Dieudonné-type theorems for set functions with values in (l)-groups

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ABSTRACT. Some versions of Dieudonné theorems are given for set functions, not necessarily positive, taking values in Dedekind complete (l)-groups, relatively to the "(D)-convergence".

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1 Introduction.

In a previous paper ([4]), we gave some versions of Vitali - Hahn - Saks and Nikodým theorems for set functions with values in suitable Dedekind complete (l)-groups. In this paper, we prove some versions of Dieudonné theorems, for (l)-group-valued finitely additive regular maps. In the literature, there exist several versions of theorems of this kind, for maps taking values in topological groups and/or Banach spaces. Among the authors, we recall Brooks and Chacon ([5], [6]), Candeloro and Letta ([7], [8]).

In the previous paper [2] similar results were proved with respect to the order convergence for *positive* means taking values in spaces of the type $L^0(X, \mathcal{B}, \mu)$, where

μ is a σ-additive locally finite positive $\widetilde{I\!R}$-valued measure.

2 Preliminaries.

We begin with the following:

Definitions 2.1 An Abelian group $(R,+)$ is called (l)-*group* if it is endowed with a compatible ordering $\leq$, and is a lattice with respect to it.
An (l)-group R is said to be *Dedekind complete* if every nonempty subset of R, bounded from above, has supremum in R. A sequence $(r_n)_n$ in R is said to be *order-convergent* (or (o)-*convergent*) to r if there exists a sequence $(p_n)_n$ in R such that $p_n \downarrow 0$ and $|r_n - r| \leq p_n$, $\forall\, n \in I\!N$ (see also [11], [15]), and we will write $(o)\lim_n r_n = r$. A bounded double sequence $(a_{i,l})_{i,l}$ in R is called (D)-*sequence* or *regulator* if for all $i \in I\!N$ we have $a_{i,l} \downarrow 0$ as $l \to +\infty$. A sequence $(r_n)_n$ in R is said to be (D)-*convergent* to $r \in R$ (and we write $(D)\lim_n r_n = r$) if there exists a (D)-sequence $(a_{i,l})_{i,l}$ in R, such that $\forall\, \varphi \in I\!N^{I\!N}$, $\exists\, n_0 \in I\!N$ such that $|r_n - r| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\, \forall\, n \in I\!N$, $n \geq n_0$. If Λ is any nonempty set, $(r_n^{(\lambda)})_n$ are sequences in R and $r^{(\lambda)}$ is in R for all $\lambda \in \Lambda$, we say that $(D)\lim_n r_n^{(\lambda)} = r^{(\lambda)}$ *uniformly with respect to* $\lambda \in \Lambda$ if there exists a (D)-sequence $(a_{i,l})_{i,l}$ in R, such that $\forall\, \varphi \in I\!N^{I\!N}$, $\exists\, n_0 \in I\!N$ such that $|r_n^{(\lambda)} - r^{(\lambda)}| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$ $\forall\, n \in I\!N$, $n \geq n_0$ and $\forall\, \lambda \in \Lambda$. The sequence $(r_n)_n$ is said to be (D)-*Cauchy* if $(D)\lim_n (r_n - r_{n+p}) = 0$ uniformly with respect to $p \in I\!N$.

An intermediate condition is the so-called (RD)-convergence: if Λ is any nonempty set, $(r_n^{(\lambda)})_n$ are sequences in R and $r^{(\lambda)}$ is in R for all $\lambda \in \Lambda$, we say that $r_n^{(\lambda)}$ *(RD)-converges* to $r^{(\lambda)}$ (convergence *with respect to the same regulator)* if there exists a (D)-sequence $(a_{i,l})_{i,l}$ in R, such that $\forall\, \varphi \in I\!N^{I\!N}$, and every $\lambda \in \Lambda$ there exists $n_0 \in I\!N$ such that $|r_n^{(\lambda)} - r^{(\lambda)}| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$ $\forall\, n \in I\!N$, $n \geq n_0$. In Lemma 2.3 there will be stated a relationship between simple D-convergence and (RD)-convergence, at least when Λ is countable.

