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## Sobczyk-Hammer decompositions and convergence theorems for measures with values in $l$ -groups

**Abstract.** We find a decomposition of the type of Sobczyk-Hammer for measures with values in  $l$ -groups, and also deduce some convergence theorems for such decompositions. Our procedure is based on some theorems of the type of Vitali-Hahn-Saks, and on the so-called Stone extension method.

### 1 Introduction

In [3] and [4], we obtained some versions of the Lebesgue decomposition theorem, and of the Vitali-Hahn-Saks theorem for finitely additive measures with values in (super) Dedekind complete  $l$ -groups. In the quoted papers, the notion of convergence was related to  $(D)$ -sequences and therefore many of the concepts and proofs appear somewhat complicated. In this paper we investigate a different kind of decomposition, and require that the involved  $l$ -group is super Dedekind complete and weakly  $\sigma$ -distributive: this allows us to avoid the machinery of  $(D)$ -convergence (see [2]), also thanks to a powerful result concerning suitable subsequences of an  $(O)$ -sequence (see Lemma 2.7), and hence all the relevant convergence properties can be formulated and proved only by means of  $(O)$ -sequences, which are a more natural tool.

The core of our research here are continuous and atomic  $l$ -group-valued measures: we obtain sufficient conditions for continuity and uniform continuity; then, by means of these results, we deduce a decomposition of a measure

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Key Words:  $l$ -groups,  $(RO)$ -convergence, uniform  $s$ -boundedness, uniform continuity, Vitali-Hahn-Saks theorems, Sobczyk-Hammer decomposition.

Mathematical Reviews subject classification: 28B15, 28B05, 28A39, 42C10, 42C25, 46G10.

in the sense of Sobczyk-Hammer, that is into its continuous and atomic part. We finally give a convergence theorem for such decompositions.

In Section 2 we give the definitions and state some preliminary results about the techniques to be used later. In Section 3 we introduce and study continuity properties of measures, obtaining a sufficient condition for a sequence of finitely additive measures to be uniformly continuous, and also some useful relations between continuity and absolute continuity. Finally, in Section 4 we introduce the sectional decompositions, and, thanks also to the previous theorems, we find some results of existence and of convergence for Sobczyk-Hammer decompositions.

## 2 Preliminary definitions and results

We shall introduce now the main definitions we need, together with some results.

**Definition 2.1.** An Abelian group  $(R, +)$  is called *l-group* if it is endowed with a compatible ordering  $\leq$ , and is a lattice with respect to it. An *l-group*  $R$  is said to be *Dedekind complete* if every nonempty subset of  $R$ , bounded from above, has supremum in  $R$ .

One important consequence of this definition is that convergence of *series* can be defined, at least when the terms are in  $R_0^+ = \{r \in R : r \geq 0\}$ .

**Definition 2.2.** Given any sequence  $(a_n)_n$  in  $R_0^+$ , we say that the *series*  $\sum_{n=1}^{\infty} a_n$  is *convergent* if the set of all partial sums  $\{s_n : n \in \mathbb{N}\}$  is bounded in  $R$ , where  $s_n = \sum_{i=1}^n a_i$  for all  $n$ . If this is the case, we set  $\sum_{n=1}^{\infty} a_n = \sup\{s_n : n \in \mathbb{N}\}$ .

Convergence of series is also related to the so-called *(O)*-convergence, according with the following definition.

**Definition 2.3.** Given a sequence  $(r_n)_n$  in  $R$ , we say that  $(r_n)_n$  *(O)-converges* to an element  $r \in R$  if there exists a sequence  $(p_n)_n$  in  $R$ , such that  $p_n \downarrow 0$  (such a sequence will be called an *(O)-sequence*), and satisfying

$$|r_n - r| \leq p_n \quad \forall n \in \mathbb{N}.$$

It is not difficult to see that a series  $\sum_{n=1}^{\infty} a_n$  ( $a_n \geq 0$ ) is convergent to some element  $s$  if and only if the sequence  $(s_n)_n$  *(O)*-converges to  $s$ .

We now introduce a first concept of  $\sigma$ -additivity, similar to the classical one (in the sequel we will slightly sharpen this concept).

**Definition 2.4.** Let  $R$  be a Dedekind complete  $l$ -group,  $\mathcal{F}$  be an algebra of subsets of a nonempty set  $X$  and  $m : \mathcal{F} \rightarrow R_0^+$  be a finitely additive measure. We say that  $m$  is *order  $\sigma$ -additive* if

$$m\left(\bigcup_{n=1}^{\infty} H_n\right) = \sum_{n=1}^{\infty} m(H_n)$$

whenever  $(H_n)_n$  is a disjoint sequence of elements of  $\mathcal{F}$ , such that  $\bigcup_{n=1}^{\infty} H_n \in \mathcal{F}$ .

In the sequel we shall assume further properties in our  $l$ -group, so we introduce now some definitions.

**Definitions 2.5.** A bounded double sequence  $(a_{i,j})_{i,j}$  in  $R$ , such that  $a_{i,j} \downarrow 0$  for each  $i \in \mathbb{N}$ , is called a *regulator* or *(D)-sequence*.

For every *(D)*-sequence  $(a_{i,j})_{i,j}$ , and every mapping  $\phi : \mathbb{N} \rightarrow \mathbb{N}$ , the element  $\bigvee_{i=1}^{\infty} a_{i,\phi(i)}$  is called a *domination* of the *(D)*-sequence.

From now on, we shall denote by  $\Phi$  the set of all mappings  $\phi : \mathbb{N} \rightarrow \mathbb{N}$ .

Now we can introduce the conditions we shall impose on  $R$ .

**Definitions 2.6.** We say that  $R$  is *weakly  $\sigma$ -distributive* if, for every *(D)*-sequence  $(a_{i,j})_{i,j}$ , the greatest lower bound of its dominations is 0, i.e.:

$$\bigwedge_{\phi \in \Phi} \left( \bigvee_{i=1}^{\infty} a_{i,\phi(i)} \right) = 0. \quad (1)$$

A Dedekind complete  $l$ -group  $R$  is said to be *super Dedekind complete* if for any nonempty set  $A \subset R$ , bounded from above, there exists a countable subset  $A^* \subset A$ , such that  $\sup A = \sup A^*$ .

From now on, we shall always assume that  $R$  is a super Dedekind complete and weakly  $\sigma$ -distributive  $l$ -group.

The following Lemma is a version of the *Fremlin Lemma* in the context of *(O)*-sequences: though it is possible to prove it as a consequence of the Fremlin-type Theorem 3.2.3, p. 42 of [6], we give here a direct proof, because it looks somewhat easier.

