

The Rickart Integral and Yosida-Hewitt Decompositions

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ABSTRACT. We discuss the Rickart integral for group-valued and l -group-valued finitely additive measures: in both cases we show that the Rickart integral gives the σ -additive part of the so-called Yosida-Hewitt decomposition.

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1 Introduction

The Yosida-Hewitt decomposition theorem [18], published in 1952, is a classic theorem of measure theory. It was shown recently [4], that this classic theorem can be very easily established by using the Rickart integral [11], which was developed in 1942. The representation of the countably additive part of the decomposition led to a similar representation (see [4]) of the normal component of a weakly compact operator $T : M \rightarrow X$, where M is a von Neumann algebra and X is a Banach space. The first abstract extension of the Yosida-Hewitt decomposition was given in

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a Banach setting (see [3]). Many different extensions have been given in succeeding years – see for example [9, 8, 13, 17].

In this paper we develop the Rickart integral for group-valued measures and various aspects of this integral are examined. The Rickart integral, as shown in section 3, is intimately related to the Yosida-Hewitt decomposition: it is the countably additive component. The topological group-valued case is treated in section 4 along with a Lebesgue decomposition theorem. Finally, the topological case along with absolute continuity in connection with the Rickart integral is presented in section 5. We mention that a similar Yosida-Hewitt representation is proved in [17] by means of a complicated Caratheodory process, which in our view further emphasizes the ease and utility the Rickart integral offers in an unexpected setting.

2 Definitions

We shall introduce the main definitions we need, together with some preliminary results. We first recall the concept of an l -group (see also [12]).

Definition 2.1 An abelian group $(R, +)$ is called an l -group if it is endowed with a compatible ordering \leq , and is a lattice with respect to it.

An l -group R is said to be *Dedekind complete* if every nonempty subset of R , bounded from above, has supremum in R .

One first consequence of Dedekind completeness of an l -group R is that convergence of *series* can be defined, at least in the positive cone of R .

Definition 2.2 Given any sequence (a_n) in the positive cone R^+ of R , we say that the *series* $\sum_{n=1}^{\infty} a_n$ is *convergent* if the set of all partial sums $\{s_n : n \in \mathbb{N}\}$ is bounded in R , where $s_n = \sum_{i=1}^n a_i$ for all n . If this is the case, we set $\sum_{n=1}^{\infty} a_n = \sup\{s_n : n \in \mathbb{N}\}$. More generally, if I is any index set, and $A = \{a_i : i \in I\}$ is any nonempty subset of R^+ , we say that A is *summable* if the set $S(A) := \{\sum_{i \in J} a_i : J \subset I, J \text{ finite}\}$ is bounded in R , and again we set $\sum_{i \in I} a_i := \sup S(A)$.

Remark 2.3 A useful remark concerns *double series*: assume that $(a_{i,j})$ is a double sequence in R^+ , and that R is Dedekind complete. We can see that it is summable if and only if all the series $\sum_{i=1}^{\infty} a_{i,j}$, $\sum_{j=1}^{\infty} a_{i,j}$ are convergent, and also the series

$$\sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{i,j} \right), \quad \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{i,j} \right)$$

both converge, and have the same sum (which coincides with the sum of the double sequence $(a_{i,j})$).

Convergence of series allows us to define σ -additivity, at least for positive measures m .

Definition 2.4 Let (Ω, \mathcal{A}) be any measurable space, and R be any Dedekind complete l -group. We say that a finitely additive measure $m : \mathcal{A} \rightarrow R^+$ is σ -additive if, for every sequence $(H_n)_n$ of pairwise disjoint elements from \mathcal{A} , the series $\sum_n m(H_n)$ is convergent, and its sum coincides with $m(\cup_n H_n)$.

More general limits can be defined in Dedekind complete l -groups.

Definition 2.5 Let $(\Lambda, <)$ be any partially ordered directed set, and assume that $(a_\lambda)_{\lambda \in \Lambda}$ is any R -valued bounded net. If R is Dedekind complete, we can define

$$\limsup_{\lambda} a_{\lambda} = \inf_{\lambda' \in \Lambda} \sup_{\lambda > \lambda'} a_{\lambda}$$

and

$$\liminf_{\lambda} a_{\lambda} = \sup_{\lambda' \in \Lambda} \inf_{\lambda > \lambda'} a_{\lambda}.$$

Of course, we say that the net $(a_\lambda)_\lambda$ converges whenever $\limsup_{\lambda} a_{\lambda} = \liminf_{\lambda} a_{\lambda}$ in which case the *limit* of the net is the common value a of the two quantities: this will be written as

$$\lim_{\lambda} a_{\lambda} = a.$$

Given a finitely additive measure $m : \mathcal{A} \rightarrow R^+$, the *Rickart Integral* of m is the limit of a particular kind of net, according with the following definition (see [11]).

