Weak Stochastic Integration in Banach Spaces [*]

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Summary. We define here a concept of weak Stochastic Integral, for processes taking values in Banach Spaces, by using the tools already developed in [3] for the "strong" integral. The integral here introduced turns out to be the stochastic counterpart of the Pettis Integral for deterministic Banach-valued functions.

Sunto. Si definisce un concetto di Integrale Stocastico in senso debole, per processi a valori in spazi di Banach, utiklizzando le tecniche giá sviluppate in [3] per l'integrale "forte". L'integrale introdotto risulta essere il corrispondente dell'Integrale di Pettis per funzioni deterministiche a valori in spazi di Banach.

0 Introduction

In this paper we shall develop a weak stochastic integral $w\int_0^t HdX = Z_t$, where X is a real valued summable process (as defined below) and H is a predictable process taking its values in a Banach space F. Our weak integral will have the following properties:

For each $t > 0$, $Z_t \in L^1_F(P)$ and $x'Z_t = \int_0^t x'H\ dX$ for every $x' \in F'$, where the last integral is the classic stochastic integral. If X is a local martingale, then $(Z_t)_{t\geq 0}$ has a cadlag modification.

The weak stochastic integral presented here is a new stochastic integral. In order to define it we use the "strong" stochastic theory in Banach spaces developed by Brooks and Dinculeanu in [3], which in turn was based on the bilinear vector

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integration theory developed in [2]. The particular strong stochastic integral $\int HdX$ in [3], used to define $w - \int HdX$, is a special case of the general bilinear stochastic integral in the sense that X is real valued. Thus $w - \int HdX$ extends $\int HdX$ in the sense that the Pettis integral extends the Bochner integral. In our construction of the weak stochastic integral, we assume that the family of measures $\mathcal{M}$ associated with the stochastic measure I_X is uniformly σ-additive -a condition which is automatically satisfied if $c_0 \not\subset F$ (see section 3). Once we have this condition we can take advantage of a *pointwise* dominated convergence theorem in [3] which is used in certain key stages in our theory. An example is given in section 3 that shows if F is infinite dimensional, then there exist weakly integrable processes H which are not strongly integrable.

1 Preliminaries

The setting

Throughout this paper, $(\Omega, \mathcal{F}, P)$ is a probability space and $(\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions.

The terminology of [6] will be used in descriptions concerning classical stochastic processes. F will denote a Banach space with dual space F'. If G is a Banach space, G_1 denotes the closed unit ball of G; by $L^1_F = L^1_F(P)$, we denote the space of Bochner integrable functions. $L(F,G)$ is the space of continuous linear operators from F into G. We shall regard $L^1(P)$ as continuously injected into $L(I\!R, L^1(P))$ or $L(F, L^1_F(P))$, depending on the setting required.

The vector integration theory presented in [2] will be used in addition to the stochastic integration theory in Banach spaces developed in [3] (the appendix in this reference serves as a brief introduction to the vector integration; see also [7]).

The stochastic measure

Let $\mathcal{R}$ denote the ring of subsets of $I\!R_+ \times \Omega$ generated by the predictable rectangles $[0_A] := \{0\} \times A$, with $A \in \mathcal{F}_0$, and $(s,t] \times A$, with $0 \leq s < t < +\infty$, and $A \in \mathcal{F}_s$. The σ-field $\mathcal{P}$ generated by $\mathcal{R}$ is called the σ-field of *predictable* sets.

Let E, G and F be Banach spaces, with $E \subset L(F,G)$. Let X be a process $X : \Omega\times]0,+\infty[\to E\subset L(F,G)$ such that $X_t \in L^1_E$ for each t. Also assume X is a cadlag, adapted process. We shall define the finitely additive stochastic measure I_X, first on predictable rectangles, by

$$I_X([0_A]) = 1_A X_0, \quad \text{and} \quad I_X((s,t]\times A) = 1_A(X_t - X_s),$$

and then extend it in an additive fashion to $\mathcal{R}$. Thus I_X is L^1_E-valued. Frequently, for notational convenience, we shall denote I_X by m. Since $E \subset L(F,G)$, we can consider $L^1_E \subset L(F, L^1_G)$ and thus the semivariation of $m = I_X$ can be computed relative to the pair (F, L^1_G). Explicitly, this semivariation, denoted by $\tilde{m}_{(F,G)}$ rather than $\tilde{m}_{(F,L^1_G)}$, is given by