**Lemma 2.7.** *Let  $(r_n)_n$  be any *(O)*-sequence in  $R_0^+$ . Then, for every  $U \in R_0^+$  there exists an element  $\omega \in \Phi$  such that the mapping*

$$N \mapsto U \wedge \sum_{n=N}^{\infty} r_{\omega(n)}$$

*is an *(O)*-sequence.*

**Proof.** For any couple  $(i, k)$  of positive integers, let us define:

$$A_{i,k} := U \wedge \left( \sum_{n=k}^{k+2^i-1} r_n \right).$$

Clearly,  $(A_{i,k})_{i,k}$  is a  $(D)$ -sequence. Next, for every element  $\phi \in \Phi$ , let us set

$$d_\phi := \bigvee_{i=1}^{\infty} A_{i,\phi(i)}.$$

Since  $R$  is weakly  $\sigma$ -distributive and super Dedekind complete, there exists a sequence  $(\phi_h)_h$  in  $\Phi$ , such that

$$\inf_h d_{\phi_h} = 0.$$

Without loss of generality, we shall assume that

$$\phi_h(n) < \phi_h(n+1), \quad \text{and} \quad \phi_h(n) < \phi_{h+1}(n)$$

for every  $h$  and  $n$  in  $\mathbb{N}$ , so that

$$N \mapsto g_N := d_{\phi_N}$$

defines an  $(O)$ -sequence. Thus we have:

$$g_1 \geq U \wedge \{r_{\phi_1(1)} + r_{\phi_1(1)+1}\}$$

and also

$$g_1 \geq U \wedge \{r_{\phi_1(2)} + r_{\phi_1(2)+1} + r_{\phi_1(2)+2} + r_{\phi_1(2)+3}\},$$

and so on. Now we observe that

$$U \wedge 2r_{\phi_1(1)+1} \leq g_1, \quad U \wedge (r_{\phi_1(1)+1} + 2r_{\phi_1(2)+3}) \leq g_1,$$

$$U \wedge (r_{\phi_1(1)+1} + r_{\phi_1(2)+3} + 2r_{\phi_1(3)+7}) \leq g_1$$

and so on. Then

$$U \wedge \left( \sum_{n=1}^k r_{\phi_1(n)+2^n-1} \right) \leq g_1$$

holds, for every positive integer  $k$ , hence

$$U \wedge \left( \sum_{n=1}^{\infty} r_{\phi_1(n)+2^n-1} \right) \leq g_1.$$

In a similar way, one proves that

$$U \wedge \left( \sum_{n=1}^{\infty} r_{\phi_N(n)+2^n-1} \right) \leq g_N$$

holds, for every positive integer  $N$ . Now, we set

$$\omega(N) := \phi_N(N) + 2^N - 1.$$

For every natural number  $k$ , we have

$$\begin{aligned} U \wedge \left( \sum_{n=N}^{N+k} r_{\omega(n)} \right) &= U \wedge \left( \sum_{n=N}^{N+k} r_{\phi_n(n)+2^n-1} \right) \leq \\ &\leq U \wedge \left( \sum_{n=N}^{N+k} r_{\phi_N(n)+2^n-1} \right) \leq U \wedge \left( \sum_{n=1}^{\infty} r_{\phi_N(n)+2^n-1} \right) \leq g_N. \end{aligned}$$

From arbitrariness of  $k$ , we obtain the assertion.  $\square$

The next result expresses the fact that, as soon as  $\{(r_n^{(k)})_n : k \in \mathbb{N}\}$  is an equibounded countable family of  $(O)$ -sequences, there exists a single  $(O)$ -sequence, which can replace them all. More precisely, we have the following:

**Lemma 2.8.** *Let  $\{(r_n^{(k)})_n : k \in \mathbb{N}\}$  be an equibounded countable family of  $(O)$ -sequences. Then there exists an  $(O)$ -sequence  $(b_j)_j$  with the following property:*

*For every  $j, k \in \mathbb{N}$ , an integer  $n = n(j, k) > 0$  exists, such that*

$$r_n^{(k)} \leq b_j.$$

**Proof.** Let us define

$$a_{k,n} = r_n^{(k)}$$

for each  $k, n$ . Clearly,  $(a_{k,n})_{k,n}$  is a  $D$ -sequence, hence there exists a sequence  $(\phi_j)_j$  in  $\Phi$ , such that

$$j \mapsto b_j := \bigvee_{n=1}^{\infty} a_{n, \phi_j(n)}$$

defines an  $(O)$ -sequence. As above, we can assume that

$$\phi_j(n) < \phi_j(n+1), \quad \phi_j(n) < \phi_{j+1}(n)$$

for all  $j, n$ . Now, fix arbitrarily  $j$  and  $k$ , and choose  $n = n(j, k) = \phi_j(k)$ . We have then

$$r_n^{(k)} = a_{k, \phi_j(k)} \leq \bigvee_{k=1}^{\infty} a_{k, \phi_j(k)} = b_j,$$

i.e. the assertion.  $\square$

From now on, we denote by  $\mathcal{F}$  an algebra and by  $\mathcal{A}$  a  $\sigma$ -algebra of subsets of a nonempty arbitrary set  $X$ .

We now introduce the concept of  $s$ -boundedness in the context of  $(O)$ -convergence.

**Definition 2.9.** A finitely additive measure  $m : \mathcal{F} \rightarrow R$  is said to be  $s$ -bounded if there exists an  $(O)$ -sequence  $(b_j)_j$  such that, for every disjoint sequence  $(H_k)_k$  in  $\mathcal{F}$  and every index  $j \in \mathbb{N}$ , one can find an integer  $k_0$  satisfying

$$|m(H_k)| \leq b_j \text{ for each } k \in \mathbb{N}, k \geq k_0. \quad (2)$$

We say that the measures  $m_n : \mathcal{F} \rightarrow R$ ,  $n \in \mathbb{N}$ , are *uniformly  $s$ -bounded* if the integer  $k_0$  in (2) can be chosen independently on  $n$ .

We now introduce the notion of  $\sigma$ -additivity in the context of  $(O)$ -convergence: this concept is in general stronger than the classical one of order  $\sigma$ -additivity.

**Definition 2.10.** Let  $m : \mathcal{F} \rightarrow R$  be a finitely additive measure. We say that  $m$  is  $\sigma$ -additive if there exists an  $(O)$ -sequence  $(b_j)_j$  such that, for every sequence  $(H_k)_k$  in  $\mathcal{F}$ , decreasing to the empty set, and for every positive integer  $j$ , there exists a natural number  $k_0$  satisfying

$$|m(B)| \leq b_j \text{ for every } B \in \mathcal{F}, B \subset H_{k_0}. \quad (3)$$

A similar definition concerns *uniform  $\sigma$ -additivity* for a family  $\{m_i : \mathcal{F} \rightarrow R\}_i$  of measures.

