Dieudonné-type theorems for set functions with values in \((l)\)-groups

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ABSTRACT. Some versions of Dieudonné theorems are given for set functions, not necessarily positive, taking values in Dedekind complete \((l)\)-groups, relatively to the "\((D)\)-convergence".


KEY WORDS: \((l)\)-groups, Vitali-Hahn-Saks theorems, Dieudonné theorems.

1 Introduction.

In a previous paper ([4]), we gave some versions of Vitali - Hahn - Saks and Nikodým theorems for set functions with values in suitable Dedekind complete \((l)\)-groups. In this paper, we prove some versions of Dieudonné theorems, for \((l)\)-group-valued finitely additive regular maps. In the literature, there exist several versions of theorems of this kind, for maps taking values in topological groups and/or Banach spaces. Among the authors, we recall Brooks and Chacon ([5], [6]), Candeloro and Letta ([7], [8]).

In the previous paper [2] similar results were proved with respect to the order convergence for positive means taking values in spaces of the type \(L^0(\mu)\), where
\( \mu \) is a \( \sigma \)-additive locally finite positive \( \mathcal{M} \)-valued measure.

### 2 Preliminaries.

We begin with the following:

**Definitions 2.1** An Abelian group \((R,+)\) is called \((l)\)-group if it is endowed with a compatible ordering \(\leq\), and is a lattice with respect to it.

An \((l)\)-group \(R\) is said to be Dedekind complete if every nonempty subset of \(R\), bounded from above, has supremum in \(R\). A sequence \((r_n)_n\) in \(R\) is said to be order-convergent (or \((o)\)-convergent) to \(r\) if there exists a sequence \((p_n)_n\) in \(R\) such that \(p_n \downarrow 0\) and \(|r_n - r| \leq p_n\), \(\forall n \in \mathbb{N}\) (see also \([11]\), \([15]\)), and we will write \((o)\lim_n r_n = r\). A bounded double sequence \((a_{i,l})_{i,l}\) in \(R\) is called \((D)\)-sequence or regulator if for all \(i \in \mathbb{N}\) we have \(a_{i,l} \downarrow 0\) as \(l \to +\infty\). A sequence \((r_n)_n\) in \(R\) is said to be \((D)\)-convergent to \(r \in R\) (and we write \((D)\lim_n r_n = r\)) if there exists a \((D)\)-sequence \((a_{i,l})_{i,l}\) in \(R\), such that \(\forall \phi \in \mathbb{N}^\mathbb{N}, \exists n_0 \in \mathbb{N}\) such that \(|r_n - r| \leq \bigvee_{i=1}^\infty a_{i,\phi(i)} \forall n \in \mathbb{N}, n \geq n_0\). If \(\Lambda\) is any nonempty set, \((r_n^{(\lambda)})_n\) are sequences in \(R\) and \(r^{(\lambda)}(\lambda)\) is in \(R\) for all \(\lambda \in \Lambda\), we say that \((D)\lim_n r_n^{(\lambda)} = r^{(\lambda)}\) uniformly with respect to \(\lambda \in \Lambda\) if there exists a \((D)\)-sequence \((a_{i,l})_{i,l}\) in \(R\), such that \(\forall \phi \in \mathbb{N}^\infty, \exists n_0 \in \mathbb{N}\) such that \(|r_n^{(\lambda)} - r^{(\lambda)}| \leq \bigvee_{i=1}^\infty a_{i,\phi(i)} \forall n \in \mathbb{N}, n \geq n_0\) and \(\forall \lambda \in \Lambda\).

The sequence \((r_n)_n\) is said to be \((D)\)-Cauchy if \((D)\lim_n (r_n - r_{n+p}) = 0\) uniformly with respect to \(p \in \mathbb{N}\).