$$\tilde{m}_{(F,G)}(A) = \sup\{|\sum m(A_i)x_i|_{L^1_G}\}, \quad A \in \mathcal{R}$$

where the supremum is extended over all (finite) families of vectors $x_i \in F_1$ and disjoint sets A_i from $\mathcal{R}$ contained in A. If $m = I_X$ can be extended to $\mathcal{P}$, we define $\tilde{m}$ in an analogous fashion. I_X is said to have *bounded semivariation* relative to (F, L^1_G) if I_X has a σ-additive L^1_E-valued extension (which is unique), still denoted by I_X, to $\mathcal{P}$, and in addition I_X has bounded semivariation on $\mathcal{P}$ relative to (F, L^1_G); in this case we call I_X *summable* relative to (F, L^1_G). An important family $\mathcal{M}$ of positive measures will now be described.

Let $m : \mathcal{P} \to L(F, L^1_G)$ be fixed.
For $z \in (L^1_G)'_1$, define $m_z : \mathcal{P} \to F'$ by

$$< m_z(A), x >=< z, m(A)x >,$$

where $x \in F$ and $A \in \mathcal{P}$ (set $F = G = I\!R$ if we regard m as taking values in $L(I\!R, L^1)$). We note that if m is σ-additive with finite semivariation, then m_z is also σ-additive, and the total variation $|m_z|$ is finite. Set

$$\mathcal{M} = \{|m_z| : z \in (L^1_G)'_1\}.$$

It turns out that $\tilde{m}_{F,G}(A) = \sup\{|m_z|(A) : |m_z| \in \mathcal{M}\}$.

The stochastic integral

In [3] the stochastic integral $\int_0^t HdX$, which belongs to L^1_G, for $t \geq 0$, is developed for predictable F-valued processes H. For the purposes of this paper, X will be real valued, hence $E = I\!R$, and $F = G$; hence I_X takes values in L^1, which is injected into $L(I\!R, L^1)$ or $L(F, L^1_F)$, depending on whether H is $I\!R-$ or $F-$valued.

We shall now assume X is real-valued and we shall always assume that I_X has finite semivariation relative to (F, L^1_F) and that I_X is summable (hence the semivariation relative to $(I\!R, L^1)$ is also finite). If H is F-valued or $I\!R$-valued and predictable, we define the norm of H to be

$$\| H \| = \sup\{\int |H| d|m_z| : |m_z| \in \mathcal{M} \},$$

if $\| H \|$ is finite. In this case we say H is *integrable* with respect to X if $\int_0^t HdI_X \in L^1_F$ for each t and the process $\left(\int_0^t HdI_X\right)_t$ has a cadlag modification.

Any such process is called the *stochastic integral* of H with respect to X and is denoted by $\int HdX$ or $H \bullet X$, that is

$$(H \bullet X)_t = (\int HdX)_t = \int 1_{[0,t]} HdX \quad \text{a.s.}$$

$\mathcal{L}^1_F(X)$ will denote the Banach space of such H.

If $\mathcal{M}$ is uniformly σ-additive, then $\mathcal{L}^1_F(X)$ will contain the family $\mathcal{B}$ of all bounded predictable processes. This is the case when $F \not\supset c_0$ (see section 3). In this case, we denote by $\mathcal{L}^1_F(\mathcal{B}, X)$ the closure of $\mathcal{B}$ in $\mathcal{L}^1_F(X)$. Call H *strongly integrable* (with respect to X) if $H \in \mathcal{L}^1_F(\mathcal{B}, X)$.

2 The weak stochastic integral and its properties

In this section, X is real valued, with the properties described in section 1. We shall always assume that X is summable relative to (F, L^1_F) and $\mathcal{M}$ is uniformly σ-additive. Let H be an F-valued predictable process. Since H is strongly measurable, we can, and shall, write H in the form

$$H = \sum_{i=1}^{\infty} x_i 1_{E_i} + K, \tag{1}$$

where $x_i \in F, E_i \in \mathcal{P}$, the E_i are disjoint, and K is a bounded predictable F-valued process (see [1]).