We observe that, in case of an equibounded sequence  $(m_n)_n$  of  $\sigma$ -additive measures, it is possible to find a unique  $(O)$ -sequence  $(b_j)_j$  which is related to  $\sigma$ -additivity of all  $m'_n$ 's: this is a consequence of 2.8. The same holds also for other properties, like continuity, that will be introduced later.

From the extension theorems found in [4], it is possible to deduce that any  $s$ -bounded positive order  $\sigma$ -additive measure is also  $\sigma$ -additive.

Some of these extension theorems will now be recalled, for completeness, and also for further reference.

We first introduce a notation: given an algebra  $\mathcal{F}$  of subsets of any non-empty set  $X$ , we shall denote by  $\sigma(\mathcal{F})$  the  $\sigma$ -algebra generated by  $\mathcal{F}$ .

A stronger notion of convergence is also needed.

**Definition 2.11.** Let  $T$  be any nonempty set, and  $(f_n : T \rightarrow R)_n$  be any sequence of functions. We say that the sequence  $(f_n)_n$  is *(RO)-convergent* to a limit function  $f$  if there exists an *(O)*-sequence  $(p_j)_j$  in  $R$  such that, for every positive integer  $j$  and every element  $t \in T$ , a natural number  $n_0$  can be found,  $n_0 = n_0(j, t)$ , for which

$$|f_n(t) - f(t)| \leq p_j \quad \forall n \geq n_0.$$

This definition will be mainly used for sequences of  $R$ -valued *measures*.

The theorems we shall list deal mainly with the so-called *Stone Isomorphism* technique: the well-known Stone Isomorphism Theorem asserts that any Boolean algebra  $\mathcal{F}$  is algebraically isomorphic with the algebra  $\Sigma$  of clopen sets in a suitable compact, totally disconnected, Hausdorff space  $S$ . Denoting by  $\psi : \mathcal{F} \rightarrow \Sigma$  such isomorphism, any finitely additive measure  $m : \mathcal{F} \rightarrow R$  can be associated with the measure  $m \circ \psi^{-1} : \Sigma \rightarrow R$ . Since  $\Sigma$  turns out to be *perfect*, any finitely additive measure on  $\Sigma$  is also order  $\sigma$ -additive.

Thus a suitable extension procedure, inspired at Carathéodory's construction, yields the following theorem (see [4]).

**Theorem 2.12.** *Let  $m : \mathcal{F} \rightarrow R_0^+$  be any finitely additive,  $s$ -bounded measure. Then there exists a  $\sigma$ -additive measure  $\tilde{m} : \sigma(\Sigma) \rightarrow R_0^+$  such that*

$$\tilde{m}|_{\Sigma} = m \circ \psi^{-1}.$$

Moreover, there exists a suitable *(O)*-sequence  $(b_j)_j$  in  $R$ , such that, for every element  $A \in \sigma(\Sigma)$  and each positive integer  $j$ , an element  $F \in \mathcal{F}$  can be found, satisfying

$$\tilde{m}(A \Delta \psi(F)) \leq b_j.$$

Combining this theorem with a convergence theorem of the type of Vitali-Hahn-Saks (see [3], [4]), the following result can be deduced:

**Theorem 2.13.** *Let  $(m_n : \mathcal{A} \rightarrow R)_n$  be an equibounded sequence of  $s$ -bounded finitely additive measures, defined on a  $\sigma$ -algebra  $\mathcal{A}$ . Assuming that the sequence  $(m_n(A))_n$  is *(RO)-convergent* to a limit  $m(A)$ ,  $\forall A \in \mathcal{A}$ , then*

*i) the sequence  $(m_n)_n$  is uniformly  $s$ -bounded, and therefore  $m$  is  $s$ -bounded too;*

*ii) the sequence  $(\tilde{m}_n(B))_n$  is *(RO)-convergent* to  $\tilde{m}(B)$ ,  $\forall B \in \sigma(\Sigma)$ , where as usual  $\Sigma$  denotes the Stone algebra isomorphic with  $\mathcal{A}$ , and  $\tilde{m}_n, \tilde{m}$  are the Stone extensions to the  $\sigma$ -algebra  $\sigma(\Sigma)$  of  $m_n$  and  $m$  respectively.*

### 3 Continuous and atomic measures

We now introduce the concept of continuity for  $l$ -group-valued measures (see also [7], [1]).

**Definition 3.1.** We say that a finitely additive measure  $m : \mathcal{F} \rightarrow R_0^+$  is *continuous* if

$$\inf_{P \in \Pi} \left[ \sup_{D \in P} m(D) \right] = 0,$$

where  $P$  is any finite partition of  $X$ , and the infimum is taken with respect to the totality  $\Pi$  of such partitions. The finitely additive measures  $m_n : \mathcal{F} \rightarrow R_0^+$ ,  $n \in \mathbb{N}$ , are said to be *uniformly continuous* if

$$\inf_{P \in \Pi} \left[ \sup_{D \in P} \left( \sup_n m_n(D) \right) \right] = 0.$$

The following result is a characterization of continuity for finitely additive positive measures.

**Proposition 3.2.** *A finitely additive measure  $m : \mathcal{F} \rightarrow R_0^+$  is continuous if and only if there exists an  $(O)$ -sequence  $(b_j)_j$  in  $R$  such that for all  $j \in \mathbb{N}$  there exists a finite partition  $Q_j$  of  $X$  into sets  $D_1, \dots, D_{h_j}$ , for which*

$$m(D_i) \leq b_j \quad \forall i = 1, \dots, h_j. \quad (4)$$

*Analogously, the finitely additive measures  $m_n : \mathcal{F} \rightarrow R_0^+$ ,  $n \in \mathbb{N}$ , are uniformly continuous if and only if there exists an  $(O)$ -sequence  $(b_j)_j$  in  $R$  such that for all  $j \in \mathbb{N}$  there exists a finite partition  $Q_j$  of  $X$  into sets  $D_1, \dots, D_{h_j}$ , satisfying*

$$\sup_n m_n(D_i) \leq b_j \quad \forall i = 1, \dots, h_j.$$

**Proof.** We give the proof only for the first part of the proposition. Let  $m$  be a positive continuous finitely additive measure. Since  $R$  is super Dedekind complete, then there exists a sequence  $(Q_j)_j$  of finite partitions of  $X$ , such that

$$\inf_j \left[ \sup_{D \in Q_j} m(D) \right] = 0.$$

Without loss of generality, we can assume that the sequence  $(Q_j)_j$  is increasing with respect to the refinement order: i.e. we assume that, for every positive integer  $j$ , each element in  $Q_j$  is the union of some elements from  $Q_{j+1}$ .