An intermediate condition is the so-called \((RD)\)-convergence: if \(\Lambda\) is any nonempty set, \((r_n^{(\lambda)})_n\) are sequences in \(R\) and \(r^{(\lambda)}(\lambda)\) is in \(R\) for all \(\lambda \in \Lambda\), we say that \(r_n^{(\lambda)}\) \((RD)\)-converges to \(r^{(\lambda)}(\lambda)\) (convergence with respect to the same regulator) if there exists a \((D)\)-sequence \((a_{i,l})_{i,l}\) in \(R\), such that \(\forall \phi \in \mathbb{N}^\infty, \exists n_0 \in \mathbb{N}\) such that \(|r_n^{(\lambda)} - r^{(\lambda)}| \leq \bigvee_{i=1}^\infty a_{i,\phi(i)} \forall n \in \mathbb{N}, n \geq n_0\). In Lemma 2.3 there will be stated a relationship between simple \(D\)-convergence and \((RD)\)-convergence, at least when \(\Lambda\) is countable.
In general, the limit of a sequence (with respect to \((D)-convergence\)) is not unique. However, there are some conditions on \(R\), which are equivalent to uniqueness of the limit: for example, weak \(\sigma\)-distributivity, whose definition we report here below:

**Definition 2.2** An \((l)\)-group \(R\) is said to be **weakly \(\sigma\)-distributive** if for every \((D)\)-sequence \((a_{i,l})_{i,l}\) we have:

\[
\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left( \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) = 0.
\]

We note that \((o)\)-convergence of sequences always implies \((D)\)-convergence, and these two convergences are equivalent when \(R\) is weakly \(\sigma\)-distributive (see [9]). We now recall the following result (see [14], pp. 42-43), which will be useful in the sequel.

**Lemma 2.3** Let \(R\) be a Dedekind complete \((l)\)-group (not necessarily weakly \(\sigma\)-distributive), \((a_{i,l}^{(n)})_{i,l}, n \in \mathbb{N}\), be a sequence of regulators in \(R\). Then for every \(u \in R, u \geq 0\) there exists a \((D)\)-sequence \((a_{i,l})_{i,l}\) in \(R\) such that:

\[
u \left[ \sum_{n=1}^{\infty} \left( \bigvee_{i=1}^{\infty} a_{i,\varphi(i)+n}^{(n)} \right) \right] \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \quad \forall \varphi \in \mathbb{N}^{\mathbb{N}}.
\]

We now introduce the following:

**Definitions 2.4** Let \(\Omega\) be any infinite set, \(\mathcal{A} \subset \mathcal{P}(\Omega)\) be an algebra, \(R\) be a Dedekind complete weakly \(\sigma\)-distributive \((l)\)-group. We say that \(m : \mathcal{A} \to R\) is **bounded** if \(\exists w \in R, w \geq 0: |m(A)| \leq w, \forall A \in \mathcal{A}\). The maps \(m_j : \mathcal{A} \to R, j \in \mathbb{N}\), are **equibounded** if there exists an element \(u \in R, u \geq 0\), such that

\[
|m_j(A)| \leq u \quad \forall j \in \mathbb{N}, \forall A \in \mathcal{A}. \tag{1}
\]

Given a finitely additive bounded measure (or, in short, **mean**) \(m : \mathcal{A} \to R\), define the **semivariation** of \(m\),

\[
v_{\mathcal{A}}(m) : \mathcal{A} \to R,
\]

