

# Sandwich Theorems, extension principles and amenability

A. BOCCUTO - D. CANDELORO

## 1. Introduction.

In the literature, a lot of results have been found, dealing with Hahn-Banach extension principles, sandwich-type theorems, and so on; some of them concern just the existence of suitable additive or linear maps, while other results ensure invariance of such mappings with respect to amenable groups of transformations.

Among the recent works on this topic, we mention here Dawidowicz ([5]), Giroto and Holzer ([8]), Plappert ([14]), Sasvári ([15]). Richer references can be found in [10] and [11]. Our research in this area has given raise to [2], [3], [4]. In [4] we introduced an extension property (property (E)) for invariant homomorphisms. Silverman and Yen, in [17], proved that property (E) follows from amenability, in case of functions defined on a vector space  $X$ , and taking values in a Dedekind complete Riesz space  $R$ . Moreover, Silverman in [16] proved that property (E) is equivalent to the Hahn-Banach extension principle. These results were both obtained by means of complicated constructions and techniques. Here we give a shorter (and completely different) proof of the Hahn-Banach extension theorem, in case of amenable semigroups of homomorphisms. We make use of a well-known representation theorem of Dedekind complete Riesz spaces (see [1], [13]): this result is used in a rather technical way, while the underlying ideas are quite intuitive. Moreover, we characterize all the possible extensions of an increasing invariant homomorphism, from a subspace  $Z$  of  $X$  to the least invariant subspace containing  $Z$  and some fixed exterior point  $x_0$ : this is in the spirit of [4]. We show also, in full generality, that the Hahn-Banach invariant principle is equivalent to the existence of sandwich invariant homomorphisms, and this

is in turn equivalent to amenability: in this context we recall that Dedekind completeness is a necessary condition. Finally, we show that amenability is equivalent to the existence of additive invariant measures extending set functions which satisfy suitable coherence conditions.

## 2. Preliminaries.

We begin with some definitions and notations.

Throughout the paper,  $R$  will denote any Dedekind complete Riesz space (DRS) ;  $X$  will be either an ordered Abelian group, or an ordered vector space (over the field  $\mathbf{R}$  ).

### 2.1. DEFINITIONS.

By a *homomorphism*  $g : X \rightarrow X$  we shall mean an increasing group-homomorphism (in case  $X$  is a group) or an increasing linear map (in case  $X$  is a vector space).

Given a subset  $Z \subset X$  , we say that  $Z$  is a *variety* in  $X$  if it is a subgroup (or a subspace) of  $X$ .

$Z$  is *cofinal* if, for each  $x \in X$ , there exists  $z \in Z$  such that  $x \leq z$ .

Given any variety  $Z \subset X$  , a *subfunctional*  $p : Z \rightarrow R$  is any increasing subadditive map, such that  $p(0) = 0$ ; in case  $X$  is a linear space, we also require that  $p$  is positively homogeneous.

If  $p$  is a subfunctional, then  $-p$  will be called a *superfunctional* . In case  $\mu : Z \rightarrow R$  is both a subfunctional and a superfunctional, we say that it is a *functional* .

### 2.2. DEFINITIONS.

Given any monoid  $G$ , with unit  $g_0$  , we shall say that  $G$  *acts* on  $X$  if  $G$  is embedded in the space  $\mathcal{H}$  of all homomorphisms  $g : X \rightarrow X$ .

$G$  is said to be (*right-*)*amenable* if there exists a (right-)invariant mean  $P : \mathcal{P}(G) \rightarrow [0, 1]$  , i.e. a finitely additive measure  $P$  , satisfying  $P(G) = 1$ , and  $P(g^{-1}(A)) = P(A)$  for all  $g \in G$  and  $A \subset G$ , where  $g^{-1}(A) = \{h \in G : hg \in A\}$ .

As no confusion can arise, we shall usually drop the term "right" in the previous expressions.

