Vitali and Schur-type theorems for Riesz-space-valued set functions

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1 Introduction.

In the literature, there exist several versions of Vitali-Hahn-Saks, Brooks-Jewett and Nikodým convergence theorems, for set functions, taking values in topological groups and Banach spaces. Among the authors, we recall Brooks and Jewett ([7], [9]), Candeloro and Letta ([10], [11]), de Lucia, Fox and Morales ([12], [13], [20]), Drewnowski ([15], [16]), Orlicz and Urbański ([24]), Pap ([25], [26], [27]) and the related bibliography, Weber ([33]). Moreover, among the recent works on this topic, we mention here Habil ([21]).
In a previous paper ([5]) we proved a version of these kinds of theorems for σ-additive absolutely continuous set functions, not necessarily positive, with values in (l)-groups. In this note we will investigate an integral for Riesz space-valued maps with respect to $\widetilde{\mathbb{R}}$-valued set functions and we will give some versions of Vitali-type theorems, which use some results of [5], and Schur-type theorems for set functions defined on all subsets of $\mathbb{N}$. We observe that there exist some Dedekind complete Riesz spaces such that order convergence is not generated by *any* topology: for example, $L^0(X, \mathcal{B}, \mu)$, where μ is a σ-additive and σ-finite non-atomic positive $\widetilde{\mathbb{R}}$-valued measure. We recall that, in such space, order convergence coincides with almost everywhere convergence (see also [32]).
In the previous paper [3] we proved similar results for *positive* set functions taking values in spaces of the type $L^0(X, \mathcal{B}, \mu)$, where μ is a σ-additive locally finite positive $\widetilde{\mathbb{R}}$-valued measure. In [1] a version of the Nikodým convergence theorem is proved, for particular Riesz spaces and with respect to another type of convergence, which in the spaces of type $L^0(X, \mathcal{B}, \mu)$ coincides with convergence in measure.

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2 Preliminaries.

We begin with the following:

Definitions 2.1 A Riesz space R is said to be *Dedekind complete* if every nonempty subset of R, bounded from above, has supremum in R. A sequence $(r_n)_n$ in R is said to be *order-convergent* (or (o)*-convergent*) to r if there exists a sequence $(p_n)_n$ in R such that $p_n \downarrow 0$ and $|r_n - r| \le p_n$, $\forall\, n \in I\!\!N$ (see also [23], [32]), and we will write $(o)\lim_n\, r_n = r$. A bounded double sequence $(a_{i,l})_{i,l}$ in R is called (D)*-sequence* or *regulator* if for all $i \in I\!\!N$ we have $a_{i,l} \downarrow 0$ as $l \to \infty$. A sequence $(r_n)_n$ in R is said to be (D)*-convergent* to $r \in R$ (and we write $(D)\lim_n\, r_n = r$) if there exists a (D)-sequence $(a_{i,l})_{i,l}$ in R, such that $\forall\, \varphi \in I\!\!N^{I\!\!N}$, $\exists\, n_0 \in I\!\!N$ such that $|r_n - r| \le \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\, \forall\, n \in I\!\!N$, $n \ge n_0$. If Λ is any nonempty set, $(r_n^{(\lambda)})_n$ are sequences in R and $r^{(\lambda)}$ is in R for all $\lambda \in \Lambda$, we say that $(D)\lim_n\, r_n^{(\lambda)} = r^{(\lambda)}$ *uniformly with respect to* $\lambda \in \Lambda$ if there exists a (D)-sequence $(a_{i,l})_{i,l}$ in R, such that $\forall\, \varphi \in I\!\!N^{I\!\!N}$, $\exists\, n_0 \in I\!\!N$ such that $|r_n^{(\lambda)} - r^{(\lambda)}| \le \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$ $\forall\, n \in I\!\!N$, $n \ge n_0$ and $\forall\, \lambda \in \Lambda$. The sequence $(r_n)_n$ is said to be (D)*-Cauchy* if $(D)\lim_n (r_n - r_{n+p}) = 0$ uniformly with respect to $p \in I\!\!N$.

In general, the limit of a sequence (with respect to (D)-convergence) is not unique. However, there are some conditions on R, which are equivalent to uniqueness of the limit: for example, weak σ-distributivity, whose definition we report here below:

Definition 2.2 A Riesz space R is said to be *weakly σ-distributive* if for every (D)-sequence $(a_{i,l})_{i,l}$ we have:

$$\bigwedge_{\varphi \in I\!\!N^{I\!\!N}} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) = 0.$$

We note that (o)-convergence of sequences always implies (D)-convergence, and these two convergences are equivalent when R is weakly σ-distributive (see [17]). We now recall the following result (see [28], pp. 42-43), which will be useful in the sequel.

Lemma 2.3 *Let R be a Dedekind complete Riesz space (not necessarily weakly σ-distributive), $(a_{i,l}^{(n)})_{i,l}$, $n \in I\!N$, be a sequence of (D)-sequences in R. Then for every $u \in R$, $u \geq 0$ there exists a (D)-sequence $(a_{i,l})_{i,l}$ in R such that:*

$$u \bigwedge \left[\sup_q \left(\sum_{n=1}^{q} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i+n)}^{(n)} \right) \right) \right] \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \qquad \forall \varphi \in I\!N^{I\!N}.$$

We now recall the famous Maeda-Ogasawara-Vulikh representation theorem (see also [2]).