or simply

\[
v(m) : \mathcal{A} \to R,
\]
by setting
\[ v_{A}(m)(A) = \sup_{B \in A, B \subseteq A} |m(B)|, \forall A \in A, \]
and, if \( \emptyset \neq E \subset A \), define \( v_{E}(m) : A \to R \), by setting
\[ v_{E}(m)(A) = \sup_{B \in E, B \subset A} |m(B)|, \forall A \in A. \]
A mean \( m : A \to R \) is said to be \( \sigma \)-additive (or, in short, measure) if there exists a \( (D) \)-sequence \( (u_{i,l})_{i,l} \) such that, \( \forall \varphi \in N^{N} \) and for every decreasing sequence \( (H_{s})_{s} \) in \( A, H_{s} \downarrow \emptyset \), there exists \( \exists \):
\[ v_{A}(m)(H_{s}) \leq \bigvee_{i=1}^{\infty} u_{i,\varphi(i)}. \]
If a sequence of measures \( m_{j} : A \to R, j \in N \), is given, uniform \( \sigma \)-additivity is defined as above, but with \( \exists \) independent of \( j \) (See also [4]).
A finitely additive measure \( m : A \to R \) is said to be \( (s) \)-bounded in \( \emptyset \neq E \subset A \), or simply \( E-(s) \)-bounded, if there exists a \( (D) \)-sequence \( (w_{i,l})_{i,l} \) such that, \( \forall \varphi \in N^{N} \) and for every disjoint sequence \( (H_{s})_{s} \) in \( E \) there exists \( \exists \): \( \forall s \geq \exists \),
\[ v_{E}(m)(H_{s}) \leq \bigvee_{i=1}^{\infty} w_{i,\varphi(i)}. \]
If \( E \) is as above, we say that the maps \( m_{j} : A \to R, j \in N \), are \( E \)-uniformly \( (s) \)-bounded if the above condition holds, but with \( \exists \) independent of \( j \) (see also [4]).
When \( E = A \) we simply speak of \( (s) \)-boundedness or uniform \( (s) \)-boundedness.
A typical consequence of \( (s) \)-boundedness of a mean \( m \) is that there exists the limit \( m(A_{n}) \) for monotone sequences \( (A_{n})_{n} \) in \( A \) (see [4]). As to uniformly \( (s) \)-bounded measures, we shall report here a slight modification of Proposition 3.4 of [4], which will be used later.

**Proposition 2.5** Assume that \( (m_{j})_{j} \) is a sequence of \( R \)-valued uniformly \( (s) \)-bounded finitely additive measures on \( A \), and let \( (e_{i,l})_{i,l} \) be a regulator related to this property. For every decreasing sequence \( (A_{n})_{n} \) in \( A \), the limit
\[ r_{j} := (RD) \lim_{n \to \infty} v(m_{j})(A_{n}) \]
exists uniformly in \( j \), and the regulator \( (e_{i,l})_{i,l} \) works for this property.
Given a sequence of means \((m_j)_{j \in \mathbb{N} \cup \{0\}}, m_j : A \to R\), and a nonempty subfamily \(E \subset A\), we say that the \(m_j\)'s (\(D\))-converge to \(m_0\) pointwise with respect to the same regulator, or in short \((RD)\lim_m m_j = m_0\) in \(\emptyset \neq E \subset A\), if there exists a \((D)\)-sequence \((b_{i,l})_{i,l}\) such that
\[
\forall \varphi \in \mathbb{N}^\mathbb{N}, \forall A \in E, \exists j_0 \in \mathbb{N} \text{ such that } \\
|m_j(A) - m_0(A)| \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \quad \forall j \in \mathbb{N}, j \geq j_0.
\] (2)

We note that the condition (2) is equivalent to the classical pointwise convergence of the involved set functions in the case of metrizable groups.

Let now \(\Omega, R\) and \(A\) be as above. From now on, we assume that \(R\) is weakly \(\sigma\)-distributive, and \(F, G \subset A\) are two fixed lattices, such that the complement (with respect to \(\Omega\)) of every element of \(F\) belongs to \(G\). In the sequel we will not say it explicitly. If \(\Omega\) is a topological normal space [resp. locally compact Hausdorff space], examples of \(A, F\) and \(G\), satisfying the above properties, are the following: \(A = \{\text{Borelian subsets of } \Omega\}, F = \{\text{closed sets}\} [\text{resp.}\{\text{compact sets}\}], G = \{\text{open sets}\}\).