Definition 2.6 Let Λ denote the family of all *decompositions* λ of Ω into a countable family of pairwise disjoint elements of \mathcal{A} : in other words, every element λ is a family $\{F_n : n \in \mathbb{N}\}$ of pairwise disjoint measurable sets, whose union is Ω . The set Λ will be endowed with the *refinement* ordering: given two distinct decompositions, $\lambda = \{F_n : n \in \mathbb{N}\}$ and $\lambda' = \{E_k : k \in \mathbb{N}\}$, we have $\lambda \prec \lambda'$ whenever every set F_n in λ is the union of some elements E_k in λ' .

Let now $m : \mathcal{A} \rightarrow R^+$ be any positive finitely additive measure. For each decomposition $\lambda \in \Lambda$, $\lambda = \{F_n : n \in \mathbb{N}\}$, we set

$$s(m, \lambda) = \sum_{n=1}^{\infty} m(F_n).$$

We can see easily that the series converges, and that its sum is smaller than $m(\Omega)$.

We shall show in a moment the existence of the limit

$$\lim_{\lambda \in \Lambda} s(m, \lambda),$$

which is called the *Rickart Integral* of m , and is denoted by $\int_{\Omega} dm$.

Proposition 2.7 *Whenever $m : \mathcal{A} \rightarrow R^+$ is a finitely additive measure, defined on a σ -algebra and taking values in a Dedekind complete l -group, there exists $\int_{\Omega} dm$.*

Proof. Existence of the limit

$$\lim_{\lambda \in \Lambda} s(m, \lambda)$$

is an easy consequence of the following monotonicity property:

$$\lambda \prec \lambda' \implies s(m, \lambda') \leq s(m, \lambda).$$

To show monotonicity, it is enough to observe that, whenever a set $F \in \mathcal{A}$ is the union of a disjoint sequence $(E_k)_k$ from \mathcal{A} , one has $m(F) \geq \sum_k m(E_k)$. An application of the remark 2.3 completes the proof. ■

It is quite easy to see that $m \rightarrow \int_{\Omega} dm$ defines a monotone operator, additive and positively homogeneous. Moreover, it can be used to define a new measure, related to m in the following way.

Definition 2.8 Let $\Omega, \mathcal{A}, m, R$ be as above. For every element $F \in \mathcal{A}$, we define $m|_F$ as the measure on \mathcal{A} given by:

$$m|_F(E) = m(E \cap F).$$

We shall denote by $\int_F dm$ the Rickart integral of $m|_F$.

It is quite easy now to deduce that, fixed m , the mapping $F \rightarrow \int_F dm$ is a finitely additive measure on \mathcal{A} , and that $\int_F dm \leq m(F)$ for all $F \in \mathcal{A}$. This measure will be denoted by $R(m)$, and called the *Rickart indefinite integral* of m .

3 The Yosida-Hewitt decomposition

We shall always assume that (Ω, \mathcal{A}) is any measurable space, and that m is a finitely additive measure, defined on \mathcal{A} and taking values in a Dedekind complete l -group R .

In this section our aim is to show that the Rickart indefinite integral $R(m)$ is intimately related to the Yosida-Hewitt decomposition of m (see [7], [14]): in particular, we shall prove that the measure $R(m)$ is σ -additive, and that $m - R(m)$ is *purely finitely additive*, i.e. the only σ -additive measure $\nu \leq m - R(m)$ is the null measure.