For every index  $j \in \mathbb{N}$ , define  $b_j := \sup_{D \in Q_j} m(D)$ . Thus,  $(b_j)_j$  turns out to be an  $(O)$ -sequence, and the sequence  $(Q_j)_j$  is the required one.

Conversely, let  $(Q_j)_j$  be a sequence of finite partitions of  $X$ , satisfying (4). Then we get:

$$\inf_j \left[ \sup_{D \in Q_j} m(D) \right] = 0$$

and, a fortiori,

$$\inf_{P \in \Pi} \left[ \sup_{D \in P} m(D) \right] = 0,$$

that is continuity of  $m$ .  $\square$

**Definition 3.3.** A finitely additive measure  $m : \mathcal{A} \rightarrow R_0^+$  is said to be *atomic* if 0 is the unique finitely additive and continuous measure  $\nu : \mathcal{A} \rightarrow R_0^+$ , satisfying  $\nu \leq m$ .

We note that the given definition of atomic measure is coherent with the classical one of atomic  $\mathbb{R}_0^+$ -valued measure, by virtue of the Sobczyk-Hammer decomposition theorem for scalar measures (see [7]).

In the sequel, a suitable notion of absolute continuity is needed. We shall introduce it now, only for  $\sigma$ -additive measures, and shall then deduce a useful result about continuity of measures.

**Definition 3.4.** Let  $\nu : \mathcal{A} \rightarrow R_0^+$  be any  $\sigma$ -additive measure, defined on a  $\sigma$ -algebra  $\mathcal{A}$ . Given any other  $\sigma$ -additive measure  $m : \mathcal{A} \rightarrow R_0^+$ , we say that  $m$  is *absolutely continuous* with respect to  $\nu$  if the following implication holds:

$$\nu(A) = 0 \Rightarrow m(A) = 0.$$

Here is the theorem concerning continuity.

**Theorem 3.5.** Let  $m, \nu : \mathcal{A} \rightarrow R_0^+$ , and assume that  $m, \nu$  are  $\sigma$ -additive,  $m$  is absolutely continuous with respect to  $\nu$  and  $\nu$  is continuous. Then  $m$  is continuous.

**Proof.** Set  $U := m(X) + \nu(X)$ . Let us fix an  $(O)$ -sequence  $(b_j)_j$  in  $R$ , according with  $\sigma$ -additivity of  $m$ , with  $\sigma$ -additivity and with continuity of  $\nu$ . Thanks to Lemma 2.7, there exists a subsequence  $(b_{j_k})_k$  such that  $(\rho_N)_N$  is an  $(O)$ -sequence, where

$$\rho_N := U \wedge \sum_{k \geq N} b_{j_k}$$

for all  $N$ . Let us set  $b'_N := b_{j_N}$ , and define

$$\alpha_N := \sup\{m(A) : \nu(A) \leq b'_N\}, \quad \forall N \in \mathbb{N} :$$

we shall prove that  $(\alpha_N)_N$  is an  $(O)$ -sequence. Clearly,  $(\alpha_N)_N$  is decreasing, so all we must show is that  $\alpha := \inf_N \alpha_N$  coincides with 0.

If this is not the case, then there exists an integer  $N$  such that  $\alpha \not\leq b_N$ , hence  $\alpha_k \not\leq b_N \forall k$ . This means that, for every natural number  $k$ , an element  $A_k \in \mathcal{A}$  can be found, such that

$$\nu(A_k) \leq b'_k, \text{ but } m(A_k) \not\leq b_N.$$

For all positive integers  $s$ , let us define

$$A_s^* := \bigcup_{k \geq s} A_k :$$

thus we get

$$\nu(A_s^*) \leq U \wedge \sum_{k \geq s} b'_k \leq \rho_s,$$

but

$$m(A_s^*) \not\leq b_N, \tag{5}$$

for all  $s$ . The sequence  $(A_s^*)_s$  is decreasing: denoting by  $A$  its limit, we get  $\nu(A) = 0$  by  $\sigma$ -additivity, and then  $m(A) = 0$ . Hence, the sequence  $(A_s^* \setminus A)_s$  is decreasing to  $\emptyset$ , and  $m(A_s^* \setminus A) = m(A_s^*)$  for all  $s$ . Thus, by  $\sigma$ -additivity, a positive integer  $\tau_N$  can be found, satisfying  $m(A_{\tau_N}^*) \leq b_N$ : but this is contrary to (5), then we must conclude that  $\alpha = 0$ .

Now, we can easily prove continuity of  $m$ . Indeed, for each  $N \in \mathbb{N}$ , a partition  $P$  of  $X$  exists, such that  $\nu(D) \leq b'_N$  for all  $D \in P$ . Then  $m(D) \leq \alpha_N$  for all  $D \in P$ , and the assertion is proved.  $\square$

We now prove that, for continuous equibounded finitely additive measures, uniform  $s$ -boundedness is a sufficient condition for uniform continuity. To this aim, we begin with the following definition.

**Definition 3.6.** Let  $m : \mathcal{F} \rightarrow R_0^+$  be any set function, and fix  $u \in R_0^+$ . Given any set  $A \in \mathcal{F}$ , we say that  $A$  is *u-decomposable* (with respect to  $m$ ) if there exists a finite partition of  $A$  into sets  $D_1, \dots, D_k$  of  $\mathcal{F}$ , such that  $m(D_i) \leq u$  for all  $i = 1, \dots, k$ . Thus we get that a positive finitely additive measure  $m$  is continuous if and only if there exists an  $(O)$ -sequence  $(r_j)_j$  in  $R$  such that, for each  $j \in \mathbb{N}$ , the set  $X$  is  $r_j$ -decomposable (see also Proposition 3.2).

Let  $(m_n : \mathcal{F} \rightarrow R_0^+)_n$  be a sequence of finitely additive equibounded measures. For every  $A \in \mathcal{F}$ , set

$$M(A) := \sup_{n \in \mathbb{N}} m_n(A).$$

Given  $A \in \mathcal{F}$ , we say that  $A$  is *uniformly  $u$ -decomposable* if it is  $u$ -decomposable with respect to  $M$ .

Thus we get that the  $m_n$ 's are uniformly continuous if and only if there exists an  $(O)$ -sequence  $(r_j)_j$  in  $R$  such that, for each  $j \in \mathbb{N}$ , the set  $X$  is uniformly  $r_j$ -decomposable.

We also need the following lemma.