**Definitions 2.6** A mean \(m : A \to R\) is said to be regular if there exists a (\(D\))-sequence \((\gamma_{i,l})_{i,l}\) in \(R\) such that for each \(A \in A\), \(\forall n \in \mathbb{N}, \exists F_n \in F, G_n \in G\) such that
\[
F_n \subset F_{n+1} \subset A \subset G_{n+1} \subset G_n \quad \forall n,
\] (3)
and \(\forall \varphi \in \mathbb{N}^\mathbb{N}\) there exists \(n_0(A, \varphi) \in \mathbb{N}\) such that \(\forall n \geq n_0\),
\[
v_A(m)(G_n \setminus F_n) \leq \bigvee_{i=1}^{\infty} \gamma_{i,\varphi(i)}; \quad (4)
\]
and if \(\forall W \in F, \forall n \in \mathbb{N}, \exists F'_n \in F, G'_n \in G\) such that
\[
W \subset F'_{n+1} \subset G'_n \subset F'_n \quad \forall n,
\] (5)
and \(\forall \varphi \in \mathbb{N}^\mathbb{N}\) there exists \(n_0(A, \varphi) \in \mathbb{N}\) such that \(\forall n \geq n_0\),
\[
v_A(m)(G'_n \setminus W) \leq \bigvee_{i=1}^{\infty} \gamma_{i,\varphi(i)}.
\] (6)
The means \( m_j : A \to R, j \in \mathbb{N} \), are said to be \textit{uniformly regular} if there exists a \((D)\)-sequence \((\gamma_{i,l})_{i,l}\) in \( R \) such that \( \forall A \in A \) and \( \forall W \in F \) there exist sequences \((F_n)_n, (G_n)_n, (F'_n)_n, (G'_n)_n\), as above, such that for each \( \varphi \in \mathbb{N}^{\mathbb{N}} \), (4) and (6) are satisfied, with \( n_0 \) independent of \( j \).

The following proposition shows that, if \( (m_j : A \to R)_j \) is a sequence of equibounded regular means, even if they are not uniformly regular, the sequences \((\gamma_{i,l})_{i,l}\), \((F_n)_n\), \((G_n)_n\), \((F'_n)_n\), \((G'_n)_n\) above can be taken independently of \( j \), satisfying (3), (4), (5) and (6) \( \forall j \in \mathbb{N} \).

**Proposition 2.7** Let \( R \) be as above, \( A \) be any algebra, and \( (m_j : A \to R)_j \) be a sequence of equibounded regular means. Then there exists a regulator \((p_{i,l})_{i,l}\) such that, for every \( A \in A \) and every \( W \in F \) there exist sequences \((F_n)_n\), \((F'_n)_n\) in \( F \), \((G_n)_n\), \((G'_n)_n\) in \( G \), satisfying (3) and (5) and such that \( \forall \varphi \in \mathbb{N}^{\mathbb{N}} \) and \( j \in \mathbb{N} \) there exists \( n_0(A, W, \varphi, j) \in \mathbb{N} \) such that \( \forall n \geq n_0 \),

\[
v_A(m_j)(G_n \setminus F_n) \leq \bigvee_{i=1}^{\infty} p_{i,\varphi(i)};
\]

(7)

and

\[
v_A(m_j)(G'_n \setminus W) \leq \bigvee_{i=1}^{\infty} p_{i,\varphi(i)}.
\]

(8)

**Proof:** Set

\[
u \equiv \sup_j \left[ \sup_{A \in A} |m_j(A)| \right].
\]

(9)

By hypothesis, for every \( A \in A \) and \( j \in \mathbb{N} \), there exist a regulator \((\gamma_{i,l})_{i,l}\) and two sequences \((G_{n,j})_n\), \((F_{n,j})_n\) such that \( F_{n,j} \in F \), \( G_{n,j} \in G \) \( \forall j, n \in \mathbb{N} \), and

\[
F_{n,j} \subset F_{n+1,j} \subset A \subset G_{n+1,j} \subset G_{n,j} \quad \forall j, n \in \mathbb{N},
\]

(10)

and \( \forall \varphi \in \mathbb{N}^{\mathbb{N}} \) and \( j \in \mathbb{N} \) there exists \( n_0 \in \mathbb{N} \), \( n_0(\varphi, j) \), such that \( \forall n \geq n_0 \) we have:

\[
v_A(m_j)(G_{n,j} \setminus F_{n,j}) \leq \bigvee_{i=1}^{\infty} \gamma_{i,\varphi(i)}.
\]

(11)
Moreover, by Lemma 2.3, there exists a regulator \((p_{i,l})_{i,l}\) such that

\[
\left( \bigwedge_{j=1}^{\infty} \sum_{i=1}^{\infty} \left( \bigvee_{i=1}^{\varphi(i)} \gamma_{i,\varphi(i+j)}^{(j)} \right) \right) \leq \bigvee_{i=1}^{\varphi(i)} p_{i,\varphi(i)} \quad \forall \varphi \in \mathbb{N}^{\mathbb{N}},
\]

where \(u\) is as in (9).

