\int_{x_0}^{x_0+h} F'(t) \, dt = \int_{x_0}^{x_0+h} \frac{(x_0 + h - t)^n}{n!} f^{(n+1)}(t) \, dt$$

$$= \int_{x_0}^{x_0+h} \frac{(x_0 + h - t)^n}{n!} f^{(n+1)}(t) \, dt - \int_{x_0}^{x_0+h} \frac{(x_0 + h - t)^n}{n!} f^{(n+1)}(t) \, dt,$$

and hence

$$|F(x_0)| \leq M \left( \int_{x_0}^{x_0+h} \frac{|x_0 + h - t|^n}{n!} \, dt + \int_{x_0}^{x_0+h} \frac{|x_0 + h - t|^n}{n!} \, dt \right)$$

$$\leq M \frac{h^n |h| + h^n |h|}{n!} = M \frac{|h|^{n+1}}{n!},$$

since $|x_0 + h - t| \leq |h|$. Thus the assertion follows.  

\begin{flushright} $\Box$ \end{flushright}

### 5 Sequences of differentiable functions

In this section we give some conditions, under which it is possible to exchange the order between limits and derivatives. First of all we intro-
duce the concept of uniform convergence for sequences of functions. We always suppose that \([a, b] \subset R_1\) is an order interval.

**Definition 5.1** A sequence \((f_n : [a, b] \rightarrow R_2)_n\) is said to be uniformly convergent to \(f : [a, b] \rightarrow R_2\) if \(\lim_{n} \sup_{t \in [a, b]} |f_n(t) - f(t)| = 0\).

We now give two fundamental properties of uniform convergence, which will be useful in the sequel. The proofs are straightforward.

**Theorem 5.2** Let \((f_n : [a, b] \rightarrow R_2)_n\) be a sequence of integrable functions, uniformly convergent to a map \(f : [a, b] \rightarrow R_2\). Then \(f\) is integrable and

\[
\lim_n \int_a^b f_n(t) \, dt = \int_a^b f(t) \, dt.
\]

**Theorem 5.3** Let \((f_n)_n\) be a sequence of uniformly continuous functions \(f_n : [a, b] \rightarrow R_2\), uniformly convergent to a mapping \(f : [a, b] \rightarrow R_2\). Then \(f\) is uniformly continuous.

Thanks to Theorems 3.4, 5.2, 5.3 and 3.3, it is possible to use a classical technique in order to prove the next result.

**Theorem 5.4** Let \((f_n : [a, b] \rightarrow R_2)_n\) be a sequence of uniformly differentiable functions, with derivatives \(f'_n\), \(n \in \mathbb{N}\). Moreover, assume that the sequence \((f'_n)_n\) is uniformly convergent in \([a, b]\) and that there
exists \( \lim_n f_n(a) \) in \( R_2 \). Then the sequence \((f_n)_n\) is uniformly convergent in \([a,b]\) to a uniformly differentiable function \( f : [a,b] \to R_2 \), and 
\[ f' = \lim_n f'_n \text{ in } [a,b]. \]

We recall that, analogously as in the classical case, it is possible to give the concept of series of elements of any Riesz space \( R \) and the ones of convergence and absolute convergence, and to deduce an analogue of the Cauchy criterion, together with its usual consequences. Thus, Theorem 5.4 implies the analogue of the classical result concerning differentiation term-by-term of a series of functions. We shall not write it down here, however we shall use it later.

**6 Power series and applications**

In this section we deal with power series: this will be the main tool in the subsequent applications.

**Definition 6.1** Let \( R \) be any commutative algebra. We shall suppose that there exists a *multiplicative* unit in \( R \), which will be denoted by \( 1 \).