Theorem 2.4 *Given a Dedekind complete Riesz space R, there exists a compact extremely disconnected topological space Ω, unique up to homeomorphisms, such that R can be embedded as a solid subspace of $\mathcal{C}_\infty(\Omega) = \{f \in \widetilde{I\!R}^{\Omega} : f$ is continuous, and $\{\omega : |f(\omega)| = +\infty\}$ is nowhere dense in $\Omega\}$. Moreover, if $(a_\lambda)_{\lambda \in \Lambda}$ is any family such that $a_\lambda \in R\ \forall \lambda$, and $a = \sup_\lambda a_\lambda \in R$ (where the supremum is taken with respect to R), then $a = \sup_\lambda a_\lambda$ with respect to $\mathcal{C}_\infty(\Omega)$, and the set $\{\omega \in \Omega : (\sup_\lambda a_\lambda)(\omega) \neq \sup_\lambda a_\lambda(\omega)\}$ is meager in Ω.*

We now introduce the following:

Definitions 2.5 Let G be any infinite set, $\mathcal{A} \subset \mathcal{P}(G)$ be a σ-algebra, $\nu : \mathcal{A} \to I\!R \bigcup \{+\infty\}$ be a positive σ-additive measure, R be a Dedekind complete Riesz space. We say that a set function $\mu : \mathcal{A} \to R$ is *bounded* if $\exists\, w \in R$, $w \geq 0$: $|\mu(A)| \leq w$, $\forall\, A \in \mathcal{A}$. The maps μ_n, $n \in I\!N$, are *equibounded* if there exists an element $u \in R$, $u \geq 0$, such that

$$|\mu_n(A)| \leq u \quad \forall\, n \in I\!N, \forall\, A \in \mathcal{A}. \tag{1}$$

Given a finitely additive bounded set function $\mu : \mathcal{A} \to R$, define $v(\mu) : \mathcal{A} \to R$, by setting

$$v(\mu)(A) = \sup_{B \in \mathcal{A}, B \subset A} |\mu(B)|, \ \forall\, A \in \mathcal{A}.$$

A finitely additive set function $\mu : \mathcal{A} \to R$ is said to be *σ-additive,* if there exists a (D)-sequence $(u_{i,l})_{i,l}$ such that, $\forall \varphi \in I\!N^{I\!N}$ and for every disjoint sequence $(H_s)_s$ in $\mathcal{A}$ there exists $\overline{s}$: $\forall s \geq \overline{s}$,

$$v(\mu)\left(\bigcup_{k=s}^{\infty} H_k\right) \leq \bigvee_{i=1}^{\infty} u_{i,\varphi(i)}.$$

If a sequence of set functions $\mu_n : \mathcal{A} \to R$, $n \in I\!N$, is given, *uniform σ-additivity* is defined as above, but with $\overline{s}$ *independent* of n.(See also [5]). A map $\mu : \mathcal{A} \to R$ is said to be *ν-absolutely continuous* if

$$v(\mu)(E) = 0 \text{ whenever } E \in \mathcal{A} \text{ and } \nu(E) = 0.$$

The maps $\mu_n : \mathcal{A} \to R$, $j \in I\!N$, are said to be *uniformly ν-absolutely continuous,* if there exists a (D)-sequence $(w_{i,l})_{i,l}$ such that, $\forall \varphi \in I\!N^{I\!N}$ and for every sequence $(E_s)_s$ in $\mathcal{A}$ with $\lim_s \nu(E_s) = 0$, there exists $\overline{\overline{s}}$: $\forall s \geq \overline{\overline{s}}$, $\forall n \in I\!N$,

$$v(\mu_n)(E_s) \leq \bigvee_{i=1}^{\infty} w_{i,\varphi(i)}.$$

A finitely additive set function $\mu : \mathcal{A} \to R$ is said to be *(s)-bounded,* if there exists a (D)-sequence $(w_{i,l})_{i,l}$ such that, $\forall \varphi \in I\!N^{I\!N}$ and for every disjoint sequence $(H_s)_s$ in $\mathcal{A}$ there exists $\overline{s}$: $\forall s \geq \overline{s}$,

$$v(\mu)(H_s) \leq \bigvee_{i=1}^{\infty} w_{i,\varphi(i)}.$$

We say that the maps $\mu_n : \mathcal{A} \to R$, $n \in I\!N$, are *uniformly (s)-bounded,* if the above condition holds, but with $\overline{s}$ independent of n (see also [5]).

Remark 2.6 We note that any (s)-bounded finitely additive map μ is bounded (See [5]).

Also the next proposition is proved in [5].

Proposition 2.7 *Under the same hypotheses and notations as above, if m is σ-additive and ν-absolutely continuous, then there exists a (D)-sequence*

$(z_{i,l})_{i,l}$ *such that,* $\forall\, \varphi \in I\!N^{I\!N}$ *and for every sequence* $(E_s)_s$ *in* $\mathcal{A}$ *with* $\lim_s \nu(E_s) = 0$, *there exists* $\overline{s}$: $\forall\, s \geq \overline{s}$,

$$v(\mu)(E_s) \leq \bigvee_{i=1}^{\infty} z_{i,\varphi(i)}. \qquad (2)$$

Conversely, if R *is weakly* σ*-distributive and* μ *satisfies* (2)*, then* μ *is* ν*-absolutely continuous.*

Definitions 2.8 Given a sequence of set functions $(\mu_n)_{n \in I\!N \bigcup \{0\}}$, we say that the μ_n's (D)*-converge to* μ_0 *pointwise with respect to the same regulator,* or in short $(RD) \lim_n \mu_n = \mu_0$, if there exists a (D)-sequence $(b_{i,l})_{i,l}$ such that $\forall\, \varphi \in I\!N^{I\!N}$, $\forall\, A \in \mathcal{A}$, $\exists\, n_0 \in I\!N$ such that

$$|\mu_n(A) - \mu_0(A)| \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \quad \forall\, n \in I\!N, n \geq n_0. \qquad (3)$$

We note that the condition (3) is equivalent to the classical pointwise convergence of the involved set functions in the case of metrizable groups.