For every \(n \in \mathbb{N}\), set \(G_n \equiv \cap_{j \leq n} G_n^{(j)}\); \(F_n \equiv \cup_{j \leq n} F_n^{(j)}\). Then \(G_n \in \mathcal{G}\), \(F_n \in \mathcal{F}\), and \(A \subset G_n\), \(A \supset F_n\) \(\forall n \in \mathbb{N}\). Furthermore it is easy to check that \(G_{n+1} \subset G_n\) and \(F_{n+1} \supset F_n\) \(\forall n\). Since \(G_n \setminus F_n \subset G_n^{(j)} \setminus F_n^{(j)} \forall j, n \in \mathbb{N}\), then \(\forall \varphi \in \mathbb{N}^{\mathbb{N}}\) and \(\forall j\) there exists \(n_0 \in \mathbb{N}\), \(n_0(\varphi, j)\), such that

\[
v_A(m_j)(G_n \setminus F_n) \leq \bigvee_{i=1}^{\varphi(i)} p_{i,\varphi(i)} \quad \forall n \geq n_0,
\]

and thus (3) and (7) are proved.

The proof of (5) and (8) is analogous to the one of (3) and (7): indeed, by hypothesis and Lemma 2.3, the regulator \((p_{i,l})_{i,l}\) in (12) is such that for all \(W \in \mathcal{F}\) and \(j \in \mathbb{N}\) there exist two sequences \((G_n^{(j)})_n\), \((F_n^{(j)})_n\) such that \(F_n^{(j)} \in \mathcal{F}\), \(G_n^{(j)} \in \mathcal{G}\) \(\forall j, n \in \mathbb{N}\), and

\[
W \subset F_{n+1}^{(j)} \subset G_n^{(j)} \subset F_n^{(j)} \quad \forall j, n \in \mathbb{N},
\]

and \(\forall \varphi \in \mathbb{N}^{\mathbb{N}}\), \(\forall j \in \mathbb{N}\), there exists \(n_0 \in \mathbb{N}\), \(n_0(\varphi, j)\), such that

\[
v_A(m_j)(G_n^{(j)} \setminus W) \leq \bigvee_{i=1}^{\varphi(i)} p_{i,\varphi(i)} \quad \forall n \geq n_0.
\]

It is readily seen that (5) and (8) are satisfied with \(F_n' \equiv \cap_{j \leq n} F_n^{(j)}\), \(G_n' \equiv \cap_{j \leq n} G_n^{(j)}\).

\[\square\]

We now introduce the concept of absolute continuity in our setting.

**Definition 2.8** Let \(m\) be any \(R\)-valued finitely additive measure on \(\mathcal{A}\). Given any other finitely additive measure \(\nu : \mathcal{A} \rightarrow \mathbb{R}_0^{+}\), we say that \(m\) is **absolutely continuous** with respect to \(\nu\) (and write \(m \ll \nu\)) if there exists a \((D)\)-sequence \((a_{i,l})_{i,l}\) such that, whenever \((H_k)_k\) is a sequence from \(\mathcal{A}\), satisfying \(\lim_k \nu(H_k) = 0\), for every \(\varphi \in \mathbb{N}^{\mathbb{N}}\) an integer \(k\) can be found, such that \(|m(H_k)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\), for all \(k \geq k\).
In case $\nu$ is fixed, and $(m_j)_j$ is a sequence of finitely additive measures on $\mathcal{A}$, *uniform absolute continuity* of the $m_j$'s with respect to $\nu$ can be defined in a similar way, but clearly the integer $\bar{k}$ must be independent of $j$.