In general, the limit of a sequence (with respect to (D)-convergence) is not unique. However, there are some conditions on R, which are equivalent to uniqueness of the limit: for example, weak σ-distributivity, whose definition we report here below:

Definition 2.2 An (l)-group R is said to be *weakly σ-distributive* if for every (D)-sequence $(a_{i,l})_{i,l}$ we have:

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) = 0.$$

We note that (o)-convergence of sequences always implies (D)-convergence, and these two convergences are equivalent when R is weakly σ-distributive (see [9]). We now recall the following result (see [14], pp. 42-43), which will be useful in the sequel.

Lemma 2.3 *Let R be a Dedekind complete (l)-group (not necessarily weakly σ-distributive), $(a_{i,l}^{(n)})_{i,l}$, $n \in \mathbb{N}$, be a sequence of regulators in R. Then for every $u \in R$, $u \geq 0$ there exists a (D)-sequence $(a_{i,l})_{i,l}$ in R such that:*

$$u \bigwedge \left[\sum_{n=1}^{\infty} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i+n)}^{(n)} \right) \right] \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \qquad \forall\, \varphi \in \mathbb{N}^{\mathbb{N}}.$$

We now introduce the following:

Definitions 2.4 Let Ω be any infinite set, $\mathcal{A} \subset \mathcal{P}(\Omega)$ be an algebra, R be a Dedekind complete weakly σ-distributive (l)-group. We say that $m : \mathcal{A} \to R$ is *bounded* if $\exists\, w \in R$, $w \geq 0$: $|m(A)| \leq w$, $\forall\, A \in \mathcal{A}$. The maps $m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, are *equibounded* if there exists an element $u \in R$, $u \geq 0$, such that

$$|m_j(A)| \leq u \quad \forall\, j \in \mathbb{N}, \forall\, A \in \mathcal{A}. \tag{1}$$

Given a finitely additive bounded measure (or, in short, *mean*) $m : \mathcal{A} \to R$, define the *semivariation* of m,

$$v_{\mathcal{A}}(m) : \mathcal{A} \to R,$$

or simply

$$v(m) : \mathcal{A} \to R,$$

by setting

$$v_{\mathcal{A}}(m)(A) = \sup_{B\in\mathcal{A}, B\subset A} |m(B)|, \ \forall\, A \in \mathcal{A},$$

and, if $\emptyset \neq \mathcal{E} \subset \mathcal{A}$, define $v_{\mathcal{E}}(m) : \mathcal{A} \to R$, by setting

$$v_{\mathcal{E}}(m)(A) = \sup_{B\in\mathcal{E}, B\subset A} |m(B)|, \ \forall\, A \in \mathcal{A}.$$

A mean $m : \mathcal{A} \to R$ is said to be *σ-additive* (or, in short, *measure*) if there exists a (D)-sequence $(u_{i,l})_{i,l}$ such that, $\forall\, \varphi \in I\!N^{I\!N}$ and for every decreasing sequence $(H_s)_s$ in $\mathcal{A}$, $H_s \downarrow \emptyset$, there exists $\overline{s}$:

$$v_{\mathcal{A}}(m)(H_{\overline{s}}) \leq \bigvee_{i=1}^{\infty} u_{i,\varphi(i)}.$$

If a sequence of measures $m_j : \mathcal{A} \to R$, $j \in I\!N$, is given, *uniform σ-additivity* is defined as above, but with $\overline{s}$ *independent* of j (See also [4]).
A finitely additive measure $m : \mathcal{A} \to R$ is said to be *(s)-bounded in $\emptyset \neq \mathcal{E} \subset \mathcal{A}$*, or simply *$\mathcal{E} - (s)$-bounded,* if there exists a (D)-sequence $(w_{i,l})_{i,l}$ such that, $\forall\, \varphi \in I\!N^{I\!N}$ and for every disjoint sequence $(H_s)_s$ in $\mathcal{E}$ there exists $\overline{s}$: $\forall\, s \geq \overline{s}$,

$$v_{\mathcal{E}}(m)(H_s) \leq \bigvee_{i=1}^{\infty} w_{i,\varphi(i)}.$$

If $\mathcal{E}$ is as above, we say that the maps $m_j : \mathcal{A} \to R$, $j \in I\!N$, are *$\mathcal{E}$-uniformly (s)-bounded* if the above condition holds, but with $\overline{s}$ independent of j (see also [4]).