**Lemma 3.7.** *Let  $(m_n : \mathcal{F} \rightarrow R_0^+)_n$  be a sequence of continuous, equibounded and uniformly  $s$ -bounded measures, and let  $(r_j)_j$  be an  $(O)$ -sequence, according with uniform  $s$ -boundedness and continuity of each measure  $m_n$ . Assume that, for some  $j \in \mathbb{N}$ , there exists  $A \in \mathcal{F}$ , with  $M(A) \not\leq 2r_j$ . Then there exists  $H \subset A$ ,  $H \in \mathcal{F}$ , uniformly  $r_j$ -decomposable and such that  $M(H) \not\leq 2r_j$ .*

**Proof.** Let  $(r_j)_j$  be as in the hypotheses: we note that such a sequence does exist, by virtue of Lemma 2.8 and equiboundedness. Without loss of generality, we can suppose  $j = 1$ , and set  $r = r_1$ . By contradiction, suppose that there exists  $A \in \mathcal{F}$ , with  $M(A) \not\leq 2r$ , such that there are no uniformly  $r$ -decomposable subsets  $H \in \mathcal{F}$ , with  $M(H) \not\leq 2r$ . Without loss of generality, we can suppose that  $m_1(A) \not\leq 2r$ . By continuity of  $m_1$ , there exists a partition  $\mathcal{D}_A$  of  $A$  into sets  $D_1, \dots, D_k$ , such that  $m_1(D_i) \leq r$  for all  $i = 1, \dots, k$ . By virtue of the assumed contradiction, we get  $M(D_{k_1}) \not\leq 2r$  for some index  $k_1 \leq k$ , and thus, in correspondence with  $k_1$ , there exists an integer  $n_1 > 1$  such that  $m_{n_1}(D_{k_1}) \not\leq 2r$ . Set  $A_1 := D_{k_1}$  and  $B_1 := A \setminus A_1$ . By difference, we get:

$$m_1(B_1) \not\leq r.$$

By continuity of  $m_1, m_2, \dots, m_{n_1}$ , there exists a partition  $\mathcal{D}$  of  $A_1$  into sets  $D'_1, D'_2, \dots, D'_s$ , such that

$$m_1(D'_i) \vee m_2(D'_i) \vee \dots \vee m_{n_1}(D'_i) \leq r$$

for each  $i = 1, \dots, s$ . Again by contradiction, it follows:  $M(D'_{k_2}) \not\leq 2r$  for a suitable index  $k_2 \leq s$ , and hence there exists an integer  $n_2 > n_1$  such that  $m_{n_2}(D'_{k_2}) \not\leq 2r$ . Put  $A_2 := D'_{k_2}$ , and  $B_2 := A_1 \setminus A_2$ . By difference, we get:

$$m_{n_1}(B_2) \not\leq r.$$

By virtue of continuity of  $m_1, m_2, \dots, m_{n_2}$ , we find a partition of  $A_2$  and a set  $A_3 \subset A_2$ ,  $A_3 \in \mathcal{F}$ , such that  $M(A_3) \not\leq 2r$  and  $m_{n_2}(A_3) \leq r$ . Setting  $B_3 := A_2 \setminus A_3$ , by difference we have:

$$m_{n_2}(B_3) \not\leq r;$$

proceeding in this way, we find an increasing sequence of natural numbers  $(n_h)_h$  and a decreasing sequence of sets  $(A_h)_h$  in  $\mathcal{F}$ , such that

$$m_{n_h}(B_{h+1}) \not\leq r$$

for all  $h \in \mathbb{N}$ , where  $(B_h := A_h \setminus A_{h+1})_h$  is a disjoint sequence in  $\mathcal{F}$ . This contradicts uniform  $s$ -boundedness, and thus the lemma is completely proved.  $\square$

We finally turn to the announced useful result.

**Theorem 3.8.** *Under the same notations as in Lemma 3.7, if the  $m_n$ 's are positive, continuous, equibounded and uniformly  $s$ -bounded finitely additive  $\mathbb{R}$ -valued measures, then they are uniformly continuous.*

**Proof.** By virtue of equiboundedness and thanks to Lemma 2.8, there exists an  $(O)$ -sequence  $(r_j)_j$ , according both with uniform  $s$ -boundedness and with continuity of each measure  $m_n$ . To prove the theorem, we shall show that, for every  $j \in \mathbb{N}$ ,  $X$  is uniformly  $2r_j$ -decomposable. Fix any positive integer  $j$ , and set  $r = r_j$ . Suppose by contradiction that  $X$  is not uniformly  $2r$ -decomposable. (From now on in this proof, the term "decomposable" will always mean "uniformly decomposable"). Then  $M(X) \not\leq 2r$ , and thus, by Lemma 3.7, there exists a set  $A_1 \in \mathcal{F}$ , with  $M(A_1) \not\leq 2r$ , but  $r$ -decomposable. By the assumed contradiction,  $X \setminus A_1$  is not  $2r$ -decomposable, and hence we have  $M(X \setminus A_1) \not\leq 2r$ . Again by Lemma 3.7, there exists an  $r$ -decomposable set  $A_2 \subset X \setminus A_1$ , with  $A_2 \in \mathcal{F}$ , such that  $M(A_2) \not\leq 2r$ . Since  $A_1$  and  $A_2$  are  $r$ -decomposable and  $X$  is not, then  $X \setminus (A_1 \cup A_2)$  is not  $2r$ -decomposable, and thus, again by Lemma 3.7, there exists an  $r$ -decomposable set  $A_3$  in  $\mathcal{F}$ , disjoint both from  $A_1$  and from  $A_2$ , such that  $M(A_3) \not\leq 2r$ . Thus we get the existence of a sequence  $(A_k)_k$  of pairwise disjoint  $r$ -decomposable sets of  $\mathcal{F}$ , such that

$$M(A_k) \not\leq 2r \quad \forall k \in \mathbb{N}.$$

This contradicts uniform  $s$ -boundedness of the measures  $m_n$ , and thus  $X$  is  $2r$ -decomposable. So the assertion follows.  $\square$

## 4 Sobczyk-Hammer decompositions

In this section, we shall deduce existence and convergence theorems for Sobczyk-Hammer decompositions, first for  $\sigma$ -additive measures, and then for finitely additive ones.

For  $\sigma$ -additive measures  $m$ , we shall obtain a decomposition of *sectional* type, i.e. we shall find a suitable set  $H \in \mathcal{A}$ , such that the restrictions of  $m$  to  $H$  and to  $H^c$  are continuous and atomic respectively.

We begin with a lemma.