### 3 The Dieudonné theorem.

We shall prove a version of Dieudonné's Theorem (see also [5], [7]). We begin with the following:

**Lemma 3.1** Let $R$, $\Omega$, $\mathcal{A}$, $\mathcal{F}$, $\mathcal{G}$ be as in Proposition 2.7, and suppose that $m : \mathcal{A} \to R$ is any regular bounded finitely additive measure. Then, for each $A \in \mathcal{A}$, and every $V \in \mathcal{G}$, one has:

$$v_A(m)(A) = v_F(m)(A),$$

(16)

$$v_A(m)(V) = v_G(m)(V).$$

(17)

**Proof:** The relation (16) is a direct consequence of regularity, and weak $\sigma$-distributivity. So, fix $V \in \mathcal{G}$. Let $(\gamma_{i,l})_{i,l}$ be the $(D)$-sequence related to regularity, let $B$ be any element from $\mathcal{A}$, $B \subset V$, and fix $\varphi \in \mathbb{N}^\mathbb{N}$. Thanks to regularity of $m$, there exists a set $G \in \mathcal{G}$, $G \supset B$, such that

$$v(m)(G \setminus B) \leq \bigvee_{i=1}^{\infty} \gamma_{i,\varphi(i)},$$

hence

$$|m(B)| \leq |m(G)| + \bigvee_{i=1}^{\infty} \gamma_{i,\varphi(i)}.$$

Without loss of generality, we may assume $G \subset V$, thus

$$|m(B)| \leq v_G(m)(V) + \bigvee_{i=1}^{\infty} \gamma_{i,\varphi(i)}.$$

As $B$ is arbitrary, we get

$$v_A(m)(V) \leq v_G(m)(V) + \bigvee_{i=1}^{\infty} \gamma_{i,\varphi(i)}.$$
Finally, as \( R \) is weakly \( \sigma \)-distributive, we deduce
\[
v_A(m)(V) \leq v_G(m)(V)
\]
and then, obviously, the two elements coincide, and so (17) is proved. \( \square \)

We now prove the following:

**Lemma 3.2** Under the same hypotheses and notations as above, let \((m_j : A \to R)_j\) be a sequence of equibounded, regular and \( G \)-uniformly \((s)\)-bounded means (with respect to a \((D)\)-sequence \((b_{i,l})_{i,l}\)). Then the \( m_j \)'s are \( A \)-uniformly \((s)\)-bounded, and uniformly regular.

**Proof:** Let \((K_n)_n\) be any disjoint sequence in \( A \). First of all, we note that the hypotheses of Proposition 2.7 are fulfilled. Let \((p_{i,l})_{i,l}\) be the same regulator as in that Proposition, define \( u \) as in (9), and let \((d_{i,l})_{i,l}\) be a \((D)\)-sequence such that:
\[
u \bigwedge \left( \sum_{h=1}^{\infty} \left( \bigvee_{i=1}^{\infty} p_{i,\varphi(i+h)} \right) \right) \leq \bigvee_{i=1}^{\infty} d_{i,\varphi(i)} \ \forall \varphi \in \mathbb{N}^\mathbb{N}.
\]
Finally, let \( e_{i,l} = 2(b_{i,l} + d_{i,l}) \), \( i, l \in \mathbb{N} \). We will prove that
\[
(D) \lim_n \left\{ \sup_j [v_A(m_j)(K_n)] \right\} = 0
\]
with respect to the regulator \((e_{i,l})_{i,l}\). If we deny this, then there exists \( \varphi \in \mathbb{N}^\mathbb{N} \) such that \( \forall k \in \mathbb{N}, \exists n_k \geq k, \exists j_k \in \mathbb{N}, \exists A_k \in A \) with \( A_k \subset K_{n_k} \) and
\[
|m_{j_k}(A_k)| \leq \bigvee_{i=1}^{\infty} e_{i,\varphi(i)}.
\]
Moreover, thanks to (16), we can assume \( A_k \in \mathcal{F} \ \forall k \).

Fix \( k \in \mathbb{N} \), and from now on let’s write \( b = \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}, \ e = \bigvee_{i=1}^{\infty} e_{i,\varphi(i)} \). We note that, by virtue of regularity of the set functions \( m_j, j \in \mathbb{N} \), there exist \( G_k \in \mathcal{G}, \ F_k \in \mathcal{F} \) such that
\[
A_k \subset G_k \subset F_k,
\]
and
\[
[v_A(m_1) \lor \ldots \lor v_A(m_{j_k})](F_k \setminus A_k) \leq \bigvee_{i=1}^{\infty} p_{i,\varphi(i+k)}.
\]
Now, we set
\[ G^*_1 = G_1, G^*_2 = G_2 \setminus F_1, \ldots, G^*_k+1 = G_{k+1} \setminus \left( \bigcup_{h=1}^{k} F_h \right), \ldots \]