When $\mathcal{E} = \mathcal{A}$ we simply speak of *$(s)-$boundedness* or *uniform $(s)-$boundedness*.

A typical consequence of (s)-boundedness of a mean m is that there exists the limit $m(A_n)$ for monotone sequences $(A_n)_n$ in $\mathcal{A}$ (see [4]). As to uniformly (s)-bounded measures, we shall report here a slight modification of Proposition 3.4 of [4], which will be used later.

Proposition 2.5 *Assume that $(m_j)_j$ is a sequence of R-valued uniformly (s)-bounded finitely additive measures on $\mathcal{A}$, and let $(e_{i,l})_{i,l}$ be a regulator related to this property. For every decreasing sequence $(A_n)_n$ in $\mathcal{A}$, the limit*

$$r_j := (RD) \lim_{n\to\infty} v(m_j)(A_n)$$

exists uniformly in j, and the regulator $(e_{i,l})_{i,l}$ works for this property.

Given a sequence of means $(m_j)_{j\in I\!N\cup\{0\}}$, $m_j : \mathcal{A} \to R$, and a nonempty subfamily $\mathcal{E} \subset \mathcal{A}$, we say that the m_j's (D)-*converge to* m_0 *pointwise with respect to the same regulator,* or in short $(RD)\lim_j m_j = m_0$ *in* $\emptyset \neq \mathcal{E} \subset \mathcal{A}$, if there exists a (D)-sequence $(b_{i,l})_{i,l}$ such that $\forall\, \varphi \in I\!N^{I\!N}$, $\forall\, A \in \mathcal{E}$, $\exists\, j_0 \in I\!N$ such that

$$|m_j(A) - m_0(A)| \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \quad \forall\, j \in I\!N, j \geq j_0. \tag{2}$$

We note that the condition (2) is equivalent to the classical pointwise convergence of the involved set functions in the case of metrizable groups.

Let now Ω, R and $\mathcal{A}$ be as above. From now on, we assume that R is weakly σ-distributive, and $\mathcal{F}$, $\mathcal{G} \subset \mathcal{A}$ are two fixed lattices, such that the complement (with respect to Ω) of every element of $\mathcal{F}$ belongs to $\mathcal{G}$. In the sequel we will not say it explicitly. If Ω is a topological normal space [resp. locally compact Hausdorff space], examples of $\mathcal{A}$, $\mathcal{F}$ and $\mathcal{G}$, satisfying the above properties, are the following: $\mathcal{A} = \{$Borelian subsets of $\Omega\}$, $\mathcal{F} = \{$closed sets $\}$[resp.$\{$compact sets$\}$], $\mathcal{G} = \{$ open sets $\}$.

Definitions 2.6 A mean $m : \mathcal{A} \to R$ is said to be *regular* if there exists a (D)-sequence $(\gamma_{i,l})_{i,l}$ in R such that for each $A \in \mathcal{A}$, $\forall\, n \in I\!N$, $\exists\, F_n \in \mathcal{F}$, $G_n \in \mathcal{G}$ such that

$$F_n \subset F_{n+1} \subset A \subset G_{n+1} \subset G_n \quad \forall\, n, \tag{3}$$

and $\forall\, \varphi \in I\!N^{I\!N}$ there exists $n_0(A, \varphi) \in I\!N$ such that $\forall\, n \geq n_0$,

$$v_{\mathcal{A}}(m)(G_n \setminus F_n) \leq \bigvee_{i=1}^{\infty} \gamma_{i,\varphi(i)}; \tag{4}$$

and if $\forall\, W \in \mathcal{F}$, $\forall\, n \in I\!N$, $\exists\, F'_n \in \mathcal{F}$, $G'_n \in \mathcal{G}$ such that

$$W \subset F'_{n+1} \subset G'_n \subset F'_n \quad \forall\, n, \tag{5}$$

and $\forall\, \varphi \in I\!N^{I\!N}$ there exists $n_0(A, \varphi) \in I\!N$ such that $\forall\, n \geq n_0$,

$$v_{\mathcal{A}}(m)(G'_n \setminus W) \leq \bigvee_{i=1}^{\infty} \gamma_{i,\varphi(i)}. \tag{6}$$