Theorem 3.1 *The measure $R(m)$ is σ -additive, and $m - R(m)$ is purely finitely additive, therefore $(R(m), m - R(m))$ is the Yosida-Hewitt decomposition of m .*

Proof. We first show σ -additivity of $R(m)$. Let $\{H_k : k \in \mathbb{N}\}$ be any disjoint sequence in \mathcal{A} , and let us denote by H their union. As $R(m)$ is positive and finitely additive, it follows clearly

$$\sum_{k=1}^{\infty} R(m)(H_k) \leq R(m)(H).$$

So, let us show the reverse inequality. To this end, let λ denote any decomposition of Ω , $\lambda = \{F_n : n \in \mathbb{N}\}$. By the mere definition, we have

$$R(m)(H) \leq s(m, \lambda, H) = \sum_{n=1}^{\infty} m(H \cap F_n).$$

Without loss of generality, we can choose λ in such a way that every element H_k is the union of some F'_n 's : hence, for each k there exists a countable subset $J(k) \subset \mathbb{N}$ such that $H_k = \bigcup_{n \in J(k)} F_n$. Hence we can write

$$R(m)(H) \leq \sum_{k=1}^{\infty} \sum_{n \in J(k)} m(F_n) = \sum_{n \in J(1)} m(F_n) + \sum_{k=2}^{\infty} \sum_{n \in J(k)} m(F_n),$$

and hence

$$R(m)(H) - \sum_{k=2}^{\infty} \sum_{n \in J(k)} m(F_n) \leq \sum_{n \in J(1)} m(F_n). \quad (1)$$

Now, we refine λ , in the following way: all the elements F_n remain fixed, except those which are contained in H_1 . If we denote by Λ_1 the set of all such decompositions λ , from (1) we get

$$R(m)(H) - \sum_{k=2}^{\infty} \sum_{n \in J(k)} m(F_n) \leq \inf_{\lambda \in \Lambda_1} \sum_{n \in J(1)} m(F_n) = R(m)(H_1), \quad (2)$$

from which

$$R(m)(H) - R(m)(H_1) \leq \sum_{k=2}^{\infty} \sum_{n \in J(k)} m(F_n) \quad (3)$$

Now, we let the elements F_n , $n \in J(2)$, vary in all possible ways, but keeping fixed the other F_n 's. From (3) we obtain

$$R(m)(H) - R(m)(H_1) - R(m)(H_2) \leq \sum_{k=3}^{\infty} \sum_{n \in J(k)} m(F_n).$$

Proceeding in this fashion, we get, for every integer N :

$$R(m)(H) - \sum_{k=1}^N R(m)(H_k) \leq \sum_{k=N+1}^{\infty} \sum_{n \in J(k)} m(F_n).$$

Taking the limits as N diverges, we obtain easily

$$R(m)(H) - \sum_{k=1}^{\infty} R(m)(H_k) \leq 0,$$

that is, the required inequality.

We now turn to the pure finite additivity of $m - R(m)$. Let $\nu : \mathcal{A} \rightarrow R^+$ be any σ -additive measure, $\nu \leq m - R(m)$.

Then we have: $\nu + R(m) \leq m$, hence $R(\nu + R(m)) \leq R(m)$.

As $\nu + R(m)$ is σ -additive, it clearly coincides with its Rickart indefinite integral, therefore we obtain $\nu + R(m) \leq R(m)$, i.e. $\nu = 0$. ■

4 The topological group-valued case

Also for measures taking values in topological groups, theorems of the kind of Yosida-Hewitt are well-known (see [9],[17]). However, we think that a description in terms of the Rickart integral would be much clearer and immediate.

We shall always assume that G is any abelian, complete, Hausdorff, topological group. We also denote by \mathcal{I} any neighbourhood basis for the neutral element $0 \in G$: we can and do assume that all elements of \mathcal{I} are closed and symmetric.

Definition 4.1 Let $m : \mathcal{F} \rightarrow G$ be any finitely additive measure, defined in some Boolean algebra \mathcal{F} . We say that m is *s-bounded* if $\lim_n m(H_n) = 0$ in G , whenever $(H_n)_n$ is a disjoint sequence from \mathcal{F} .

We shall also use the following notation, for every element $A \in \mathcal{F}$:

$$m^\#(A) := \{m(B) : B \subset A, B \in \mathcal{F}\}.$$

It is well-known that *s*-boundedness of m can be equivalently formulated as:

For every disjoint sequence $(H_n)_n$ from \mathcal{F} and every $U \in \mathcal{I}$ there exists an integer n_U such that $m^\#(H_n) \subset U$ for all $n \geq n_U$.