### 2.3. DEFINITIONS.

Let  $X$  and  $R$  be given, and  $G$  be any monoid acting on  $X$ . If  $Z \subset X$  is any variety, we say that  $Z$  is *invariant* (with respect to  $G$ ) if  $gz \in Z$  for all  $z \in Z$  and  $g \in G$ . If  $Z$  is an invariant variety, and  $\psi : Z \rightarrow R$  is any mapping, we say that  $\psi$  is *sub-invariant* if  $\psi(z) \geq \psi(gz)$  for all  $z \in Z$  and  $g \in G$ . If  $\psi$  is sub-invariant, then  $-\psi$  is said to be *super-invariant*.

If  $\psi$  is both sub- and super-invariant, we say that it is *invariant* (with respect to  $G$ ).

Given a mapping  $p : Z \rightarrow R$ , where  $Z$  is any invariant variety, we say that  $p$  is a *G-subfunctional* if it is a sub-invariant subfunctional. If  $p$  is a *G-subfunctional*,  $-p$  will be said to be a *G-superfunctional*. When a mapping  $\mu : Z \rightarrow R$  is both a *G-subfunctional* and a *G-superfunctional*, we say that  $\mu$  is a *G-functional*.

In the sequel, we shall use the following representation theorem (see [1], [6], [13]).

### 2.4. THEOREM.

*Given any DRS  $R$ , there exists a Stonian space  $\Omega$ , unique up to homeomorphisms, such that  $R$  can be embedded as a solid subspace in  $\mathcal{C}_\infty(\Omega)$ , where  $\mathcal{C}_\infty(\Omega)$  is the space of all continuous extended-real mappings  $f : \Omega \rightarrow \tilde{\mathbf{R}}$ , such that  $\{\omega \in \Omega : |f(\omega)| = +\infty\}$  is nowhere dense in  $\Omega$ .*

We now list the properties we are going to compare with amenability.

Assume that  $G$  is any monoid.

### 2.5. DEFINITION.

We say that  $G$  *satisfies property (H-B)* if, for every group (or vector space)  $X$  on which  $G$  acts, for every invariant variety  $Z \subset X$ , and any DRS  $R$ , the following extension principle holds:

"Given any  $G$ -subfunctional  $p : X \rightarrow R$  and any  $G$ -functional  $\mu : Z \rightarrow R$ , such that  $\mu(z) \leq p(z)$  for all  $z \in Z$ , there exists a  $G$ -functional  $\tilde{\mu} : X \rightarrow R$  such that  $\tilde{\mu}|_Z \equiv \mu$  and  $\tilde{\mu}(x) \leq p(x)$  for all  $x \in X$ ".

### 2.6. DEFINITION.

We say that  $G$  *satisfies property (S)* if, for all  $X$  and  $R$  as above, the following "sandwich-type" result holds:

”Given any  $G$ -superfunctional  $q : X \rightarrow R$  and any  $G$ -subfunctional  $p : X \rightarrow R$ , such that  $q \leq p$ , there exists a  $G$ -functional  $\psi : X \rightarrow R$  satisfying  $q \leq \psi \leq p$ .

### 2.7. DEFINITION.

We say that  $G$  satisfies property (E) if, for every  $X$  and  $R$  as above, and for every cofinal invariant variety  $Z \subset X$ , the following extension result holds: ”For each  $G$ -functional  $\mu : Z \rightarrow R$  there exists a  $G$ -functional  $\tilde{\mu} : X \rightarrow R$  extending  $\mu$ ”.

In the subsequent chapters, we shall prove that properties (E), (S) and (H-B) are all equivalent with amenability.

## 3. The Hahn-Banach Theorem.

Here we shall essentially prove that amenability of  $G$  implies property (H-B). This was already proved in [17], for the case that  $X$  is a linear space, but the proof we are presenting here is easier and more intuitive, though it relies on the representation Theorem, 2.4.