The means $m_j : \mathcal{A} \to R$, $j \in I\!N$, are said to be *uniformly regular* if there exists a (D)-sequence $(\gamma_{i,l})_{i,l}$ in R such that $\forall\, A \in \mathcal{A}$ and $\forall\, W \in \mathcal{F}$ there exist sequences $(F_n)_n, (G_n)_n, (F'_n)_n, (G'_n)_n$ as above , such that for each $\varphi \in I\!N^{I\!N}$, (4) and (6) are satisfied, with n_0 independent of j.

The following proposition shows that, if $(m_j : \mathcal{A} \to R)_j$ is a sequence of equibounded regular means, even if they are not uniformly regular, the sequences $(\gamma_{i,l})_{i,l}$, $(F_n)_n$, $(G_n)_n$, $(F'_n)_n$, $(G'_n)_n$ above can be taken independently of j, satisfying (3),(4), (5) and (6) $\forall\, j \in I\!N$.

Proposition 2.7 *Let R be as above, $\mathcal{A}$ be any algebra, and $(m_j : \mathcal{A} \to R)_j$ be a sequence of equibounded regular means. Then there exists a regulator $(p_{i,l})_{i,l}$ such that, for every $A \in \mathcal{A}$ and every $W \in \mathcal{F}$ there exist sequences $(F_n)_n$,$(F'_n)_n$ in $\mathcal{F}$, $(G_n)_n$, $(G'_n)_n$ in $\mathcal{G}$, satisfying (3) and (5) and such that $\forall\, \varphi \in I\!N^{I\!N}$ and $j \in I\!N$ there exists $n_0(A, W, \varphi, j) \in I\!N$ such that $\forall\, n \geq n_0$,*

$$v_{\mathcal{A}}(m_j)(G_n \setminus F_n) \leq \bigvee_{i=1}^{\infty} p_{i,\varphi(i)}; \tag{7}$$

and

$$v_{\mathcal{A}}(m_j)({G'}_n \setminus W) \leq \bigvee_{i=1}^{\infty} p_{i,\varphi(i)}. \tag{8}$$

Proof: Set

$$u \equiv \sup_j \left[\sup_{A \in \mathcal{A}} |m_j(A)| \right]. \tag{9}$$

By hypothesis, for every $\mathcal{A} \in \mathcal{A}$ and $j \in I\!N$, there exist a regulator $(\gamma^{(j)}_{i,l})_{i,l}$ and two sequences $(G^{(j)}_n)_n$, $(F^{(j)}_n)_n$ such that $F^{(j)}_n \in \mathcal{F}$, $G^{(j)}_n \in \mathcal{G}$ $\forall\, j, n \in I\!N$, and

$$F^{(j)}_n \subset F^{(j)}_{n+1} \subset A \subset G^{(j)}_{n+1} \subset G^{(j)}_n \quad \forall\, j, n \in I\!N, \tag{10}$$

and $\forall\, \varphi \in I\!N^{I\!N}$ and $j \in I\!N$ there exists $n_0 \in I\!N$, $n_0(\varphi, j)$, such that $\forall\, n \geq n_0$ we have:

$$v_{\mathcal{A}}(m_j)(G^{(j)}_n \setminus F^{(j)}_n) \leq \bigvee_{i=1}^{\infty} \gamma^{(j)}_{i,\varphi(i)}. \tag{11}$$

Moreover, by Lemma 2.3, there exists a regulator $(p_{i,l})_{i,l}$ such that

$$u \bigwedge \left[\sum_{j=1}^{\infty} \left(\bigvee_{i=1}^{\infty} \gamma^{(j)}_{i,\varphi(i+j)} \right) \right] \leq \bigvee_{i=1}^{\infty} p_{i,\varphi(i)} \qquad \forall \varphi \in I\!N^{I\!N}, \tag{12}$$

where u is as in (9).