These sets are pairwise disjoint elements of \( \mathcal{G} \), hence there exists \( k_0 \in \mathbb{N} \) such that
\[ \sup_j v(m_j)(G^*_k) \leq b \]
for all \( k \geq k_0 \). Now, as
\[ A_{k+1} \setminus G^*_k+1 \subset \bigcup_{h=1}^{k} (F_h \setminus A_h) \]
holds for all \( k \), we get
\[ |m_{jk}(A_k)| \leq |m_{jk}(A_k \cap G^*_k)| + |m_{jk}(A_k \setminus G^*_k)| \leq b + u \bigwedge \left[ \sum_{h=1}^{k} \left( \bigvee_{i=1}^{\infty} p_{i,\varphi(i+h)} \right) \right] \leq e, \quad \forall k \geq k_0. \]
This is contrary to (18). So, the set functions \( m_j \) are \( \mathcal{A} \)-uniformly \( (s) \)-bounded.

We now turn to uniform regularity. By Proposition 2.7, the regulator \( (p_{i,l})_{i,l} \) above is such that, for every \( A \in \mathcal{A} \), two sequences can be found, \( (F_n)_n \) and \( (G_n)_n \) in \( \mathcal{F} \) and \( \mathcal{G} \) respectively, satisfying (3) and (7). As the sequence \( (G_n \setminus F_n)_n \) is decreasing, by (7), 2.5 and weak \( \sigma \)-distributivity of \( R \) we have
\[ (D) \lim_n \{ \sup_j [v_{\mathcal{A}}(m_j)(G_n \setminus F_n)] \} = 0. \quad (19) \]
Similarly, for each \( W \in \mathcal{F} \), we can find \( (F'_n)_n \) and \( (G'_n)_n \) in \( \mathcal{F} \) and \( \mathcal{G} \) respectively, satisfying (5) and (8). Since the sequence \( (G'_n \setminus W)_n \) is decreasing, by virtue of (8), 2.5 and weak \( \sigma \)-distributivity of \( R \) we get
\[ (D) \lim_n \{ \sup_j [v_{\mathcal{A}}(m_j)(G'_n \setminus W)] \} = 0. \quad (20) \]
This concludes the proof of the lemma. \( \square \)

**Theorem 3.3 (Dieudonné)** Let \( \Omega, R, \mathcal{G}, \mathcal{F} \) be as above, and assume that \( \mathcal{A} \subset \mathcal{P}(\Omega) \) is a \( \sigma \)-algebra, and \( \mathcal{G} \) is stable under countable disjoint unions. Suppose that
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\((m_j : A \to R)_j\) is a sequence of equibounded regular \(\sigma\)-additive measures such that there exists

\[ m_0 = (RD) \lim_j m_j \text{ in } G. \]

Then we have:

i) The measures \(m_j, j \in \mathbb{N}\), are \(A\)-uniformly \((s)\)-bounded and uniformly regular.

ii) There exists in \(R\) the limit \(m_0 = (RD) \lim_j m_j\) in \(A\).

iii) The \(m_j\)'s are uniformly \(\sigma\)-additive.

iv) \(m_0\) is regular and \(\sigma\)-additive.

**Proof:**

i) Thanks to [4], Theorem 5.4, the set functions \(m_j\) are \(G\)-uniformly \((s)\)-bounded; hence, from Lemma 3.2 we get \(A\)-uniform \((s)\)-boundedness and uniform regularity.

ii) Fix \(A \in A\), and let \((y_{i,l})_{i,l}\) be the regulator related with uniform regularity. For each \(\varphi \in \mathbb{N}^{\mathbb{N}}\) there exists \(G \in G\) such that \(A \subset G\) and

\[ v_A(m_j)(G \setminus A) \leq \bigvee_{i=1}^{\infty} y_{i,\varphi(i)} \quad \forall j. \]