Definition 2.1 H is said to be *weakly integrable* with respect to X if $x'H \in \mathcal{L}^1_{I\!R}(\mathcal{B}, X)$ for each $x' \in F'$ and there exists a representation (1) for H such that

$$\sum_i x_i I_X(E_i \cap A) \tag{2}$$

converges unconditionally in L^1_F for each $A \in \mathcal{P}$, and

$$\sup\{\sum_i |x'x_i||m_z|(E_i) : |m_z| \in \mathcal{M} \ , x' \in F'_1\} < \infty. \tag{3}$$

(The family $\mathcal{M}$ in (3) is defined for $z \in L^\infty_1$). In this case, we define the process $Z = (Z_t)_{t \geq 0}$, as follows

$$Z_t = \sum_i x_i I_X(E_i \cap [0,t]) + \int_0^t K dX.$$

We call Z_t the *weak stochastic integral* of H with respect to X on $[0,t]$.

We also write Z_t as $w \int_0^t H dX$ or $(H \circ X)_t$, in order to distinguish it from the strong stochastic integral.

In Theorem 2.4 we shall see that Z does not depend on the representation (1).

Definition 2.2 Assuming that H is weakly integrable, we define the *weak norm* of H by

$$\| H \|_w = \sup\{\int |x'H| d|m_z| : x' \in F'_1, |m_z| \in \mathcal{M}\}$$

which is finite by assumption. ($\mathcal{M}$ is the same family, which appears in (3)).

Remark 2.3 We note that the condition that $\mathcal{M}$ is uniformly σ-additive allows us ([3], Cor. 3.12) to obtain $\int_0^t K dX$, and obtain the L^1_F- valued process $(H \circ X)$. The weak norm corresponds to the Pettis norm, and we can obtain a normed space of weakly integrable functions H.

Properties of the weak stochastic integral

Theorem 2.4 *The definition of the weak stochastic integral does not depend upon the representation (1).*

Proof. Suppose that H has two representations:

$$H = \sum_i x_i 1_{E_i} + K \; = \; \sum_j y_j 1_{A_j} + L$$

of the type appearing in (1), and assume that they satisfy (2) and (3). In this case we have

$$g := \sum_i x_i 1_{E_i} - \sum_j y_j 1_{A_j} = L - K.$$

Hence g is bounded and if we set

$$g_n = \sum_{i=1}^{n} x_i 1_{E_i} - \sum_{j=1}^{n} x_j 1_{A_j},$$

we have $|g_n(s,\omega)| \leq |L(s,\omega)| + |K(s,\omega)|$, since the sets involved are disjoint. We can apply the Lebesgue dominated convergence for stochastic processes ([3], Theorem 3.15), since $\mathcal{M}$ is uniformly σ-additive and L and K are bounded, to conclude that $(g_n \bullet X)_t \to ((L-K) \bullet X)_t$ in L^1_F for each t.

That is, $\sum_i x_i I_X(E_i \cap [0,t]) - \sum_j y_j I_X(A_i \cap [0,t])$ equals $(L \bullet X)_t - (K \bullet X)_t$, which implies that $(H \circ X)_t$ is independent of the representation. ■

Theorem 2.5 *Suppose H is a strongly integrable process with respect to X. Then H is weakly integrable with respect to X and*

$$(H \circ X)_t = (H \bullet X)_t \;\; a.s. \; for \;\; each \; t.$$

Proof. Assume that H is strongly integrable with respect to X.

Write $H = \sum_i x_i 1_{E_i} + K$ as in (1). Then $g := \sum_i x_i 1_{E_i} \in \mathcal{L}^1_F(\mathcal{B}, X)$, and by Proposition AI.12 in [3], the indefinite integral $(\int_{\cdot} g dI_X)$ is σ-additive (this is not true for a general $g \in \mathcal{L}^1_F(X)$). Hence, for $A \in \mathcal{P}$, $\sum_i \int_{A \cap E_i} g dI_X$ converges unconditionally in L^1_F, which shows that H satisfies (2). If we set $A = [0,t]$, we see that $(g \bullet X)_t = \sum_i x_i I_X(E_i \cap [0,t])$. Note that (3) is satisfied, since $\| H \|_w \leq \| H \| < \infty$ (observe that the family $\mathcal{M}$ is different for these two norms).