It is also well-known that any *s*-bounded finitely additive measure $m : \mathcal{F} \rightarrow G$ is *bounded*, i.e. for every $U \in \mathcal{I}$ there exists an integer N_U such that $m^\#(\Omega) \subset N_U$. (See [6]).

A very useful tool for our discussion is the *Lebesgue decomposition* for a group-valued measure. The results are well-known, so we just recall the basic facts.

Definition 4.2 Let $m, \mu : \mathcal{F} \rightarrow G$ be two finitely additive *s*-bounded measures, defined in the Boolean algebra \mathcal{F} .

We say that m is *absolutely continuous* with respect to μ , and write $m \ll \mu$, if, whenever $U \in \mathcal{I}$ is fixed, there exists an element $V \in \mathcal{I}$ such that the following implication holds:

$$\mu^\#(A) \subset V \Rightarrow m(A) \in U$$

for any $A \in \mathcal{F}$.

We say that m and μ are *mutually singular* if, for every $U \in \mathcal{I}$ there exists $F \in \mathcal{F}$ such that

$$m^\#(F^c) \subset U, \quad \text{and} \quad \mu^\#(F) \subset U.$$

If this is the case, we sometimes say that m is *singular* with respect to μ (or vice-versa), without mentioning mutuality. Singularity will be denoted by: $m \perp \mu$.

Theorem 4.3 (see [17]) *Let $m, \mu : \mathcal{F} \rightarrow G$ be two s -bounded finitely additive measures, defined in some Boolean algebra \mathcal{F} .*

Then there exist two s -bounded finitely additive measures, m_a and m_s , from \mathcal{F} to G , both absolutely continuous with respect to m , such that:

- i) $m = m_a + m_s$
- ii) $m_a \ll \mu, \quad m_s \perp \mu.$

(The measures m_a and m_s are usually described as the *Lebesgue decomposition* of m with respect to μ).

Of course, as soon as an s -bounded finitely additive measure is absolutely continuous with respect to a σ -additive one, then it is σ -additive too. (The definition of σ -additivity of a G -valued measure is deduced from the topology on G).

Let now \mathcal{A} denote any σ -algebra of subsets of some fixed space Ω .

Definition 4.4 Given an s -bounded finitely additive measure $m : \mathcal{A} \rightarrow G$, we say that m is *purely finitely additive* if it is singular with respect to every σ -additive measure $\mu : \mathcal{A} \rightarrow G$.

With this definition fixed, the meaning of a Yosida-Hewitt decomposition for any s -bounded finitely additive measure $m : \mathcal{A} \rightarrow G$ is straightforward: in the literature, there are several results concerning such decomposition; our aim here is to show that the Yosida-Hewitt decomposition has a quite simple description, i.e. it consists of the couple $(R(m), m - R(m))$, where $R(m)$ has a similar meaning as in Definition 2.8.

The remaining part of the paper is devoted to show the existence of the Rickart integral, and its properties, also in the actual situation.

To this end, one important tool is the so-called *Stone extension* of a finitely additive, s -bounded measure.

We first recall the well-known *Stone representation* theorem, in its topological form (see [15]): every Boolean algebra \mathcal{F} is (algebraically) isomorphic with the algebra Σ of all *clopen* subsets of a compact, Hausdorff, totally disconnected topological space S (called *the Stone space* associated with \mathcal{F}).

Next, we mention a well-known extension theorem, for topological group-valued σ -additive measures, defined on an *algebra*: (see [16]).

Theorem 4.5 *Let G be as above, and $m : \Sigma \rightarrow G$ be any s -bounded and σ -additive measure, defined on an algebra Σ . Then there exists a (unique) σ -additive measure \tilde{m} , defined on the σ -algebra $\mathcal{A}(\Sigma)$ generated by Σ , and such that $m(F) = \tilde{m}(F)$ for all $F \in \Sigma$.*

Putting together Theorem 4.5 and the preceding remark, we obtain:

Corollary 4.6 *Let $m : \mathcal{F} \rightarrow G$ be any s -bounded, finitely additive measure, defined on some Boolean algebra \mathcal{F} . Denoting with S the Stone space associated to \mathcal{F} , and by $\phi : \mathcal{F} \rightarrow \Sigma$ the algebraic isomorphism between \mathcal{F} and the algebra Σ of all clopen subsets of S , the measure $m_1 := \phi^{-1} \circ m : \Sigma \rightarrow G$ is σ -additive and s -bounded, then it admits a σ -additive extension m_2 to the σ -algebra $\mathcal{A}(\Sigma)$.*

5 The Rickart Integral, topological case

In this section, G, Ω, \mathcal{A} will be as above, and $m : \mathcal{A} \rightarrow G$ will be a fixed s -bounded finitely additive measure.