**Lemma 4.1.** *Let  $m : \mathcal{A} \rightarrow R$  be a positive  $\sigma$ -additive measure. If  $m$  is not atomic, then there exists at least a set  $F \in \mathcal{A}$  with  $m(F) \neq 0$ , such that  $m|_F$  is continuous.*

**Proof.** Since  $m$  is not atomic, then there exists a continuous non-trivial measure  $\mu : \mathcal{A} \rightarrow R_0^+$ , such that  $\mu \leq m$ . It is readily seen that  $\mu$  is  $\sigma$ -additive. Let now  $\mathcal{H}$  be the family of all sets  $H \in \mathcal{A}$  such that  $\mu(H) = 0$ , and define

$$\alpha = \sup\{m(H) : H \in \mathcal{H}\}. \quad (6)$$

By super Dedekind completeness, there exists a sequence  $(K_n)_n$  in  $\mathcal{H}$ , such that  $\alpha = \sup_n m(K_n)$ . Without loss of generality, we can assume that the sequence  $(K_n)_n$  is non-decreasing: then,  $m(K) = \alpha$  and  $\mu(K) = 0$ , where  $K = \bigcup_{n=1}^{\infty} K_n$ .

Put  $F = K^c$ , and let us show that  $F$  is the required set. First of all, it is easy to check that the measure  $m|_F$  is absolutely continuous with respect to  $\mu$ . Indeed, if there existed a set  $G \in \mathcal{A}$  such that  $\mu(G) = 0$  and  $m|_F(G) > 0$ , then

$$m[(G \cap F) \cup K] = m(G \cap F) + m(K) > \alpha,$$

but

$$\mu[(G \cap F) \cup K] = \mu(G \cap F) + \mu(K) = 0,$$

and therefore  $\alpha$  could not be the supremum in (6), a contradiction. Thus, by virtue of Theorem 3.5,  $m|_F$  is continuous. Finally we observe that  $m(F) \neq 0$ , otherwise  $\mu \equiv 0$ , which is impossible.  $\square$

A useful consequence is the following.

**Proposition 4.2.** *Let  $m_1$  and  $m_2$  be two atomic (resp.: continuous)  $\sigma$ -additive positive  $R$ -valued measures, defined on  $\mathcal{A}$ . Then  $m_1 + m_2$  is atomic (resp.: continuous).*

**Proof.** The assertion concerning continuous measures is easy, so we deal only with the atomic case. Let us assume that  $m_1$  and  $m_2$  are atomic, and that  $m_1 + m_2$  is not. Then there exists a set  $F$  according with the previous Lemma. Thus  $m_1|_F$  is continuous, and hence null. Similarly,  $m_2|_F$  is null, hence we obtain  $(m_1 + m_2)|_F = 0$ , i.e. a contradiction.  $\square$

We now turn to the following Sobczyk-Hammer-type theorem.

**Theorem 4.3.** *Let  $m : \mathcal{A} \rightarrow R$  be a  $\sigma$ -additive positive measure. Then there exists a set  $E \in \mathcal{A}$ , such that  $m|_E$  is continuous and  $m|_{E^c}$  is atomic.*

**Proof.** If  $m$  is atomic, it is enough to take  $E = \emptyset$ . Otherwise, by virtue of Lemma 4.1, there exists  $F \in \mathcal{A}$  such that  $m(F) \neq 0$  and  $m|_F$  is continuous. Denote by  $\mathcal{K}$  the family of such sets, and write

$$\alpha := \sup\{m(F) : F \in \mathcal{K}\}.$$

By super Dedekind completeness of  $R$ , there exists an increasing sequence  $(F_n)_n$  in  $\mathcal{K}$ , such that  $\sup_n m(F_n) = \alpha$ . Put  $E = \bigcup_{n=1}^{\infty} F_n$ . We prove that  $E$  is the requested set. First of all, we note that

$$m|_E(A) = \sup_n m|_{F_n}(A) \tag{7}$$

for all  $A \in \mathcal{A}$ . Moreover, it is easy to check that the  $m|_{F_n}$ 's,  $n \in \mathbb{N}$ , are uniformly  $s$ -bounded. By Theorem 3.8, these measures are uniformly continuous, and hence, by (7),  $m|_E$  is  $\sigma$ -additive and continuous.

Finally we prove that  $m|_{E^c}$  is atomic. Otherwise, by Theorem 4.1, there exists a set  $H \in \mathcal{A}$ , such that  $m(H) \neq 0$ ,  $H \cap E = \emptyset$  and  $m|_H$  is continuous. Then  $H \cup E \in \mathcal{K}$  and  $m(H \cup E) > \alpha$ , a contradiction. This concludes the proof.  $\square$

We now prove some convergence theorems for Sobczyk-Hammer-type decompositions.

First of all, we consider the case of positive,  $\sigma$ -additive,  $(RO)$ -convergent measures. We recall that a sequence  $(m_k)_k$  of finitely additive measures, defined on an algebra  $\mathcal{F}$  and with values in  $R$ , is  $(RO)$ -convergent to a measure  $m$ , if there exists an  $(O)$ -sequence  $(r_n)_n$  such that, for all  $A \in \mathcal{F}$  and  $n \in \mathbb{N}$ , an integer  $k_0$  can be found, such that

$$|m_k(A) - m(A)| \leq r_n$$

for all  $k \geq k_0$ .

In order to give convergence theorems for decompositions, we first prove the following:

**Proposition 4.4.** *Let  $(m_n)_n$  be a sequence of  $\sigma$ -additive positive  $R$ -valued measures, defined on a  $\sigma$ -algebra  $\mathcal{A}$ . If the  $m_n$ 's are equibounded, then there exists  $H \in \mathcal{A}$  such that*

$$m_n|_H \text{ is continuous; } m_n|_{H^c} \text{ is atomic}$$

for every  $n \in \mathbb{N}$ .

**Proof.** For any positive integer  $N$ , denote by  $\mu_N$  the measure:

$$\mu_N := \sum_{i=1}^N m_i,$$

and let  $(A_N, A_N^c)$  be a Sobczyk-Hammer sectional decomposition of  $\mu_N$  (see Theorem 4.3). Set now, for positive integers  $N, L$ , with  $N \leq L$ :

$$B_{N,L} := \bigcup_{p=0}^{L-N} A_{N+p}, \quad B_N := \bigcup_{j=N}^{\infty} B_{N,j} = \bigcup_{j=N}^{\infty} A_j :$$

we have

$$m_{N|B_{N,L}} \leq \sum_{p=0}^{L-N} (\mu_{N+p})_{|A_{N+p}}$$

and hence  $m_{N|B_{N,L}}$  is continuous. Letting  $L \rightarrow \infty$ , we deduce, by uniform  $s$ -boundedness, that the measure  $m_{N|B_N}$  is continuous (see Theorem 3.8). Clearly,  $\mu_{N|B_N^c} \leq \mu_{N|A_N^c}$  is atomic, for every  $N$ . Set now

$$H := \bigcap_{N=1}^{\infty} B_N :$$

we shall show that  $H$  is the requested set. Indeed, for every  $N \in \mathbb{N}$ , we have  $m_{N|H} \leq m_{N|B_N}$ , hence  $m_{N|H}$  is continuous.