Finally, we shall show that $x'H$ is integrable with respect to X. Since H is strongly integrable, $H \in \mathcal{L}^1_F(\mathcal{B}, X)$. This implies that $x'H$ is in the closure (in the $\|\cdot\|$- norm) of all real valued bounded predictable processes. We apply the Lebesgue dominated convergence theorem ([3], Theorem 3.15) to $x'g$ and $x'g_n := \sum_{i=1}^n x'x_i 1_{E_i}$. (In the notation of that theorem, $\varphi = |x'g| \in \mathcal{F}_{I\!R}(\mathcal{B}, I_{I\!R,I\!R})$) Since $|x'g_n| \leq |x'g|$ and $x'g_n \to x'g$ pointwise and $\mathcal{M}$ is uniformly σ-additive, we can conclude that $x'g \in \mathcal{L}^1_{I\!R}(\mathcal{B}, X)$, because $x'g_n \in \mathcal{L}^1_{I\!R}(\mathcal{B}, X)$, again by the uniform σ-additivity of $\mathcal{M}$([3], Cor. 3.12). Observe that $x'K \in \mathcal{L}^1_{I\!R}(\mathcal{B}, X)$, hence $x'H \in \mathcal{L}^1_{I\!R}(\mathcal{B}, X)$, that is $x'H$ is weakly integrable. Finally

$$(H \bullet X)_t = (g \bullet X)_t + (K \bullet X)_t = \sum_i x_i I_X(E_i \cap [0,t]) + (K \bullet X)_t = (H \circ X)_t$$

■

We also observe that, in case H is finite-dimensional, then weak integrability implies strong integrability.

Theorem 2.6 *Suppose H is weakly integrable with respect to X. Then*

$$x'(H \circ X)_t = ((x'H) \bullet X)_t \ \ a.s.$$

for each t and $x' \in F'$

Proof. We know by definition that $x'H$ is in $\mathcal{L}^1_{I\!R}(\mathcal{B}, X)$. Let $H = g + K$ be a representation (1) for H, where $g = \sum_i x_i 1_{E_i}$. Set $g_n = \sum_{i=1}^n x_i 1_{E_i}$. Then $x'g = x'H - x'K \in \mathcal{L}^1_{I\!R}(\mathcal{B}, X)$. Hence $|x'g|$ is in the norm closure of the bounded real predictable processes $x'g_n \in \mathcal{L}^1_{I\!R}(\mathcal{B}, X)$ since $\mathcal{M}$ is uniformly σ-additive. Apply the Lebesgue dominated convergence theorem ([3], Theorem 3.15) in $\mathcal{L}^1_{I\!R}(\mathcal{B}, X)$ and conclude that $x'g_n \to x'g$ in this space, and for each t $((x'g_n) \bullet X)_t \to ((x'g) \bullet X)_t$ in L^1. That is

$$\sum_i x'x_i I_X(E_i \cap [0,t]) = ((x'g) \bullet X)_t, \ \text{ in } L^1.$$

Since $\sum_i x_i I_X(E_i \cap [0,t])$ converges in L^1_F, we have

$$x'(g \circ X)_t = \sum_i x'x_i I_X(E_i \cap [0,t]).$$

Thus $x'(g \circ X)_t = ((x'g) \bullet X)_t$.

Also,

$$\int x'K1_{[0,t]}dI_X = x'\int K1_{[0,t]}dI_X, \tag{4}$$

since equality holds when K is simple, and since $\mathcal{M}$ is uniformly σ-additive, so we can pass to the limit to obtain (4). Hence $x'(K \bullet X)_t = ((x'K) \bullet X)_t$, and we have finally $x'(H \circ X)_t = ((x'H) \bullet X)_t$ ■

Theorem 2.7 *Suppose H is weakly integrable with respect to the local martingale X. Then $(H \circ X)_{t\geq 0}$ has a cadlag modification.*

Proof. Since X is a local martingale, there exists a sequence of stopping times $T_n \uparrow \infty$ such that, for each n, the stopped process X^{T_n} is a martingale. Fix n, and let $T = T_n$. One can show that X^T is a summable process and that H is weakly integrable with respect to X^T and $H1_{[0,T]}$ is weakly integrable with respect to X. By [3], theorem 3.8, one can show that

$$(H \circ X)_t^T = ((H1_{[0,T]}) \circ X)_t = (H \circ X^T)_t$$

for each t. It follows from [3], theorem 3.21, since $x'H$ is integrable with respect to X^T, that $((x'H) \bullet X^T)_{t\geq 0}$ is a martingale for each $x' \in F'$.