We begin our discussion with a useful remark, due to Sion (see [16]).

Remark 5.1 *Given a partially ordered, directed set (Λ, \prec) , and a net $\varphi : \Lambda \rightarrow G$, the existence of the limit*

$$\lim_{\lambda \in \Lambda} \varphi(\lambda)$$

is equivalent to the existence of the limit

$$\lim_n \varphi(\lambda_n)$$

for every increasing sequence $(\lambda_n)_n$ in Λ .

Now, let Λ be again the set of all decompositions of Ω into a countable family of pairwise disjoint members of \mathcal{A} , exactly as in the previous sections. For every $\lambda \in \Lambda$, $\lambda = \{F_k : k \in \mathbb{N}\}$, we set:

$$s(m, \lambda) = \sum_{k=1}^{\infty} m(F_k).$$

(Convergence of the series follows from s -boundedness of m).

We set also, for every element $F \in \mathcal{A}$:

$$s(m, \lambda, F) = s(m|_F, \lambda) = \sum_{n=1}^{\infty} m(F \cap F_n)$$

Theorem 5.2 *In the setting described above, there exists the limit*

$$\int_F dm := \lim_{\lambda \in \Lambda} s(m, \lambda, F),$$

uniformly in $F \in \mathcal{A}$.

Such limit will be denoted as $R(m, F), \forall F \in \mathcal{A}$.

Proof. To start with, we consider the product group $G^{\mathcal{A}}$, with natural operation and neutral element $\underline{0}$: in this group, we define the topology of uniform convergence by the following basis of neighbourhoods for $\underline{0}$:

$$\underline{\mathcal{I}} := \{U^{\mathcal{A}} : U \in \mathcal{I}\}.$$

If we apply Remark 5.1 in this group, to achieve our goal it is sufficient to show that, for every increasing sequence $(\lambda_k)_k$ in Λ there exists the limit

$$\lim_{k \rightarrow \infty} s(m, \lambda_k, F)$$

uniformly in F .

Let us denote by S the Stone space associated with \mathcal{A} , by m_1 the finitely additive measure associated with m and defined on the algebra Σ of all clopen subsets of S , and by m_2 the σ -additive extension of m_1 to the σ -algebra $\mathcal{A}(\Sigma)$ generated by Σ . We shall also denote by $\varphi : \mathcal{A} \rightarrow \Sigma$ the Stone isomorphism. Thus, $m_1 = m \circ \varphi^{-1}$. Moreover, for every set $F \in \mathcal{A}$ and every element $\lambda \in \Lambda$, $\lambda = \{F_n : n \in \mathbb{N}\}$, the

sequence $(\varphi(F_n \cap F))_n$ is a disjoint family of elements from Σ , whose union is a set $V(\lambda, F) \in \mathcal{A}(\Sigma)$, such that $V \subset \varphi(F)$. Moreover, by the definition of m_2 , we see that $s(m, \lambda, F) = m_2(V(\lambda, F))$.

If λ_1 and λ_2 are two elements from Λ , such that $\lambda_1 \prec \lambda_2$ (i.e. λ_2 is *finer* than λ_1), then it is easy to see that the set $V(\lambda_2, F)$ is contained in $V(\lambda_1, F)$.

It follows that, as soon as $(\lambda_k)_k$ is an increasing sequence in Λ , the corresponding sequence $V_k := (V(\lambda_k, F))_k$ is decreasing in $\mathcal{A}(\Sigma)$, and therefore there exists the limit $\lim_k m_2(V_k) = \lim_k s(m, \lambda_k, F)$.

Moreover, it is clear that, for each $\lambda \in \Lambda$, we have $V(\lambda, F) = \varphi(F) \cap V(\lambda, \Omega)$.