For every positive real number \( k \), and for every positive element \( r \in R \), we denote by \( S_k(r) \) the following subset of \( R \): 
\[ S_k(r) = \{ x \in R : |x| \leq k \cdot r \}; \]
moreover, for each positive real number \( t \), we set 
\[ U_t(r) = \bigcup_{0 < k \leq t} S_k(r), \]
\[ R_r = \bigcup_{t > 0} U_t(r). \] In case \( r = 1 \), we shall simply write \( S_k \) and \( U_t \) rather
than $S_k(1)$ and $U_1(1)$. A *power series* is a series of the type

$$\sum_{n=0}^{\infty} a_n x^n,$$

with $x \in R$, $a_n \in R \ \forall \ n$, and with the convention $x^0 = 1 \ \forall \ x \in R$.

**Proposition 6.2** If the series (3) converges at some $x \in R$, then it converges uniformly and absolutely in every set $S_k(|x|)$ with $0 < k < 1$.

**Proof.** From convergence of the series at $x$, it follows that the sequence $(|a_n x^n|)_n$ is bounded in $R$: let $M$ denote any upper bound for that sequence. Now, for any real number $k \in ]0,1[$ and every element $r \in S_k(|x|)$, we have $|a_n r^n| \leq |a_n| k^n |x^n| \leq Mk^n$ for all positive integers $n$. This clearly implies the assertion. \[\square\]

The following results have many consequences: for example they show that some elements in $R$ have an inverse, and give an expression for it.

**Proposition 6.3** The geometric series $\sum_{n=0}^{\infty} x^n$ absolutely converges in the set $U_1$; moreover, for every element $x \in U_1$ there exists the inverse of $1 - x$ in the algebra $R$: such inverse is the sum of the geometric series above.

**Proof.** Let us fix $x \in U_1$, and choose any real number $\alpha \in ]0,1[$ such that $|x| \leq \alpha 1$. Then we get $|x^n| \leq \alpha^n 1$: this clearly implies convergence of the geometric series at $x$. Moreover, for every positive integer $n$ we have
(1 - x) \sum_{j=0}^{n} x^j = 1 - x^{n+1}. From convergence of the series, we deduce that \( \lim_n x^n = 0 \), and finally \( (1 - x) \sum_{n=0}^{\infty} x^n = 1 \).

**Proposition 6.4** The exponential series \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \) converges everywhere in the set \( \mathbb{R}_1 \).

**Proof.** First of all, we observe that the exponential series obviously converges at the points \( \nu_1 \), for every positive integer \( \nu \). To conclude the proof, it will suffice to apply Proposition 6.2.

As usual, the sum of the exponential series \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \) will be denoted by \( \exp(x) \), whenever it exists. Moreover, usual techniques show that this function coincides with its derivative (see also Theorem 6.7 below), and obeys the usual algebraic rules of the exponential function, therefore \( \exp(x) \) has always an inverse element, i.e. \( \exp(-x) \).

Another consequence involves Taylor series:

**Proposition 6.5** Let \( f : U_t \to \mathbb{R} \) be a function of class \( C^\infty(U_t) \). A sufficient condition for convergence to \( f \) of its Mc-Laurin series

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n
\]

is that there exists a positive element \( M \in \mathbb{R} \) such that \( |f^{(n)}(x)| \leq M^n \) holds, for every \( n \in \mathbb{N} \) and every \( x \in U_t \).
Proof. The proof is an easy consequence of Theorem 4.2 and of convergence of the exponential series. □

Now, given the power series (3), it is possible to associate to it its "derivative series"

\[ \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad x \in R. \] (5)

We now prove the following:

**Theorem 6.6** Fix \( t > 0 \). The series (3) converges at every \( x \in U_t \) if and only if the series (5) converges at every \( x \in U_t \).

**Proof.** We begin with the "if" part. Fix arbitrarily \( x \in U_t \), and let \( k \in [0, t[ \) be any real number such that \( x \in S_k \). By hypotheses and by Proposition 6.2, it follows that the series (5) absolutely converges at \( x \).