For every $n \in I\!N$, set $G_n \equiv \cap_{j \le n} G_n^{(j)}$; $F_n \equiv \cup_{j \le n} F_n^{(j)}$: then $G_n \in \mathcal{G}$, $F_n \in \mathcal{F}$, and $A \subset G_n$, $A \supset F_n$ $\forall n \in I\!N$. Furthermore it is easy to check that $G_{n+1} \subset G_n$ and $F_{n+1} \supset F_n$ $\forall n$. Since $G_n \setminus F_n \subset G_n^{(j)} \setminus F_n^{(j)}$ $\forall j, n \in I\!N$, then $\forall \varphi \in I\!N^{I\!N}$ and $\forall j$ there exists $n_0 \in I\!N$, $n_0(\varphi, j)$, such that

$$v_{\mathcal{A}}(m_j)(G_n \setminus F_n) \leq \bigvee_{i=1}^{\infty} p_{i,\varphi(i)} \quad \forall n \geq n_0, \tag{13}$$

and thus (3) and (7) are proved.

The proof of (5) and (8) is analogous to the one of (3) and (7): indeed, by hypothesis and Lemma 2.3, the regulator $(p_{i,l})_{i,l}$ in (12) is such that for all $W \in \mathcal{F}$ and $j \in I\!N$ there exist two sequences $({G'_n}^{(j)})_n$, $({F'_n}^{(j)})_n$ such that ${F'_n}^{(j)} \in \mathcal{F}$, ${G'_n}^{(j)} \in \mathcal{G}$ $\forall j, n \in I\!N$, and

$$W \subset {F'_{n+1}}^{(j)} \subset {G'_n}^{(j)} \subset {F'_n}^{(j)} \quad \forall j, n \in I\!N, \tag{14}$$

and $\forall \varphi \in I\!N^{I\!N}$, $\forall j \in I\!N$, there exists $n_0 \in I\!N$, $n_0(\varphi, j)$, such that

$$v_{\mathcal{A}}(m_j)(G_n^{(j)} \setminus W) \leq \bigvee_{i=1}^{\infty} p_{i,\varphi(i)} \quad \forall n \geq n_0. \tag{15}$$

It is readily seen that (5) and (8) are satisfied with $F'_n \equiv \cap_{j \le n} {F'_n}^{(j)}$, $G'_n \equiv \cap_{j \le n} {G'_n}^{(j)}$.
□

We now introduce the concept of absolute continuity in our setting.

Definition 2.8 Let m be any R-valued finitely additive measure on $\mathcal{A}$. Given any other finitely additive measure $\nu : \mathcal{A} \to I\!R_0^+$, we say that m is *absolutely continuous* with respect to ν (and write $m \ll \nu$) if there exists a (D)-sequence $(a_{i,l})_{i,l}$ such that, whenever $(H_k)_k$ is a sequence from $\mathcal{A}$, satisfying $\lim_k \nu(H_k) = 0$, for every $\varphi \in I\!N^{I\!N}$ an integer $\overline{k}$ can be found, such that $|m(H_k)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$, for all $k \geq \overline{k}$.

In case ν is fixed, and $(m_j)_j$ is a sequence of finitely additive measures on $\mathcal{A}$, *uniform absolute continuity* of the m_j's with respect to ν can be defined in a similar way, but clearly the integer $\overline{k}$ must be independent of j.

3 The Dieudonné theorem.

We shall prove a version of Dieudonné 's Theorem (see also [5], [7]).We begin with the following:

Lemma 3.1 *Let R, Ω, $\mathcal{A}$, $\mathcal{F}$, $\mathcal{G}$ be as in Proposition 2.7, and suppose that $m : \mathcal{A} \to R$ is any regular bounded finitely additive measure. Then, for each $A \in \mathcal{A}$, and every $V \in \mathcal{G}$, one has:*