We finally prove that  $m_{N|H^c}$  is atomic, for all  $N$ . To this aim, fix any integer  $N \in \mathbb{N}$ , and let  $\beta$  be any positive measure,  $\beta \leq m_{N|H^c}$ : we deduce easily that  $\beta_{|B_{N+p}^c} \leq m_{N|B_{N+p}^c} \leq (\mu_{N+p})_{|B_{N+p}^c}$  for all  $p \in \mathbb{N}$ , and hence  $\beta_{|B_{N+p}^c}$  is null, by atomicity of  $(\mu_{N+p})_{|B_{N+p}^c}$ . But  $\beta_{|H^c} = (RO) \lim_{p \rightarrow \infty} \beta_{|B_{N+p}^c}$ , hence  $\beta$  is null. This concludes the proof, by arbitrariness of  $N$ .  $\square$

We now prove a first convergence theorem for Sobczyk-Hammer-type decompositions.

**Theorem 4.5.** *Let  $(m_n)_n$  be a sequence of  $\sigma$ -additive positive  $R$ -valued measures, defined on a  $\sigma$ -algebra  $\mathcal{A}$ . Suppose that the  $m_n$ 's are equibounded and  $(RO)$ -convergent to a measure  $m$ . Then  $m$  is  $\sigma$ -additive, and the sequences  $(m_n^1)_n$  and  $(m_n^2)_n$  are  $(RO)$ -convergent to the measures  $m^1$  and  $m^2$ , where  $(m_n^1, m_n^2), (m^1, m^2)$  are the sectional Sobczyk-Hammer decompositions of  $m_n$  and  $m$  respectively,  $n \in \mathbb{N}$ .*

**Proof.** By virtue of the Vitali-Hahn-Saks theorem (see [4]), the measures  $m_n$  are uniformly  $s$ -bounded, and hence the limit measure  $m$  is  $\sigma$ -additive. By applying Proposition 4.4 to the sequence  $(m_n)_n$ , we get the existence of a set  $H \in \mathcal{A}$ , which yields a sectional decomposition of the measures  $m_n$ ,  $n \in \mathbb{N}$ , and of the limit measure  $m$ . The assertion follows immediately from  $(RO)$ -convergence.  $\square$

We now turn to the finitely additive case.

**Theorem 4.6.** *Let  $m : \mathcal{F} \rightarrow R$  be any positive finitely additive  $s$ -bounded measure, defined on an algebra  $\mathcal{F}$ . Then there exists a decomposition  $m = m^1 + m^2$  of  $m$  into two positive finitely additive measures, such that  $m^1$  is continuous and  $m^2$  is atomic.*

**Proof.** We make use of the Stone Isomorphism, namely we consider the Stone space  $S$  associated with  $\mathcal{F}$ , and the algebraic isomorphism  $\psi$  from  $\mathcal{F}$  to the algebra  $\Sigma$  of all clopen subsets of  $S$ . From Theorem 2.12, we know that the measure  $m^0 := m \circ \psi^{-1}$  can be extended to a  $\sigma$ -additive measure  $\tilde{m}$  on  $\sigma(\Sigma)$ . Using Theorem 4.4, we can decompose  $\tilde{m}$  into the sum  $\tilde{m}^1 + \tilde{m}^2$  in the sense of Sobczyk-Hammer, where  $\tilde{m}^1$  is continuous and  $\tilde{m}^2$  atomic. Now, we restrict  $\tilde{m}^1$  and  $\tilde{m}^2$  to the algebra  $\Sigma$ , thus obtaining two measures, which we denote by  $m_S^1$  and  $m_S^2$  respectively.

We shall see that  $m_S^1$  and  $m_S^2$  are continuous and atomic respectively: from this, it will follow immediately that the measures

$$m^1 := m_S^1 \circ \psi \quad \text{and} \quad m^2 := m_S^2 \circ \psi$$

give the requested decomposition of  $m$ . Let us prove that  $m_S^1$  is continuous. By continuity of  $\tilde{m}^1$ , there exists an  $(O)$ -sequence  $(a_j)_j$  such that, for all  $j \in \mathbb{N}$ , a finite partition  $\{D_1, \dots, D_{h_j}\}$  in  $\sigma(\Sigma)$  can be found, satisfying

$$\tilde{m}^1(D_k) \leq a_j \quad \forall k = 1, \dots, h_j.$$

Now, by virtue of Theorem 2.12, there exists an  $(O)$ -sequence  $(b_j)_j$  such that, for all  $D \in \sigma(\Sigma)$  and  $j \in \mathbb{N}$ , it is possible to find  $E \in \Sigma$  such that  $\tilde{m}^1(E \Delta D) \leq b_j$ .

Moreover, thanks to Lemma 2.7, and denoting by  $U$  any majorant for all elements  $m_n(X)$ ,  $n \in \mathbb{N}$ , there exists a subsequence  $(b_{j_l})_l$  such that

$$N \mapsto U \wedge \sum_{l=N}^{\infty} b_{j_l}$$

is still an  $(O)$ -sequence. Let us call  $(\rho_N)_N$  such a sequence.

The  $(O)$ -sequence  $(a_N + \rho_N)_N$  can be used to prove continuity of  $m_S^1$ . To this aim, fix any positive integer  $N$ : we can find a partition  $\{D_1, \dots, D_{h_N}\}$  in  $\sigma(\Sigma)$ , satisfying  $\tilde{m}^1(D_k) \leq a_N$  for all  $k$ . For each index  $k$ , from 1 to  $h_N$ , choose an element  $E_k \in \Sigma$  such that  $\tilde{m}^1(E_k \Delta D_k) \leq b_{j_{(N+k)}}$ . We have

$$\tilde{m}^1(E_k) \leq a_N + \rho_N,$$

for all  $k$ .

Let now  $F_1 = E_1$ ,  $F_2 = E_2 \setminus E_1$ ,  $\dots$ ,  $F_{h_N} = E_{h_N} \setminus (E_1 \cup \dots \cup E_{h_N-1})$ . Finally, set

$$F_{h_N+1} := X \setminus \bigcup_{k=1}^{h_N} F_k = X \setminus \bigcup_{k=1}^{h_N} E_k.$$

Of course, all the sets  $F_k$ ,  $k \in \mathbb{N}$ , belong to  $\Sigma$ , and

$$m_S^1(F_k) = \tilde{m}^1(F_k) \leq b_N + \rho_N \quad \forall k = 1, \dots, h_N.$$

As to  $F_{h_N+1}$ , we see that

$$\begin{aligned} m_S^1(F_{h_N+1}) &= \tilde{m}^1(X) - \sum_{k=1}^{h_N} \tilde{m}^1(F_k) = \\ &= \tilde{m}^1\left(\bigcup_{k=1}^{h_N} D_k \setminus \bigcup_{k=1}^{h_N} E_k\right) \leq \tilde{m}^1\left(\bigcup_{k=1}^{h_N} (D_k \Delta E_k)\right) \leq U \wedge \sum_{k=1}^{h_N} b_{j_{(N+k)}} \leq \rho_N. \end{aligned}$$

Hence, the partition  $\{F_1, \dots, F_{h_N}, F_{h_N+1}\}$  satisfies the condition

$$m_S^1(F_k) \leq \rho_N + a_N$$

for all  $k = 1, \dots, h_N + 1$ , and thus  $m_S^1$  is continuous.