So, let  $X$  be any ordered Abelian group (or vector space),  $G$  any amenable monoid acting on  $X$ ,  $R$  any DRS, and  $Z$  any invariant variety of  $X$ .

Assume also that  $\mu : Z \rightarrow R$  is any  $G$ -functional, and  $p : X \rightarrow R$  is any  $G$ -subfunctional, such that  $\mu(z) \leq p(z)$  for all  $z \in Z$ .

Finally, we shall denote by  $T$  the ring  $\mathbf{Z}$  (if  $X$  is a group) or the field  $\mathbf{R}$  (in case  $X$  is a linear space).

### 3.1. DEFINITIONS.

We call *block* any list  $b \equiv (k, z, \mathbf{t}, \mathbf{g})$ , where  $k \in \mathbf{N}$ ,  $z \in Z$ ,  $\mathbf{t} \equiv (t_0, t_1, \dots, t_k) \in T^{k+1}$ ,  $\mathbf{g} \equiv (g_0, g_1, \dots, g_k) \in G^{k+1}$  (we recall here that  $g_0$  is the unit of  $G$ ).

The set of all blocks is denoted by  $B$ .

A block  $b \equiv (k, z, \mathbf{t}, \mathbf{g})$  is *positive* if  $S(\mathbf{t}) = \sum_{i=0}^k t_i > 0$ . The set of all positive blocks is denoted by  $B^+$ .

Now, for all  $x \in X$ , and any block  $b \in B^+$ , we set

$$(3.2.) \quad s(b, x) = \frac{p(z + \sum_{i=0}^k t_i g_i x) - \mu(z)}{S(\mathbf{t})} .$$

For all  $x \in X$ , we also set

$$(3.3.) \quad \begin{cases} \beta^*(x) = \inf \{ s(b, x) : b \in B^+ \} \\ \beta_*(x) = -\beta^*(-x) = \sup \{ -s(b, -x) : b \in B^+ \} \end{cases}$$

provided the involved inf ( and sup ) exist in  $R$ .

#### 3.4. PROPOSITION.

For all  $x \in X$ ,  $\beta^*(x)$  and  $\beta_*(x)$  are well-defined, and we have

$$(3.4.1.) \quad -p(-x) \leq \beta_*(x) \leq \beta^*(x) \leq p(x).$$

PROOF. In view of classical extension results (see [7] for instance) there exists a functional  $\varphi : X \rightarrow R$  such that  $\varphi|_Z \equiv \mu$  and satisfying  $-p(-x) \leq \varphi(x) \leq p(x)$  for all  $x \in X$ .

Fix  $x \in X$ . In order to show that  $\beta^*(x)$  and  $\beta_*(x)$  exist, and satisfy  $\beta_*(x) \leq \beta^*(x)$ , it will suffice to prove that

$$(3.4.2.) \quad s(b, x) \geq -s(b', -x)$$

holds true, whenever  $b \equiv (k, z, \mathbf{t}, \mathbf{g})$  and  $b' \equiv (k', z', \mathbf{t}', \mathbf{g}')$  are positive blocks.

So, fix  $b$  and  $b'$  in  $B^+$ .

Now, by virtue of 2.4.,  $R$  can be viewed as a solid subspace of  $\mathcal{C}_\infty(\Omega)$ , where  $\Omega$  is a suitable Stonian space.

Depending on  $b, b', x$ , a dense open set  $S \subset \Omega$  can be found, in such a way that

$$p(z + \sum_{i=0}^k t_i g_i x) - \mu(z), \quad \mu(z') - p(z' - \sum_{j=0}^{k'} t'_j g'_j x), \quad p(x) \quad \text{and} \quad -p(-x)$$

are real-valued continuous functions on  $S$ .