$$v_{\mathcal{A}}(m)(A) = v_{\mathcal{F}}(m)(A), \tag{16}$$

$$v_{\mathcal{A}}(m)(V) = v_{\mathcal{G}}(m)(V). \tag{17}$$

Proof: The relation (16) is a direct consequence of regularity, and weak σ-distributivity. So, fix $V \in \mathcal{G}$. Let $(\gamma_{i,l})_{i,l}$ be the (D)-sequence related to regularity, let B be any element from $\mathcal{A}$, $B \subset V$, and fix $\varphi \in I\!N^{I\!N}$. Thanks to regularity of m, there exists a set $G \in \mathcal{G}$, $G \supset B$, such that

$$v(m)(G \setminus B) \leq \bigvee_{i=1}^{\infty} \gamma_{i,\varphi(i)},$$

hence

$$|m(B)| \leq |m(G)| + \bigvee_{i=1}^{\infty} \gamma_{i,\varphi(i)}.$$

Without loss of generality, we may assume $G \subset V$, thus

$$|m(B)| \leq v_{\mathcal{G}}(m)(V) + \bigvee_{i=1}^{\infty} \gamma_{i,\varphi(i)}.$$

As B is arbitrary, we get

$$v_{\mathcal{A}}(m)(V) \leq v_{\mathcal{G}}(m)(V) + \bigvee_{i=1}^{\infty} \gamma_{i,\varphi(i)}.$$

Finally, as R is weakly σ-distributive, we deduce

$$v_{\mathcal{A}}(m)(V) \leq v_{\mathcal{G}}(m)(V)$$

and then, obviously, the two elements coincide, and so (17) is proved. $\square$

We now prove the following:

Lemma 3.2 *Under the same hypotheses and notations as above, let* $(m_j : \mathcal{A} \to R)_j$ *be a sequence of equibounded, regular and* $\mathcal{G}$*-uniformly* (s)*-bounded means (with respect to a* (D)*-sequence* $(b_{i,l})_{i,l}$*). Then the* m_j*'s are* $\mathcal{A}$*-uniformly* (s)*-bounded, and uniformly regular.*

Proof: Let $(K_n)_n$ be any disjoint sequence in $\mathcal{A}$. First of all, we note that the hypotheses of Proposition 2.7 are fulfilled. Let $(p_{i,l})_{i,l}$ be the same regulator as in that Proposition, define u as in (9), and let $(d_{i,l})_{i,l}$ be a (D)-sequence such that:

$$u \bigwedge \left[\sum_{h=1}^{\infty} \left(\bigvee_{i=1}^{\infty} p_{i,\varphi(i+h)} \right) \right] \leq \bigvee_{i=1}^{\infty} d_{i,\varphi(i)} \quad \forall \varphi \in I\!N^{I\!N}.$$

Finally, let $e_{i,l} = 2(b_{i,l} + d_{i,l})$, $i, l \in I\!N$. We will prove that

$$(D) \lim_n \{ \sup_j [v_{\mathcal{A}}(m_j)(K_n)] \} = 0$$

with respect to the regulator $(e_{i,l})_{i,l}$. If we deny this, then there exists $\varphi \in I\!N^{I\!N}$ such that $\forall\, k \in I\!N$, $\exists\, n_k \geq k$, $\exists\, j_k \in I\!N$, $\exists\, A_k \in \mathcal{A}$ with $A_k \subset K_{n_k}$ and

$$|m_{j_k}(A_k)| \not\leq \bigvee_{i=1}^{\infty} e_{i,\varphi(i)}. \tag{18}$$

Moreover, thanks to (16), we can assume $A_k \in \mathcal{F}\ \forall\, k$.

Fix $k \in I\!N$, and from now on let's write $b = \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}$, $e = \bigvee_{i=1}^{\infty} e_{i,\varphi(i)}$. We note that, by virtue of regularity of the set functions m_j, $j \in I\!N$, there exist $G_k \in \mathcal{G}$, $F_k \in \mathcal{F}$ such that

$$A_k \subset G_k \subset F_k,$$

and

$$[v_{\mathcal{A}}(m_1) \vee \ldots \vee v_{\mathcal{A}}(m_{j_k})](F_k \setminus A_k) \leq \bigvee_{i=1}^{\infty} p_{i,\varphi(i+k)}.$$

Now, we set

$$G_1^* = G_1, G_2^* = G_2 \setminus F_1, \ldots, G_{k+1}^* = G_{k+1} \setminus \left(\bigcup_{h=1}^{k} F_h \right), \ldots$$