We say that $(D) \lim_n \mu_n = \mu_0$ *uniformly,* or briefly $(U) \lim_n \mu_n = \mu_0$, if there exists a (D)-sequence $(c_{i,l})_{i,l}$ such that $\forall\, \varphi \in I\!N^{I\!N}$, $\exists\, n_0 \in I\!N$ such that

$$|\mu_n(A) - \mu_0(A)| \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)} \quad \forall\, A \in \mathcal{A}, \forall\, n \in I\!N, n \geq n_0.$$

We now recall some results which we proved in [5] and which we will apply in the sequel in order to develop our integration theory. From now on, let G be any infinite set, $\mathcal{A} \subset \mathcal{P}(G)$ be a σ-algebra, R be a Dedekind complete Riesz space, $\nu : \mathcal{A} \to I\!R \bigcup \{+\infty\}$ be a positive σ-additive measure.

Theorem 2.9 *Let* $(\mu_n : \mathcal{A} \to R)_n$ *be a sequence of equibounded* σ*-additive set functions. If the* μ_n*'s are* (RD)*-convergent, then they are uniformly* (s)*-bounded.*

Theorem 2.10 *Under the same hypotheses and notations as above, let* $\mu_n : \mathcal{A} \to R$, $n \in I\!N$, *be* σ*-additive set functions. Then uniform* (s)*-boundedness of the* μ_n*'s implies uniform* σ*-additivity.*

Theorem 2.11 *In the same situation as above, if* $(\mu_n : \mathcal{A} \to R)_n$ *is a sequence of uniformly* σ*-additive and* ν*-absolutely continuous set functions, then the* μ_n*'s are uniformly* ν*-absolutely continuous. Conversely, if the* μ_n*'s are uniformly* ν*-absolutely continuous, then they are uniformly* σ*-additive.*

3 The Vitali and the Schur theorems

In this section we introduce a "Bochner-type" integral for Riesz space-valued functions with respect to a σ-additive positive $\widetilde{\mathbb{R}}$-valued measure ν. In [4] similar integrals were investigated with respect to set functions which could take values even in Riesz spaces, but which were *finite.* The integral here studied will be useful for proving a Vitali-type theorem for integrals with respect to measures which can take also the value $+\infty$. Our Vitali theorem is a consequence of Theorems 2.10 and 2.11. Moreover, we will prove that the integral here investigated does coincide with the one of [4], at least when ν is finite. Furthermore, we give a version of Dunford-Pettis-type theorems and, as a consequence of Theorems 2.9, 2.10, 2.11 and our Vitali theorem, we prove a version of Schur-type theorem for σ-additive set functions defined on $\mathcal{P}(\mathbb{N})$.

From now on, assume that R is a weakly σ-distributive Dedekind complete Riesz space. With the same notations as in the previous section, we say that a function $f : G \to R$ is *integrable simple* if it admits a representation of the type $f = \sum_{i=1}^{n} a_i \chi_{A_i}$, where $n \in \mathbb{N}$, $A_i \in \mathcal{A}$ and $\nu(A_i) < +\infty$, $i = 1, 2, \ldots, n$.

If f is an integrable simple function, $f = \sum_{i=1}^{n} a_i \chi_{A_i}$, then we set, as usual,

$$\int_E f \, d\nu = \sum_{i=1}^{n} a_i \, \nu(A_i \bigcap E), \quad \forall \, E \in \mathcal{A}.$$

It is easy to check that the integral is well-defined and for every integrable simple function f the function $\int_{\cdot} |f| \, d\nu$ is ν-absolutely continuous. We say that a sequence of functions $(f_n)_n$ in R^G is *convergent in measure* to $f \in R^G$ if there exists a (D)-sequence $(a_{i,l})_{i,l}$ in R such that, $\forall \, \varphi \in \mathbb{N}^{\mathbb{N}}$, there exists a sequence $(C_n)_n$ of elements of $\mathcal{A}$ such that

$$\{g \in G : |f_n(g) - f(g)| \not\leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\} \subset C_n \quad \forall \, n \in \mathbb{N} \tag{4}$$

and such that

$$\lim_{n \to +\infty} \nu(C_n) = 0. \tag{5}$$

A sequence $(f_n)_n$ of integrable simple functions is said to be *convergent in* L^1 to an integrable simple function f_0 if

$$(D) \lim_n \int_G |f_n - f_0| \, d\nu = 0,$$

and it is said to be *Cauchy in* L^1 if

$$(D) \lim_n \int_G |f_n - f_{n+p}| \, d\nu = 0$$

uniformly with respect to $p \in \mathbb{N}$. We say that a sequence $(f_n)_n$ of integrable simple functions is *uniformly integrable* if there exists a (D)-sequence $(z_{i,l})_{i,l}$ such that, $\forall \varphi \in \mathbb{N}^{\mathbb{N}}$, $\exists \delta_0 \in \mathbb{R}^+$: $\forall E \in \mathcal{A}$ with $\nu(E) \leq \delta_0$, we have:

$$\int_E |f_n| \, d\nu \leq \bigvee_{i=1}^{\infty} z_{i,\varphi(i)},$$

that is if the integrals $\int_\cdot |f_n| \, d\nu$ are uniformly ν-absolutely continuous. Now, our aim is to define our integral and to prove that it is well-defined. To this end, by means of some arguments similar to the ones of [22], pp. 99-101, we begin with proving the following two technical propositions.