Let $A \in \mathcal{F}_t$, and suppose $x' \in F'$. Then

$$E(1_A((x'H) \bullet X^T)_\infty) = E(1_A((x'H) \bullet X^T)_t).$$

In view of Theorem 2.6, we have

$$x'E(1_A(H \circ X^T)_\infty) = x'E(1_A(H \circ X)_t^T).$$

Since this holds for all $x' \in F'$, we conclude that the process $(H \circ X^T)_{t\geq 0}$ is a martingale, that is, the process $(H \circ X)_{t\geq 0}^T$ is a martingale. The existence of cadlag modifications for Banach valued martingales was established in [4]. We denote the modification of the last martingale by $(H \circ X)_{t\geq 0}^T$ again. Since cadlag modifications are unique up to evanescent sets, this means that

$$(H \circ X)_t^{T_{n+1}} = (H \circ X)_t^{T_n}$$

for all n and $t \geq 0$ outside an evanescent set, and so we conclude that $(H \circ X)$ has a cadlag modification. ■

3 The uniformity of $\mathcal{M}$ and an example

The uniformity of $\mathcal{M}$

We have seen that the uniform σ-additivity of $\mathcal{M}$ plays an important role in the vector integration theory used in this paper. The reason for its importance lies in the fact that $\int H dI_X$ is in a sense an integral structurally determined by the integrals $\int H d(I_X)_z$, for $z \in (L^1_F)'_1$. Fortunately in our setting, the condition $F \not\supset c_0$ yields the desired uniformity of $\mathcal{M}$ as we shall indicate below. Note that a similar condition was used to imply the "Beppo Levi property" used in [2] to establish completeness and weak compactness in the Lebesgue space of functions integrable with respect to a vector measure. With $\mathcal{M}$ uniform, we have at our disposal a *pointwise* dominated convergence theorem, which also yields that bounded measurable functions belong to our Lebesgue space. This is due to the fact that $\mathcal{M}$ in this case has a control measure.

We state a theorem from [5] which extends a result of Dobrakov, [8].

Theorem 3.1 *Let $m : \mathcal{R} \to L(F, G)$ be finitely additive, with bounded semivariation relative to (F, G). If $G \not\supset c_0$ then $\mathcal{M} = \{|m_z| : z \in G'_1\}$ is uniformly strongly additive on $\mathcal{R}$.*

In particular, if m is σ-additive, then $\mathcal{M}$ is uniformly σ-additive.

In our setting, in section 2, $G = L^1_F$. If $F \not\supset c_0$, then by Kwapien's theorem ([9]), we have $L^1_F \not\supset c_0$ and our family $\mathcal{M}$ is uniformly σ-additive.

An example

We give an example to show that for any infinite dimensional Banach space F there are weakly integrable F-valued processes H that are not strongly integrable. Fix an element ω. Set $\Omega = \{\omega\}$. Define $\mathcal{F}_t = \{\{\omega\}, \emptyset\}$ for each $t \in [\,0, \infty]$. Define $P(\{\omega\}) = 1$. Then $\mathcal{P} = \{\omega\} \times$ (Borel subsets of $\mathbb{R}_+$). Identify $L^1(P)$ and L^1_F with $\mathbb{R}$ and F respectively, where F is any infinite dimensional Banach space.

Define $X_t(\omega)$ to be t, if $0 \leq t < 1$, and 0 if $t \geq 1$. Choose a sequence (x_n) from F and disjoint intervals $I_i \subset [0, 1[$, having lenght c_i such that $\sum_i c_i x_i$ converges unconditionally but $\sum_i c_i |x_i| = \infty$. Define $H = \sum x_i 1_{\{\omega\} \times I_i}$.

For $t < 1$, denoting I_X by m, then $m(\{\omega\} \times B) = 1_{\{\omega\}}\lambda(B \cap [0,1[)$, where B is any Borel subset of $I\!\!R_+$, and λ is Lebesgue measure.

If $z \in (L^1_F)' = F'$, and $B \subset [0,1[$, then $m_z(\{\omega\} \times B)x = \lambda(B) < z, x >$ for each $x \in F$. If $|z| = 1$, then $|m_z|(\{\omega\} \times B) = \lambda(B)$. Observe that H is weakly integrable with respect to X by the bounded multiplier theorem for unconditionally convergent series. H is not strongly integrable with respect to X, since if $|z| = 1$ we have $\int |H| d|m_z| = \sum |x_n| c_n = +\infty$.

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