These sets are pairwise disjoint elements of $\mathcal{G}$, hence there exists $k_0 \in I\!N$ such that

$$\sup_j v(m_j)(G_k^*) \leq b$$

for all $k \geq k_0$. Now, as

$$A_{k+1} \setminus G_{k+1}^* \subset \bigcup_{h=1}^{k} (F_h \setminus A_h)$$

holds for all k, we get

$$\begin{aligned} |m_{j_k}(A_k)| &\leq |m_{j_k}(A_k \cap G_k^*)| + |m_{j_k}(A_k \setminus G_k^*)| \\ &\leq b + u \bigwedge \left[\sum_{h=1}^{k} \left(\bigvee_{i=1}^{\infty} p_{i,\varphi(i+h)} \right) \right] \leq e, \quad \forall\, k \geq k_0. \end{aligned}$$

This is contrary to (18). So, the set functions m_j are $\mathcal{A}$-uniformly (s)-bounded.

We now turn to uniform regularity. By Proposition 2.7, the regulator $(p_{i,l})_{i,l}$ above is such that, for every $A \in \mathcal{A}$, two sequences can be found, $(F_n)_n$ and $(G_n)_n$ in $\mathcal{F}$ and $\mathcal{G}$ respectively, satisfying (3) and (7). As the sequence $(G_n \setminus F_n)_n$ is decreasing, by (7), 2.5 and weak σ-distributivity of R we have

$$(D) \lim_n \{ \sup_j [v_{\mathcal{A}}(m_j)(G_n \setminus F_n)] \} = 0. \tag{19}$$

Similarly, for each $W \in \mathcal{F}$, we can find $(F'_n)_n$ and $(G'_n)_n$ in $\mathcal{F}$ and $\mathcal{G}$ respectively, satisfying (5) and (8). Since the sequence $(G'_n \setminus W)_n$ is decreasing, by virtue of (8), 2.5 and weak σ-distributivity of R we get

$$(D) \lim_n \{ \sup_j [v_{\mathcal{A}}(m_j)(G'_n \setminus W)] \} = 0. \tag{20}$$

This concludes the proof of the lemma. $\square$

Theorem 3.3 (Dieudonné) *Let* Ω, R, $\mathcal{G}$, $\mathcal{F}$ *be as above, and assume that* $\mathcal{A} \subset \mathcal{P}(\Omega)$ *is a* σ*-algebra, and* $\mathcal{G}$ *is stable under countable disjoint unions. Suppose that*

$(m_j : \mathcal{A} \to R)_j$ is a sequence of equibounded regular σ-additive measures such that there exists

$$m_0 = (RD) \lim_j m_j \quad \text{in } \mathcal{G}.$$

Then we have:

i) *The measures m_j, $j \in I\!N$, are $\mathcal{A}$-uniformly (s)-bounded and uniformly regular.*

ii) *There exists in R the limit $m_0 = (RD) \lim_j m_j$ in $\mathcal{A}$.*

iii) *The m_j's are uniformly σ-additive.*

iv) *m_0 is regular and σ-additive.*

Proof: i) Thanks to [4], Theorem 5.4, the set functions m_j are $\mathcal{G}$-uniformly (s)-bounded; hence, from Lemma 3.2 we get $\mathcal{A}$-uniform (s)-boundedness and uniform regularity.

ii) Fix $A \in \mathcal{A}$, and let $(y_{i,l})_{i,l}$ be the regulator related with uniform regularity. For each $\varphi \in I\!N^{I\!N}$ there exists $G \in \mathcal{G}$ such that $A \subset G$ and

$$v_{\mathcal{A}}(m_j)(G \setminus A) \leq \bigvee_{i=1}^{\infty} y_{i,\varphi(i)} \quad \forall\, j.$$