Hence, for every increasing sequence $(\lambda_k)_k$ in Λ ,

$$V(\lambda_k, F) = \varphi(F) \cap V(\lambda_k, \Omega)$$

for each $k \in \mathbb{N}$. From s -boundedness of m_2 we then obtain that the limit

$$\lim_{k \rightarrow \infty} m_2(V(\lambda_k, F)) = \lim_{k \rightarrow \infty} s(m, \lambda_k, F)$$

exists uniformly in F , and therefore we are finished. ■

As a natural consequence of Theorem 5.2, we can introduce the definition of the Rickart integral of a G -valued finitely additive measure m .

Definition 5.3 Given an s -bounded finitely additive measure $m : \mathcal{A} \rightarrow G$, the *Rickart integral* of m is the limit

$$\int_{\Omega} dm := \lim_{\lambda \in \Lambda} s(m, \Omega)$$

and the *indefinite Rickart integral* is the finitely additive measure

$$R(m) := F \mapsto \lim_{\lambda \in \Lambda} s(m, F),$$

$F \in \mathcal{A}$.

As above, the latter limit is denoted by $\int_F dm$, and turns out to be linearly dependent of m . Moreover, it is clear that, as soon as m is countably additive, then m coincides with its indefinite Rickart integral.

We will now prove that the indefinite Rickart integral is σ -additive, (and *a fortiori* s -bounded).

Theorem 5.4 *If $m : \mathcal{A} \rightarrow G$ is s -bounded, then its indefinite Rickart integral $R(m)$ is σ -additive.*

Proof. Let us fix any sequence $(F^j)_j$ of pairwise disjoint elements from \mathcal{A} , and denote by F their union.

Fix now any element $U \in \mathcal{I}$, and choose any decreasing sequence $(U_r)_r$ in \mathcal{I} , satisfying:

$$U_1 + U_1 \subset U, \quad U_r + U_r \subset U_{r-1}$$

for $r > 1$.

Let us choose any decomposition $\lambda^* \in \Lambda$, $\lambda^* = \{E_1, E_2, \dots, E_n, \dots\}$, such that

$$R(m)(F) - s(m, \lambda, F) \in U_1$$

for every $\lambda \succ \lambda^*$.

Without loss of generality, we may also suppose that each set F^j is the union of some elements from λ^* : for every decomposition $\lambda \succ \lambda^*$, and for every index j , we shall denote by λ_j the set of those elements $E \in \lambda$, that are contained in F^j . Thus, $\lambda = \bigcup_{j=1}^{\infty} \lambda_j$. Now, let us choose $\lambda \succ \lambda^*$ in such a way that

$$\sum_{E \in \lambda_j} m(E) - R(m)(F^j) \in U_j$$

holds, for every $j \in \mathbb{N}$. For such choice of λ , we get:

$$s(m, \lambda, F) = \sum_{j=1}^{\infty} \sum_{E \in \lambda_j} m(E).$$

From convergence of the series, an integer M exists, such that

$$s(m, \lambda, F) - \sum_{j=1}^N \sum_{E \in \lambda_j} m(E) \in U$$

for all $N \geq M$.

So we have, for $N \geq M$,

$$\begin{aligned}
& R(m)(F) - \sum_{j=1}^N R(m)(F^j) = \\
& R(m)(F) - s(m, \lambda, F) + s(m, \lambda, F) - \sum_{j=1}^N \sum_{E \in \lambda_j} m(E) + \sum_{j=1}^N \sum_{E \in \lambda_j} m(E) - \sum_{j=1}^N R(m)(F^j) \in \\
& \in U + U + \sum_{j=1}^N U_j \subset U + U + U
\end{aligned}$$

and this concludes the proof. ■

The last part of this section is the description of the Yosida-Hewitt decomposition of m . This will be achieved once we prove that $m - R(m)$ is purely finitely additive.

To this end, we shall prove a general result, concerning the Rickart indefinite integral.

Proposition 5.5 *Assume that $m, \mu : \mathcal{A} \rightarrow G$ are two s -bounded finitely additive measures, such that $m \ll \mu$. Then $R(m) \ll R(\mu)$.*

Proof. For any element $\lambda \in \Lambda$ we denote by m_λ and by μ_λ the measures defined by

$$m_\lambda(F) = s(m, \lambda, F), \quad \mu_\lambda(F) = s(\mu, \lambda, F)$$

for every $F \in \mathcal{A}$.

Clearly, both m_λ and μ_λ are finitely additive and s -bounded (indeed, $m_\lambda \ll m$ and $\mu_\lambda \ll \mu$).