Proposition 3.1 *If $(f_n)_n$ is a sequence of integrable simple functions, Cauchy in L^1, then the integrals $\int_\cdot |f_n| \, d\nu$ are uniformly bounded and uniformly ν-absolutely continuous.*

Proof: First of all, we note that the sequence $\left(\int_G |f_n| \, d\nu \right)_n$ is bounded: indeed it is Cauchy, because $(f_n)_n$ is Cauchy in L^1 and

$$\left| \int_G [|f_n| - |f_m|] \, d\nu \right| \leq \int_G \big| \, |f_n| - |f_m| \, \big| \, d\nu \leq \int_G |f_n - f_m| \, d\nu. \tag{6}$$

Moreover, it is easy to check that for every $n \in \mathbb{N}$ the set function $\int_\cdot |f_n| \, d\nu$ is ν-absolutely continuous, and hence σ-additive. We now notice that the Cauchy condition implies (RD)-convergence of the sequence $\left(\int_A f_n \, d\nu \right)_n$ for every $A \in \mathcal{A}$. Now, theorem 2.9 ensures uniform (s)-boundedness of the measures $\int_\cdot f_n \, d\nu$, and therefore the assertion follows from theorems 2.10 and 2.11. $\square$

We now prove the following:

Corollary 3.2 *Let $(f_n)_n$ be a sequence of integrable simple functions, Cauchy in L^1, and define $\lambda : \mathcal{A} \to R$ as follows:*

$$\lambda(E) \equiv (D) \lim_n \int_E |f_n| \, d\nu, \quad E \in \mathcal{A}.$$

Then λ is σ-additive.

Proof: As already observed, the measures $\left(\int_\cdot f_n \, d\nu \right)_n$ are σ-additive, and uniformly (s)-bounded. So, they are uniformly σ-additive, and this, together with weak σ-distributivity of R, implies the σ-additivity of λ. □

We now are in position to formulate our definition of integrability.

Definition 3.3 We say that $f_0 \in R^G$ is *integrable* (with respect to ν) if there exists a sequence $(f_n)_n$ of integrable simple functions, convergent in measure to f_0 and Cauchy in L^1 (we call such a sequence a *defining sequence* for f_0), and in this case we set:

$$\int_E f_0 \, d\nu = (D) \lim_n \int_E f_n \, d\nu \quad \forall \, E \in \mathcal{A}. \tag{7}$$

It is easy to check that the limit in (7) exists uniformly with respect to $E \in \mathcal{A}$: this is a consequence of the fact that, if R is a Dedekind complete Riesz space, then, for any nonempty set Λ, every family of sequences, which is (D)-Cauchy uniformly with respect to $\lambda \in \Lambda$, is (D)-convergent uniformly with respect to $\lambda \in \Lambda$: see also [4], Theorem 2.16; [17], [23], [31]. Moreover, it is easy to see that, if f_0 is integrable, then $|f_0|$ is integrable too: indeed, if $(f_n)_n$ is a defining sequence for f_0, then $(|f_n|)_n$ is a defining sequence for $|f_0|$. We now prove that the integral in (7) is well-defined, that is it does not depend on the choice of the defining sequence $(f_n)_n$:

Theorem 3.4 *Under the same hypotheses and notations as above, let $f_0 \in R^G$ be an integrable function, and $(f_n)_n$ and $(h_n)_n$ be two defining sequences for f_0. Then*

$$(D) \lim_n \int_E f_n \, d\nu = (D) \lim_n \int_E h_n \, d\nu \quad \forall \, E \in \mathcal{A}.$$

Proof: It's easy to see that $(f_n - h_n)$ is Cauchy in L^1 and convergent in measure to 0; moreover, thanks to proposition 3.1, the sequence $(|f_n - h_n|)$ is uniformly integrable. So we shall show that (D)-$\lim_n \int_G |f_n - h_n| d\nu = 0$.

Setting $g_n = f_n - h_n$, let $(z_{i,l})_{i,l}$ be a (D)-sequence, related with uniform integrability of the $|g_n|$'s. Let $(a_{i,l})_{i,l}$ be the (D)-sequence, corresponding to the condition of convergence in measure of the sequences $(g_n)_n$ to 0. Fix arbitrarily $\varphi \in I\!N^{I\!N}$, and let $(C_n)_n$ be corresponding to condition (4) relatively to convergence in measure of $(g_n)_n$ to 0. We note that, for every $n \in I\!N$ and for all $x \in G \setminus C_n$, we have:

$$|g_n(x)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}. \tag{8}$$

Fix arbitrarily $H \in \mathcal{A}$, with $\nu(H) < +\infty$. For all $\varphi \in I\!N^{I\!N}$, $\exists\, n_0 \in I\!N$: $\forall\, n \geq n_0$ we get:

$$\int_H |g_n|\, d\nu \leq \int_{H \setminus C_n} |g_n|\, d\nu + \int_{H \cap C_n} |g_n|\, d\nu \leq \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) \cdot \nu(H) + \bigvee_{i=1}^{\infty} z_{i,\varphi(i)},$$