Corresponding to \(G\), there exists \(j_0 \in \mathbb{N}\) such that

\[ |m_j(G) - m_{j+p}(G)| \leq \bigvee_{i=1}^{\infty} \alpha_{i,\varphi(i)} \quad \forall j \geq j_0, \quad \forall p \in \mathbb{N}, \]

where \((\alpha_{i,l})_{i,l}\) is the regulator for \((RD)\)-convergence in \(G\). So we have:

\[ |m_j(A) - m_{j+p}(A)| \leq 2 \bigvee_{i=1}^{\infty} y_{k,\varphi(i)} + \bigvee_{i=1}^{\infty} \alpha_{i,\varphi(i)} \quad \forall j \geq j_0, \quad \forall p \in \mathbb{N}. \quad (21) \]

From (21) it follows that the sequence \((m_j(A))_j\) is \((D)\)-Cauchy in \(R\). Since \(R\) is a Dedekind complete \((l)\)-group, then the sequence \((m_j(A))_j\) is \((D)\)-convergent (see also [3], Theorem 2.16; [9]). Thus ii) is proved.

iii) follows from ii) and [4], Corollary 5.5.

iv) follows easily from i), ii), iii) and weak \(\sigma\)-distributivity of \(R\). □

Under suitable additional conditions, it’s also possible to state a finitely additive version of Dieudonné’s theorem.
Theorem 3.4 Let $\Omega$, $R$, $A$, $G$, $F$ be as in Proposition 2.7, and assume that $G$ is stable under countable disjoint unions. Suppose that $(m_j : A \to R)_j$ is a sequence of equibounded regular finitely additive measures, absolutely continuous with respect to a real-valued, nonnegative, finitely additive measure $\nu$ on $A$. Assume that there exists

$$m_0 = (RD) \lim_j m_j \text{ in } G.$$ 

Then we have:

i) The means $m_j$, $j \in \mathbb{N}$, are $A$-uniformly $(s)$-bounded, uniformly regular and uniformly absolutely continuous with respect to $\nu$.

ii) There exists in $R$ the limit $m_0 = (RD) \lim_j m_j$ in $A$.

iii) $m_0$ is $(s)$-bounded, regular and absolutely continuous with respect to $\nu$.

Proof: (i) Let $(\alpha_{i,l})_{i,l}$ be the regulator related to $(RD)$-convergence in $G$, and let $(\beta_{i,l})_{i,l}$ be a regulator such that, for every disjoint sequence $(H_k)_k$ in $A$, for every $j \in \mathbb{N}$ and every $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exists $k_0 \in \mathbb{N}$ such that

$$v(m_j)(H_k) \leq \sum_{i=1}^{\infty} \beta_{i,\varphi(i)}$$

as soon as $k \geq k_0$: such a regulator exists, because of absolute continuity and Lemma 2.3. Setting $c_{i,l} = \alpha_{i,l} \vee \beta_{i,l}$, we claim that $(6c_{i,l})_{i,l}$ works as a regulator for $G$-uniform $(s)$-boundedness of the means $m_j$. Indeed, if this is not the case, there exist: a disjoint sequence $(G_k)_k$ in $G$, a mapping $\varphi \in \mathbb{N}^{\mathbb{N}}$ and a subsequence $(j_k)_k$ in $\mathbb{N}$ such that

$$|m_{j_k}(G_k)| \leq 6 \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}$$

(22)

for each $k \in \mathbb{N}$. Now, denote by $V$ the union of all $G_k$'s, and by $B$ the $\sigma$-algebra in $V$ generated by the sets $G_k$: hence the measures $m_j$ $(RD)$-converge to $m_0$ in $B$. Then, we can apply Corollary 5.7 of [4], and deduce $B$-uniform $(s)$-boundedness of the $m_j$'s, with respect to the regulator $(6c_{i,l})_{i,l}$, and this clearly is contrary to (22). Thus, the $m_j$'s are $G$-uniformly $(s)$-bounded, and therefore they are $A$-uniformly
(s)-bounded and uniformly regular, by 3.2, and uniformly absolutely continuous with respect to $\nu$, by virtue of [4], Theorem 4.8.

(ii) can be proved as in the previous theorem.

(iii) The properties of (s)-boundedness, regularity and absolute continuity are easy consequences of the previous ones and of weak $\sigma$-distributivity of $R$. □

References


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