We now prove atomicity of  $m_S^2$ . Let  $\nu : \Sigma \rightarrow \mathcal{R}$  be a continuous positive finitely additive measure, such that

$$0 \leq \nu(A) \leq m_S^2(A) \quad \forall A \in \Sigma.$$

Then  $\nu$  admits a Carathéodory-type extension  $\tilde{\nu}$  to the whole of  $\sigma(\Sigma)$ . Continuity of  $\tilde{\nu}$  follows immediately from continuity of  $\nu$ . Thus we get that  $\tilde{\nu}$  is a continuous finitely additive measure, such that

$$0 \leq \tilde{\nu}(A) \leq \tilde{m}^2(A) \quad \forall A \in \sigma(\Sigma).$$

Thus, thanks to atomicity of  $\tilde{m}^2$ , we get:  $\tilde{\nu} \equiv 0$  on  $\sigma(\Sigma)$ , and thus, *a fortiori*,  $\nu \equiv 0$  on  $\Sigma$ . This concludes the proof.  $\square$

Before stating our final convergence theorem, we introduce some definitions, in order to consider also measures taking values not necessarily positive.

**Definition 4.7.** Let  $\mathcal{F}$  be any algebra of subsets of a nonempty set  $X$ . Assume that  $m : \mathcal{F} \rightarrow R$  is any finitely additive bounded measure. We put:

$$m^+(A) = \sup\{m(B) : B \in \mathcal{F}, B \subset A\},$$

$$m^-(A) = -\inf\{m(B) : B \in \mathcal{F}, B \subset A\},$$

$$v(m)(A) = \sup\{|m(B)| : B \in \mathcal{F}, B \subset A\}$$

for all  $A \in \mathcal{F}$ . The set functions  $m^+$ ,  $m^-$ ,  $v(m)$  are called the *positive variation*, *negative variation* and *semivariation* of  $m$  respectively. It is easy to see that:

$m^+$  and  $m^-$  are positive finitely additive measures,  
 $m^+ - m^- = m$ , and  $v(m) \leq m^+ + m^- \leq 2v(m)$ .

For any bounded finitely additive measure  $m : \mathcal{F} \rightarrow R$ , we shall say that  $m$  is *continuous* (resp. *atomic*) if the measure  $m^+ + m^-$  is.

From these definitions, it turns out immediately that any  $s$ -bounded finitely additive measure  $m : \mathcal{F} \rightarrow R$  admits a Sobczyk-Hammer decomposition: it suffices to decompose  $m^+$  and  $m^-$ , and then apply Proposition 4.2.

We now state our final theorem.

**Theorem 4.8.** *Let  $(m_n)_n$  be any sequence of  $s$ -bounded, equibounded, finitely additive measures, defined on a  $\sigma$ -algebra  $\mathcal{A}$  and taking values in  $R$ , and assume that*

$$(RO) \lim_n m_n(A) = m(A)$$

*exists, for all  $A \in \mathcal{A}$ . Then,  $m$  is  $s$ -bounded, and the sequences  $(m_n^1)_n$  and  $(m_n^2)_n$  are  $(RO)$ -convergent to the measures  $m^1$  and  $m^2$ , where  $(m_n^1, m_n^2)$ ,  $(m^1, m^2)$  are the continuous and the atomic part which form the Sobczyk-Hammer decompositions of  $m_n$ ,  $n \in \mathbb{N}$  and  $m$ , respectively.*

**Proof.** Again, we make use of the Stone Isomorphism technique. We shall denote by  $\Sigma$  the algebra of clopen sets, which is isomorphic to  $\mathcal{A}$ , and denote by  $\psi : \mathcal{A} \rightarrow \Sigma$  such isomorphism. Moreover, we shall denote respectively by  $\widetilde{m}_n, \widetilde{m}_n^+$ , and so on, the countably additive extensions of  $m_n, m_n^+$  and so on, to the  $\sigma$ -algebra  $\sigma(\Sigma)$  (we observe that  $m_n, n \in \mathbb{N}$ , and  $m$  are  $s$ -bounded, and hence their *Stone extensions*  $\widetilde{m}_n, \widetilde{m}$  do exist).

Thanks to Theorem 2.13, the sequence  $(\widetilde{m}_n)_n$  is  $(RO)$ -convergent to  $\widetilde{m}$  in  $\sigma(\Sigma)$ . Now, we apply Proposition 4.4 to the sequences  $(\widetilde{m}_n^+)_n$  and  $(\widetilde{m}_n^-)_n$ , thus obtaining a set  $H \in \sigma(\Sigma)$  such that:

- 1)  $\widetilde{m}_n^+|_H$  is continuous,  $\widetilde{m}_n^+|_{H^c}$  is atomic (and the same for  $\widetilde{m}_n^-$ );
- 2)  $(RO) \lim_n \widetilde{m}_n|_H = \widetilde{m}|_H$ ,  $(RO) \lim_n \widetilde{m}_n|_{H^c} = \widetilde{m}|_{H^c}$ .

If we denote by  $m_{nS}^1, m_{nS}^2, m^1_S, m^2_S$  the restrictions to  $\Sigma$  of the measures  $\widetilde{m}_n|_H, \widetilde{m}_n|_{H^c}, \widetilde{m}|_H, \widetilde{m}|_{H^c}$  respectively, then  $m_n^1 := m_{nS}^1 \circ \psi, m_n^2 := m_{nS}^2 \circ \psi$  are the Sobczyk-Hammer decompositions of the measures  $m_n$ , and  $m^1 := m^1_S \circ \psi, m^2 := m^2_S \circ \psi$  give the Sobczyk-Hammer decomposition of  $m$ : clearly,  $(RO)$ -convergence of the measures  $\widetilde{m}_n|_H$  and  $\widetilde{m}_n|_{H^c}$  implies  $(RO)$ -convergence of  $m_n^1$  to  $m^1$  and of  $m_n^2$  to  $m^2$ .  $\square$

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