and hence

$$(D) \lim_n \int_H |g_n|\, d\nu = 0. \tag{9}$$

We note that, by Corollary 3.2, the set functions $(D) \lim_n \int_{\cdot} |g_n|\, d\nu$ are σ-additive. From this it follows that

$$(D) \lim_n \int_F |g_n|\, d\nu = 0 \tag{10}$$

even for every $F \in \mathcal{A}$ with σ-finite measure ν.
We now observe that, since the functions g_n are integrable simple, then each of them vanishes up to the complement of a set of finite measure ν. So there exists a set $E_0 \in \mathcal{A}$ with σ-finite measure ν such that

$$\int_{G \setminus E_0} |g_n|\, d\nu = 0, \tag{11}$$

and thus, from (10) and (11), we get

$$(D) \lim_n \int_G |g_n|\, d\nu = (D) \lim_n \int_{E_0} |g_n|\, d\nu = 0$$

because E_0 is with σ-finite measure ν. Thus our integral defined in (7) does not depend on the choice of the defining sequence. This concludes the proof of Theorem 3.4. $\square$

It is possible to formulate the concepts of convergence in L^1 and Cauchy in L^1 for sequences of integrable maps in the same fashion as they were introduced for integrable simple functions. Moreover, it is easy to check that, if f and h are integrable functions, then $f+h$ is integrable too and

$$\int_E (f+h)\, d\nu = \int_E f\, d\nu + \int_E h\, d\nu \quad \forall\, E \in \mathcal{A}.$$

Furthermore, an immediate consequence of Proposition 3.2 and Theorem 3.4 is that, if f is integrable, then the set function $\int_{\cdot} f\, d\nu$ is σ-additive.

Remark 3.5 We observe that in [4] we gave a definition of integral for Riesz-space-valued functions with respect to *finite* set function, which could be real- or Riesz-space-valued. For the sake of simplicity, we report this definition in our context: a map $f \in R^G$ is said to be *integrable* if there exists a sequence $(f_n)_n$ of simple function, convergent in measure to f and uniformly integrable. As a consequence of Proposition 3.1, we get that, if a function f is integrable according to the definition in 3.3, that is if there exists a sequence $(f_n)_n$ of simple functions, convergent in measure to f and Cauchy in L^1, then $(f_n)_n$ is uniformly integrable, and hence, if ν is finite, f is also integrable according to the definition formulated in [4]. In the case in which ν is finite, the converse is true too, by virtue of the Vitali theorem of [4], and thus the definition of integral in 3.3 and the one of [4] do coincide.

The following lemma will be useful in order to prove our version of the Vitali theorem.

Lemma 3.6 *Let f be an integrable function. Then the set function $\int_{\cdot} |f|\, d\nu$ is ν-absolutely continuous.*

Proof: Let $(f_n)_n$ be a defining sequence for f. We note that, as said above, $(|f_n|)_n$ is a defining sequence for $|f|$. Take $E \in \mathcal{A}$ with $\nu(E) = 0$. Since the integrals $\int_{\cdot} |f_n|\, d\nu$ are ν-absolutely continuous, then

$$\int_E |f_n|\, d\nu = 0 \quad \forall\, n \in I\!N. \tag{12}$$

Taking into account of weak σ-distributivity of R and taking the (D)-limit in (12) as n tends to $+\infty$ we get:

$$\int_E |f| \, d\nu = 0,$$

that is the assertion. $\square$

We now are in position to state a version of the Vitali theorem, which is a consequence of Theorems 2.10 and 2.11 (For similar results in the case $R = \mathbb{R}$, see [8], Theorem 3, pp. 167-168).

Theorem 3.7 *Let R be a Dedekind complete weakly σ-distributive Riesz space, G, $\mathcal{A}$ and ν be as above, and suppose that ν is σ-finite; assume that $(f_n)_n$ is a sequence of integrable functions in R^G, convergent in measure (with respect to ν) to an integrable map $f_0 \in R^G$; moreover suppose that the integrals $\int_\cdot |f_n| \, d\nu$ are uniformly (s)-bounded. Then the sequence $(f_n)_n$ converges in L^1 to f_0.*
Conversely, if $(f_n)_n$ is a sequence of integrable mappings, convergent in measure and in L^1 to an integrable function f_0, then the integrals $\int_\cdot |f_n| \, d\nu$ are uniformly σ-additive and uniformly ν-absolutely continuous (Nothing can be said about convergence in measure).

Proof: We start proving the first part. As f_0 is assumed to be integrable, replacing f_n by $f_n - f_0$, we shall assume that $f_0 = 0$. Now, we observe that the measures $\int_\cdot f_n \, d\nu$ are uniformly bounded (this can be proved as in Proposition 3.1); moreover, as the measures $\int_\cdot |f_n| \, d\nu$, $n \in \mathbb{N}$, are ν-absolutely continuous (thanks to Lemma 3.6) and uniformly (s)-bounded (by hypothesis), by Theorems 2.11 and 2.10 they are uniformly ν-absolutely continuous, and uniformly σ-additive.