We now prove that $m_\lambda \ll \mu_\lambda$, uniformly in λ : that is, for every element $U \in \mathcal{I}$ there exists an element $V \in \mathcal{I}$ such that the implication

$$(\mu_\lambda)^\#(A) \subset V \Rightarrow m_\lambda(A) \subset U$$

holds true, for any $A \in \mathcal{A}$ and $\lambda \in \Lambda$.

To begin with, we fix any element $U \in \mathcal{I}$: by hypotheses, there exists an element $V \in \mathcal{I}$ such that $\mu^\#(A) \subset V \Rightarrow m^\#(A) \subset U$ for $A \in \mathcal{A}$. We can choose an element $W \in \mathcal{I}$, such that $W + W \subset V$.

We shall show that W is the required neighbourhood. Fix any element $A \in \mathcal{A}$ and any decomposition $\lambda \in \Lambda$, such that $(\mu_\lambda)^\#(A) \subset W$.

By writing $\lambda = \{E_1, E_2, \dots, E_n, \dots\}$, and setting

$$F_n = \bigcup_{j=1}^n E_j$$

for every positive integer n , by s -boundedness there exists an integer k_0 such that

$$\mu^\#(F_k \setminus F_{k+p}) \subset W$$

for all $k \geq k_0$ and all $p > 0$. So from

$$\sum_{j=1}^{\infty} \mu(B \cap E_j) \in W$$

we deduce

$$\mu(B \cap F_k) \in V$$

for every $B \in \mathcal{A}$, $B \subset A$, and every $k \geq k_0$.

We now turn to $m_\lambda(A)$ and claim that it is an element of U . Indeed, from the above relations we get

$$\mu^\#(A \cap F_k) \in V \quad \text{and hence} \quad m(A \cap F_k) \in U,$$

for all $k \geq k_0$. Then

$$m_\lambda(A) = \lim_{k \rightarrow \infty} m(A \cap F_k) \in U$$

because U is closed.

This shows absolute continuity of m_λ with respect to μ_λ uniformly in λ .

The last step is to prove that $R(m) := \int f dm \ll R(\mu) := \int f d\mu$.

Keeping fixed the notations above, let $W_1 \in \mathcal{I}$ satisfy $W_1 + W_1 \subset W$.

If we choose $A \in \mathcal{A}$, satisfying $\mu_0^\#(A) \subset W_1$, we claim that $R(m)(A) \in U$.

Indeed, $R(\mu)(B)$ is the limit of $\mu_\lambda(B)$, uniformly in $B \in \mathcal{A}$, as shown in 5.2. Hence, there exists $\lambda^* \in \Lambda$ such that $R(\mu)(B) - \mu_\lambda(B) \in W_1$ for all $B \in \mathcal{A}$ and all $\lambda \succ \lambda^*$. Now, if we fix $B \in \mathcal{A}$, $B \subset A$ and any $\lambda \succ \lambda^*$, we get $\mu_\lambda(B) \in W$ and then $m_\lambda(A) \in U$. By taking the limit, as λ runs in Λ , we obtain $R(m)(A) \in U$. ■

Corollary 5.6 *For every s -bounded finitely additive measure $m : \mathcal{A} \rightarrow G$, the difference $m - R(m)$ is purely finitely additive.*

Proof. Let us denote by ν the difference $m - R(m)$, and let $\mu : \mathcal{A} \rightarrow G$ be any fixed σ -additive measure.

We want to show that $\mu \perp \nu$. By using Theorem 4.3, we obtain a decomposition $\mu = \mu_1 + \mu_2$, where $\mu_1 \ll \nu$ and $\mu_2 \perp \nu$.

From $\mu_1 \ll \nu$ we get $R(\mu_1) \ll R(\nu)$, thanks to Proposition 5.5. But $\mu_1 \ll \mu$, hence μ_1 is σ -additive, so $R(\mu_1) = \mu_1$. Moreover $R(\nu) = 0$ by linearity. Thus we find $\mu_1 \ll 0$, i.e. $\mu_1 = 0$, and $\mu = \mu_2 \perp \nu$. ■

In conclusion, we have shown that, for any s -bounded finitely additive G -valued measure m , the Yosida-Hewitt decomposition is given by

$$m = R(m) + (m - R(m))$$

where $R(m)$ is the indefinite Rickart integral of m .

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