Corresponding to G, there exists $j_0 \in I\!N$ such that

$$|m_j(G) - m_{j+p}(G)| \leq \bigvee_{i=1}^{\infty} \alpha_{i,\varphi(i)} \quad \forall\, j \geq j_0, \quad \forall\, p \in I\!N,$$

where $(\alpha_{i,l})_{i,l}$ is the regulator for (RD)-convergence in $\mathcal{G}$. So we have:

$$|m_j(A) - m_{j+p}(A)| \leq 2 \bigvee_{i=1}^{\infty} y_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} \alpha_{i,\varphi(i)} \quad \forall\, j \geq j_0, \quad \forall\, p \in I\!N. \tag{21}$$

¿From (21) it follows that the sequence $(m_j(A))_j$ is (D)-Cauchy in R. Since R is a Dedekind complete (l)-group, then the sequence $(m_j(A))_j$ is (D)-convergent (see also [3], Theorem 2.16; [9]). Thus ii) is proved.

iii) follows from ii) and [4], Corollary 5.5.

iv) follows easily from i), ii), iii) and weak σ-distributivity of R. □

Under suitable additional conditions, it's also possible to state a finitely additive version of Dieudonné's theorem.

Theorem 3.4 *Let Ω, R, $\mathcal{A}$, $\mathcal{G}$, $\mathcal{F}$ be as in Proposition 2.7, and assume that $\mathcal{G}$ is stable under countable disjoint unions. Suppose that $(m_j : \mathcal{A} \to R)_j$ is a sequence of equibounded regular finitely additive measures, absolutely continuous with respect to a real-valued, nonnegative, finitely additive measure ν on $\mathcal{A}$. Assume that there exists*

$$m_0 = (RD) \lim_j m_j \quad \text{in } \mathcal{G}.$$

Then we have:

i) *The means m_j, $j \in I\!N$, are $\mathcal{A}$-uniformly (s)-bounded, uniformly regular and uniformly absolutely continuous with respect to ν.*

ii) *There exists in R the limit $m_0 = (RD)\lim_j m_j$ in $\mathcal{A}$.*

iii) *m_0 is (s)-bounded, regular and absolutely continuous with respect to ν.*

Proof: (i) Let $(\alpha_{i,l})_{i,l}$ be the regulator related to (RD)-convergence in $\mathcal{G}$, and let $(\beta_{i,l})_{i,l}$ be a regulator such that, for every disjoint sequence $(H_k)_k$ in $\mathcal{A}$, for every $j \in I\!N$ and every $\varphi \in I\!N^{I\!N}$, there exists $k_0 \in I\!N$ such that

$$v(m_j)(H_k) \le \bigvee_{i=1}^{\infty} \beta_{i,\varphi(i)}$$

as soon as $k \ge k_0$: such a regulator exists, because of absolute continuity and Lemma 2.3. Setting $c_{i,l} = \alpha_{i,l} \vee \beta_{i,l}$, we claim that $(6c_{i,l})_{i,l}$ works as a regulator for $\mathcal{G}$-uniform (s)-boundedness of the means m_j. Indeed, if this is not the case, there exist:
a disjoint sequence $(G_k)_k$ in $\mathcal{G}$, a mapping $\varphi \in I\!N^{I\!N}$ and a subsequence $(j_k)_k$ in $I\!N$ such that

$$|m_{j_k}(G_k)| \not\le 6 \bigvee_{i=1}^{\infty} c_{i,\varphi(i)} \tag{22}$$

for each $k \in I\!N$. Now, denote by V the union of all $G_k{}'s$, and by $\mathcal{B}$ the σ-algebra in V generated by the sets G_k: hence the measures m_j (RD)-converge to m_0 in $\mathcal{B}$. Then, we can apply Corollary 5.7 of [4], and deduce $\mathcal{B}$-uniform (s)-boundedness of the $m_j{}'s$, with respect to the regulator $(6c_{i,l})_{i,l}$, and this clearly is contrary to (22). Thus, the $m_j{}'s$ are $\mathcal{G}$-uniformly (s)-bounded, and therefore they are $\mathcal{A}$-uniformly

(s)-bounded and uniformly regular, by 3.2, and uniformly absolutely continuous with respect to ν, by virtue of [4], Theorem 4.8.

(ii) can be proved as in the previous theorem.

(iii) The properties of (s)-boundedness, regularity and absolute continuity are easy consequences of the previous ones and of weak σ-distributivity of R. □

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