By hypothesis ν is σ-finite, and hence there exists a sequence $(E_s)_s$ in $\mathcal{A}$ such that $E_s \uparrow G$ and $\nu(E_s) < +\infty \ \forall\, s \in \mathbb{N}$. Set $F_s \equiv G \setminus E_s \ \forall\, s$: we have $F_s \downarrow \emptyset$. By uniform σ-additivity of the integrals $\int_\cdot |f_n| \, d\nu$, $n \in \mathbb{N}$, there exists a (D)-sequence $(z_{i,l})_{i,l}$: $\forall\, \varphi \in \mathbb{N}^{\mathbb{N}}$, $\exists\, s_0 \in \mathbb{N}$: $\forall\, n \in \mathbb{N}$,

$$\int_{F_{s_0}} |f_n| \, d\nu \le \bigvee_{i=1}^{\infty} z_{i,\varphi(i)}. \tag{13}$$

Since $(f_n)_n$ converges in measure to 0 with respect to ν, then there exists a regulator $(a_{i,l})_{i,l}$ such that, $\forall\, \varphi \in I\!N^{I\!N}$, there exists a sequence $(C_n)_n$ in $\mathcal{A}$ satisfying (4) and such that $\lim_n \nu(C_n) = 0$. Fix arbitrarily $\varphi \in I\!N^{I\!N}$, and let $(C_n)_n$ be as above. For every n and $s \in I\!N$, we get:

$$\int_{E_s} |f_n|\, d\nu \leq \int_{E_s \cap C_n} |f_n|\, d\nu + \int_{E_s \setminus C_n} |f_n|\, d\nu$$
$$\leq \int_{C_n} |f_n|\, d\nu + \nu(E_s) \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

From uniform absolute continuity of the measures $\int_{\cdot} |f_n|\, d\nu$, it follows the existence of a (D)-sequence $(u_{i,l})_{i,l}$ such that, $\forall\, \varphi \in I\!N^{I\!N}$, $\exists\, n_1 \in I\!N$: $\forall\, n \geq n_1$, $\forall\, s \in I\!N$,

$$\sup_{q \in I\!N} \left(\int_{C_n} |f_q|\, d\nu \right) \leq \bigvee_{i=1}^{\infty} u_{i,\varphi(i)}.$$

Therefore, for every $n \geq n_1$ and $s \in I\!N$ we have:

$$\int_{E_s} |f_n|\, d\nu \leq \bigvee_{i=1}^{\infty} u_{i,\varphi(i)} + \nu(E_s) \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}. \tag{14}$$

From (14) we see that the sequence (f_n) converges in L^1 to 0 on each set E_s, that is
for every $s \in I\!N$ there exists a regulator $(A^s_{i,l})_{i,l}$ such that, for every fixed $\varphi \in I\!N^{I\!N}$ there exists an integer $\overline{n}$ such that

$$\int_{E_s} |f_n|\, d\nu \;\leq \bigvee_{i=1}^{\infty} A^s_{i,\varphi(i+s)}$$

for every $n \geq \overline{n}$. Thanks to uniform boundedness of the integrals, the regulators $(A^s_{i,l})_{i,l}$ can be taken bounded by some fixed element $u \in R$, and therefore we can apply 2.3 to get a single regulator $(A_{i,l})_{i,l}$ such that, for every $s \in I\!N$, and every $\varphi \in I\!N^{I\!N}$ an integer $n_0(s, \varphi)$ can be found, in such a way that

$$\int_{E_s} |f_n|\, d\nu \;\leq \bigvee_{i=1}^{\infty} A_{i,\varphi(i)} \tag{15}$$

holds, for every $n \geq n_0$.
Now, let's take $(b_{i,l})_{i,l} = (A_{i,l} + z_{i,l})_{i,l}$, and fix $\varphi \in I\!N^{I\!N}$. Then an integer s_0 can be found, such that (13) holds, for all n. Depending on s_0 and φ, an integer $n_0(s_0, \varphi)$ can be found, such that (15) holds, for all $n \geq n_0$. Thus, depending on φ , an integer n_0 exists, such that

$$\int_G |f_n| \, d\nu \; \leq \int_{F_{s_0}} |f_n| \, d\nu + \int_{E_{s_0}} |f_n| \, d\nu \; \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}$$

for all $n \geq n_0$, and this shows the first part of the theorem.

We now turn to the second part. Assuming that f_n tends to f_0 in L^1, we see that the sequence $\left(\int_G |f_n| \, d\nu \right)_n$ is bounded. Then, by σ-additivity of each f_n, $n \in I\!N \bigcup \{0\}$ and using Lemma 2.3, we get a regulator $(\gamma_{i,l})_{i,l}$ such that, for every disjoint sequence $(H_k)_k$ in $\mathcal{A}$, $\forall \, \varphi \in I\!N^{I\!N}$, $\forall \, n \in I\!N$, $\exists \, k_0 \in I\!N$ such that

$$\int_{U_{k_0}} |f_n| d\nu \leq \bigvee_{i=1}^{\infty} \gamma_{i,\varphi(i)} \tag{16}$$

holds, where U_{k_0} is the union of all H_k, $k \geq k_0$. Moreover, from convergence in L^1, it follows that a regulator $(\beta_{i,l})_{i,l}$ exists such that, $\forall \, \varphi \in I\!N^{I\!N}$, $\exists \, n_0 \in I\!N$, satisfying

$$\int_G |f_n - f_0| \, d\nu \leq \bigvee_{i=1}^{\infty} \beta_{i,\varphi(i)} \tag{17}$$

for all $n \geq n_0$. Now, choose $a_{i,l} = 2(\gamma_{i,l} + \beta_{i,l})$, and fix arbitrarily a disjoint sequence $(H_k)_k$ in $\mathcal{A}$. For any φ, there exists an integer n_0 such that (17) holds. Corresponding to n_0, and the same φ, there exists an integer k_0 such that (16) holds, for $n = 0, 1, \ldots, n_0$. Now, if $n \leq n_0$, we have

$$\int_{U_{k_0}} |f_n| \, d\nu \leq \bigvee_{i=1}^{\infty} \gamma_{i,\varphi(i)} \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

If $n \geq n_0$, we get

$$\int_{U_{k_0}} |f_n| \, d\nu \leq \int_{U_{k_0}} |f_n - f_0| \, d\nu + \int_{U_{k_0}} |f_0| \, d\nu$$
$$\leq \bigvee_{i=1}^{\infty} \gamma_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} \beta_{i,\varphi(i)} \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

This proves uniform σ-additivity, hence uniform (s)-boundedness, and uniform absolute continuity.