Now, from \( |a_n x^n| = |x||a_n x^{n-1}| \leq |x||n a_n x^{n-1}| \) it follows that the series (3) converges at \( x \). Concerning the "only if" part, assume that the series (3) converges in \( U_t \), and fix any element \( x \in U_t \). Let \( k \) be any positive real number, \( k < t \), such that \( x \in S_k \). Set \( k' := \frac{k + t}{2}, \ k'' := \frac{k' + k}{2} \), so that \( k < k'' < k' < t \), and put \( x_1 = k' \frac{1}{2}, \ x_2 = k'' \frac{1}{2} \). Clearly, \( x_1 \in U_t \) and therefore, thanks to Proposition 6.2, the series (3) converges absolutely at \( x_1 \); now, from \( |n a_n x_2^{n-1}| = n |a_n||(k'')^{n-1}\frac{1}{2} \leq |a_n|(k')^{n}\frac{1}{2} \) (which holds at least definitely), we can deduce convergence of the series (5) at \( x_2 \).

From Proposition 6.2 there follows convergence of (5) at \( x \). □
A consequence of Theorems 6.6 and 5.4 is the following:

**Theorem 6.7** Fix any positive real number $t$. If the series \( (3) \) converges $\forall x \in U_t$ and if $f$ is its sum, that is
\[
f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \forall x \in U_t,
\]
then we get
\[
f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \forall x \in U_t.
\]

**Proof.** Fix arbitrarily $x \in U_t$ and take an order interval $[-k_1, k_1]$ containing $x$ and contained in $U_t$. Then, by Theorem 6.6, the series \( (5) \) converges (absolutely) at every $r \in U_t$, and by 6.2 the series \( (3) \) and \( (5) \) converge uniformly in $[-k_1, k_1]$. The assertion follows from this and term-by-term differentiation. $\square$

Of course, Theorem 6.7 implies that the sum of a power series $\sum_{n=0}^{\infty} a_n x^n$, convergent in $U_t$, is of class $C^\infty(U_t)$, and its Mc-Laurin series coincides with the initial power series.

A first consequence of the previous results is a fixed point theorem, of the type of Banach.

**Theorem 6.8** Let $f : [a, b] \to [a, b]$ be any mapping, satisfying
\[
|f(x_2) - f(x_1)| \leq K |x_2 - x_1|
\]
for a suitable positive element $K \in U_1$, and all $x_1, x_2 \in [a, b]$. Then there exists a unique fixed point $s$ for $f$. Moreover, $s$ is the limit of every sequence $(s_n)_n$ defined by choosing $s_0$ arbitrarily in $[a, b]$ and requiring

$$s_{n+1} = f(s_n) \quad \forall n \in \mathbb{N} \cup \{0\}.$$ 

**Proof.** Let us fix arbitrarily $s_0 \in [a, b]$, and define the sequence $(s_n)_n$ as described, by iterations of $f$. For all $n, p \in \mathbb{N}$ we get

$$|s_{n+p} - s_n| \leq |s_1 - s_0|K^n(1 - K)^{-1}$$

by means of usual techniques. As $\lim_n K^n = 0$, the sequence $(s_n)_n$ is clearly convergent, and its limit $s$ satisfies $f(s) = s$ thanks to continuity of $f$. Uniqueness can be proved by similar techniques. \(\square\)

As a remark, we can observe that a mapping $f : [a, b] \to [a, b]$ satisfies the contraction condition of 6.8 as soon as $f$ is uniformly differentiable, and its derivative satisfies $|f'(x)| \leq K$ for all $x \in [a, b]$; this follows easily from Theorem 4.2.

Further applications allow us to obtain solutions of suitable functional equations, according with the following theorems. Though these equations are nothing but examples, from them one could find wide generalizations and also formulations in different contexts, for example in Stochastic Analysis, by specializing the underlying Riesz space.
Theorem 6.9 Assume that $\kappa(u) = \sum_{n=1}^{\infty} \gamma_n u^n$ is a fixed convergent power series, with $u \in R$ and $\gamma_n \in R$ $\forall n \in \mathbb{N}$. Then, for every element $s \in U_1$, there exist functions $Y : R \to R$ such that $Y(u) - \kappa(u) = Y(su)$, for all $u \in R$.