Finally, we note that in general, in our setting, convergence in L^1 does not imply convergence in measure, even if ν is the Lebesgue measure: see [4], Example 3.23.1.

This concludes the proof of Theorem 3.7. $\square$

At this stage, we could easily obtain a Dunford-Pettis-type theorem, however we prefer first to compare "strong" convergence (i.e. convergence in L^1) with a "weak" type of convergence, in order to better clarify the connection between our formulation of the Dunford-Pettis theorem and the classic one (for similar results existing in the literature in the case $R = I\!R$, see [14]).

Lemma 3.8 *Let $f : G \to R$ be an integrable map. Define $\mu(A) = \int_A f \, d\nu$, for all $A \in \mathcal{A}$. Then*

$$\int_A |f| d\nu \leq 2v(\mu)(A) \tag{18}$$

for all $A \in \mathcal{A}$.

Proof: First, we shall assume that f is simple, i.e.

$$f = \sum_{i=1}^{n} a_i \chi_{A_i},$$

where $a_i \in R$ and $A_i \in \mathcal{A}$ for all i. Now, applying 2.4, we can find a meager set $N \subset \Omega$ such that $f(x)(\omega) = \sum_{i=1}^n a_i(\omega)\chi_{A_i}(x)$, for all $\omega \in N^c$. For each $\omega \in N^c$, we set $f_\omega(x) = f(x)(\omega)$: then f_ω is a measurable simple real-valued map, and clearly

$$\int_A f_\omega \, d\nu = \left(\int_A f \, d\nu \right)(\omega) = \mu(A)(\omega),$$

for all $A \in \mathcal{A}$. Now, for real-valued maps, the formula (18) holds true, hence we have (up to a meager set N' larger than N):

$$\int_A |f_\omega| d\nu \leq 2v(\mu_\omega)(A),$$

where of course μ_ω denotes the real-valued measure $A \to \mu(A)(\omega)$. Thanks to the characterization of 2.4, we get (18). Now, for general f, let (f_n) be a defining sequence for f (see definition 3.3).

Then,

$$\int_A |f| d\nu = \lim_n \int_A |f_n| d\nu \leq 2 \limsup_n v(\mu_n)(A), \tag{19}$$

where $\mu_n(A) = \int_A f_n \, d\nu$, for all $A \in \mathcal{A}$.
Now,

$$v(\mu_n)(A) = \sup_{B \subset A} \left| \int_B f_n d\nu \right| \leq \sup_{B \subset A} \left| \int_B (f_n - f) \; d\nu + \int_B f \; d\nu \right| \leq$$

$$\leq \sup_{B \subset A} \left\{ \int_B |f_n - f| d\nu \; + \; | \int_B f \; d\nu| \right\} \leq \int_G |f_n - f| d\nu + v(\mu)(A).$$

Hence, replacing into (19), we get

$$\int_A |f| d\nu \; \leq 2 \limsup_n \int_G |f_n - f| d\nu + 2v(\mu)(A).$$

Thanks to convergence in L^1 of (f_n) to f, and by weak σ-distributivity of R, we get (18). □

Proposition 3.9 *Let (f_n) be any sequence of integrable functions. Then f_n converges to f_0 in L^1 if and only if*

$$\lim_n \int_A f_n \; d\nu = \int_A f_0 \; d\nu$$

uniformly in A.

Proof: We have already noticed (just after Definition 3.3) that convergence in L^1 implies uniform convergence of the integrals $\int_A \; f_n \; d\nu$ to $\int_A \; f \; d\nu$. So we just show the converse. We can assume $f_0 = 0$, as usual, and $\lim_n | \int_A \; f_n \; d\nu| = 0$ uniformly in A: from this, applying Lemma 3.8, we see that

$$\lim_n \int_G |f_n| \; d\nu \leq 2 \lim_n v(\mu_n)(G) = 0,$$

i.e. the assertion. □

Thus, also for Riesz space-valued functions, convergence in L^1 can be characterized by means of uniform convergence of the integrals. It's well-known that, in the scalar case, *pointwise* convergence of the integrals is equivalent to weak convergence in L^1. For Riesz space-valued functions, this seems not to be the case, mainly because uniform boundedness of the integrals in general does not follow from pointwise boundedness (see [30]); however we shall still speak of "weak convergence" according with the following definition.

Definition 3.10 Let (f_n) be a sequence of integrable functions, such that the set $\{\int_G |f_n| \, d\nu : n \in \mathbb{N}\}$ is bounded in R. We say that (f_n) *weakly converges* to f_0 if

$$(RD) \lim_n \int_A f_n \, d\mu = \int_A f_0 \, d\nu$$

pointwise, for each $A \in \mathcal{A}$.

Now, we state our Dunford-Pettis-type theorem.

Theorem 3.11 *Under the same hypotheses as in Theorem 3.7, let $(f_n)_n$ be a sequence of integrable functions, convergent in measure to an integrable map $f_0 \in R^G$, and assume that the set $\{\int_G |f_n| \, d\nu : n \in \mathbb{N}\}$ is bounded in R. Then $(f_n)_n$ converges in L^1 to f_0 if and only if it converges weakly to f_0.*

Proof: Thanks to 3.9, we just have to prove the "if" part. Since $\left(\int_{\cdot} f_n \, d\nu\right)_n$ is a (RD)-convergent sequence of σ-additive set functions, then, by Theorem 2.9, these indefinite integrals are uniformly (s)-bounded. Hence, by virtue of the first part of the Vitali theorem 3.7, the sequence $(f_n)_n$ converges to f_0 in L^1. $\square$

We now turn to a Schur-type theorem.

Theorem 3.12 *Let R be a Dedekind complete weakly σ-distributive Riesz space, $(\mu_n : \mathcal{P}(\mathbb{N}) \to R)_n$ be a sequence of σ-additive equibounded mappings, and assume that there exists $\mu_0 : \mathcal{P}(\mathbb{N}) \to R$ such that $(RD) \lim_n \mu_n = \mu_0$. Then:*
the μ_n's are uniformly σ-additive, μ_0 is σ-additive,
$\lim_n \mu_n(A) = \mu_0(A)$ *uniformly in A, and*

$$(D) \lim_n \left[\sup_q \left(\sum_{j=1}^{q} |\mu_n(\{j\}) - \mu_0(\{j\})| \right) \right] = 0. \qquad (20)$$

Proof: By virtue of Theorems 2.9 and 2.10, the μ_n's are uniformly σ-additive, and thus, thanks to weak σ-distributivity of R, μ_0 is σ-additive (see also [5]). We now prove that $\lim_n \mu_n = \mu_0$ uniformly. By uniform σ-additivity of the μ_n's, there exists a regulator $(\alpha_{i,l})_{i,l}$ such that, $\forall\, \varphi \in I\!N^{I\!N}$, $\exists\, N^* \in I\!N$: $\forall\, N \geq N^*$, $\forall\, n \in I\!N$, $\forall\, H \subset I\!N$, we have:

$$\begin{aligned} & |\mu_n(H \bigcap \{N+1, N+2, \dots\}) - \mu_0(H \bigcap \{N+1, N+2, \dots\})| \\ \leq \; & \bigvee_{i=1}^{\infty} \alpha_{i,\varphi(i)}. \end{aligned} \tag{21}$$

Moreover, by (RD)-convergence of the μ_n's to μ_0 there exists a (D)-sequence $(\zeta_{i,l})_{i,l}$: $\forall\, N \in I\!N$, $\forall\, \varphi \in I\!N^{I\!N}$, $\exists\, \overline{n}$: $\forall\, n \geq \overline{n}$, $\forall\, H \subset I\!N$, we get:

$$|\mu_n(H \bigcap \{1, \dots, N\}) - \mu_0(H \bigcap \{1, \dots, N\})| \leq \bigvee_{i=1}^{\infty} \zeta_{i,\varphi(i)}. \tag{22}$$

Thus $\forall\, \varphi \in I\!N^{I\!N}$, $\exists\, \overline{n}$: $\forall\, n \geq \overline{n}$, $\forall\, H \subset I\!N$ we have:

$$\begin{aligned} & |\mu_n(H) - \mu_0(H)| \leq |\mu_n(H \bigcap \{1, \dots, N^*\}) - \mu_0(H \bigcap \{1, \dots, N^*\})| \\ + \; & |\mu_n(H \bigcap \{N^*+1, N^*+2, \dots\}) - \mu_0(H \bigcap \{N^*+1, N^*+2, \dots\})| \\ \leq \; & \bigvee_{i=1}^{\infty} \alpha_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} \zeta_{i,\varphi(i)}. \end{aligned} \tag{23}$$

where $N^* = N^*(\varphi)$ is as in (21). From (23) it follows that $\lim_n \mu_n = \mu_0$ uniformly. Finally, to prove (20), let $\nu : \mathcal{P}(I\!N) \to I\!R$ be the counting measure. Set now

$$f_n(j) = \mu_n(\{j\}), \quad \forall j \in I\!N, \quad \forall\, n \in I\!N \bigcup \{0\}.$$

It is easy to show that

$$\mu_n(E) = \sum_{j \in E} f_n(j) = \int_E f_n \, d\nu, \quad \forall\, E \subset I\!N, \quad \forall\, n \in I\!N \bigcup \{0\},$$

since the μ_n's are σ-additive. Now, we note that (20) is equivalent to convergence in L^1 of the sequence $(f_n)_n$ to f_0 relatively to the above considered measure ν. This last property is an immediate consequence of Proposition 3.9 and uniform convergence of the μ_n's to μ_0. This concludes the proof of the theorem. $\square$

Corollary 3.13 *Let $G = \mathbb{N}$, $\mathcal{A} = \mathcal{P}(\mathbb{N})$, and ν be any σ-finite measure on $\mathcal{A}$. Then, given a sequence $(f_n)_n$ of integrable functions on this space, such that the set $\{\int_{\mathbb{N}} |f_n| \, d\nu : n \in \mathbb{N}\}$ is bounded in R, weak convergence of $(f_n)_n$ to f_0 is equivalent to strong convergence.*

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