Proof. We note that the given series converges absolutely at every $u \in R$ (see Proposition 6.2). Moreover, as $s \in U_1$, also $s^n \in U_1$ for every positive integer $n$, and $(1 - s^n)^{-1}$ exists, thanks to 6.3. Consider the following power series:

$$Y(u) = \sum_{n=1}^{\infty} \gamma_n (1 - s^n)^{-1} u^n.$$  

From Proposition 6.3 we deduce:

$$|\gamma_n (1 - s^n)^{-1} u^n| \leq |\gamma_n u^n| \sum_{j=0}^{\infty} |s|^j = (1 - |s|)^{-1} |\gamma_n u^n| \quad \forall n \in \mathbb{N}.$$  

By means of the Cauchy criterion, we then obtain (absolute) convergence of $Y(u)$ for every $u \in R$. Now,

$$Y(u) - Y(su) = \sum_{n=1}^{\infty} (1 - s^n) \gamma_n u^n (1 - s^n)^{-1} = \kappa(u)$$  

for every $u \in R$, thus the equation is fulfilled. Clearly, if $Y$ is any solution, also $Y + a$ is a solution, for every constant element $a \in R$.  

A slightly different equation does not require analyticity of $\kappa$.

Existence and uniqueness of the solution could be deduced from some modifications of Theorem 6.8, but we have chosen a more direct approach.
**Theorem 6.10** Assume that \( \kappa : \mathbb{R} \to \mathbb{R} \) is any bounded function, and fix two elements in \( \mathbb{R} \), \( \alpha, \beta \), with \( \alpha \in U_1 \). Then there is a unique bounded function \( Y : \mathbb{R} \to \mathbb{R} \) satisfying \( Y(u) - \kappa(u) = \alpha Y(\beta u) \), for all \( u \in \mathbb{R} \).

**Proof.** Set \( Y(u) = \sum_{n=0}^{\infty} \kappa(\beta^n u) \alpha^n \). Clearly \( Y \) is well-defined, because \( \kappa \) is bounded. Moreover, \( Y \) is bounded and satisfies \( \alpha Y(\beta u) = Y(u) - \kappa(u) \), as required. Now, assume that another bounded function \( Y_1 : \mathbb{R} \to \mathbb{R} \) exists, satisfying the same equation. Then we must have \( \kappa(u) = Y_1(u) - \alpha Y_1(\beta u) \) and therefore

\[
Y(u) = \sum_{n=0}^{\infty} \kappa(\beta^n u) \alpha^n = \sum_{n=0}^{\infty} (Y_1(\beta^n u) - \alpha Y_1(\beta^{n+1} u)) \alpha^n \tag{6}
\]

for all \( u \in \mathbb{R} \) (absolute convergence of the series being ensured by boundedness of \( Y_1 \)). The conclusion is now obvious. \( \Box \)

We remark that the function \( Y \) here obtained is a generalization of the so-called *Weierstrass functions*, which are continuous but nowhere differentiable, and have self-similarity features. We also notice that uniqueness in the previous theorem is strictly related to boundedness of the function \( Y \): if we drop such condition, there may exist many different solutions. For example, let us assume \( \kappa = 0, \beta = 2 \ 1, \alpha = \frac{1}{2} \ 1 \); the equation then reduces to \( Y(2u) = 2Y(u) \): the solution given by Theorem 6.10 is identically 0, but every function of the type \( Y_1(u) = ru \) clearly is a solution.
(though unbounded), for every constant $r \in R$.

Finally, we turn to some kind of differential functional equation. For the sake of simplicity, we shall deal with a very particular type of equation, as described in the following theorem.

**Theorem 6.11** Fix any positive element $s \in R_1$. Then there exist non-trivial differentiable functions $Y : R_1 \to R$ satisfying the equation:

$$Y'(u) = Y(su)$$

for all $u \in R_1$.

**Proof.** First of all, let us observe that the space of solutions is a linear one. Next, put

$$Y(u) = \sum_{n=0}^{\infty} \frac{s^{(n^2-n)/2}}{n!} u^n, \quad u \in R_1$$

it is not difficult to deduce convergence of the series in $R_1$ and that the function $Y$ obtained in this way is non-trivial and satisfies (7).

**References**