This also entails that  $\varphi(gg_i x)$ ,  $\varphi(gg'_j x)$  are real-valued on  $S$ , for all  $g \in G$ , because of the bounds

$$-p(-x) \leq \varphi(hx) \leq p(x)$$

holding by construction of  $\varphi$ , for all  $h \in G$ .

Hence, for all  $\omega \in S$ , for all  $i$  and  $j$ , it's possible to consider the integrals

( of bounded real functions ):

$$\int_G \varphi(g g_i x) (\omega) dP(g) \quad \text{and} \quad \int_G \varphi(g g'_j x) (\omega) dP(g) ,$$

where  $P$  is any invariant mean on  $G$ .

As  $P$  is right-invariant, these integrals are too, so we have

$$\int_G \varphi(g g_i x) (\omega) dP(g) = \int_G \varphi(gx) (\omega) dP(g) = \int_G \varphi(g g'_j x) (\omega) dP(g)$$

for all  $i$  and  $j$ .

Now we remark that, for all  $\omega \in S$  and  $g \in G$ , it holds :

$$\begin{aligned} \frac{p(z + \sum_{i=0}^k t_i g_i x) (\omega) - \mu(z) (\omega)}{S(\mathbf{t})} &\geq \frac{p(gz + \sum_{i=0}^k t_i g g_i x) (\omega) - \mu(gz) (\omega)}{S(\mathbf{t})} \geq \\ &\geq \frac{\varphi(gz + \sum_{i=0}^k t_i g g_i x) (\omega) - \mu(gz) (\omega)}{S(\mathbf{t})} = \frac{\sum_{i=0}^k t_i \varphi(g g_i x) (\omega)}{S(\mathbf{t})} . \end{aligned}$$

By integrating in  $g$ , and using sub-invariance of  $p$ , invariance of  $\mu$ , we find

$$s(b, x) (\omega) = \frac{p(z + \sum_{i=0}^k t_i g_i x) (\omega) - \mu(z)(\omega)}{S(\mathbf{t})} \geq \int_G \varphi(gx) (\omega) dP(g), \quad \omega \in S.$$

By a similar argument, we also have

$$-s(b', -x) (\omega) = \frac{\mu(z') (\omega) - p(z' - \sum_{j=0}^{k'} t'_j g'_j x) (\omega)}{S(\mathbf{t}')} \leq \int_G \varphi(gx) (\omega) dP(g), \quad \omega \in S.$$

This shows that

$$-s(b', -x) (\omega) \leq s(b, x) (\omega)$$

for all  $\omega \in S$ .

As  $S$  is a dense open set, we can deduce (3.4.2.) , and so (3.4.1.) is proved.

3.5. LEMMA.

In the same hypothesis as above, fix  $x \in X$  and let  $b \equiv (k, z, \mathbf{t}, \mathbf{g})$  be any block such that  $S(\mathbf{t}) = 0$ . Then it holds:

$$\mu(z) \leq p(z + \sum_{i=0}^k t_i g_i x).$$

PROOF. Let  $b \equiv (k, z, \mathbf{t}, \mathbf{g})$  be any block such that  $S(\mathbf{t}) = 0$ . Now fix  $\varphi$  and  $S$  as in the proof of 3.4. For all  $\omega \in S$  and  $g \in G$  we find

$$\begin{aligned} p(z + \sum_{i=0}^k t_i g_i x)(\omega) &\geq p(gz + \sum_{i=0}^k t_i g g_i x)(\omega) \geq \\ &\geq \varphi(gz + \sum_{i=0}^k t_i g g_i x)(\omega) = \mu(z)(\omega) + \sum_{i=0}^k t_i \varphi(g g_i x)(\omega). \end{aligned}$$

By integrating with respect to some invariant mean  $P$ , we get

$$p(z + \sum_{i=0}^k t_i g_i x)(\omega) \geq \mu(z)(\omega) + \left( \sum_{i=0}^k t_i \right) \int_G \varphi(gx)(\omega) dP(g) = \mu(z)(\omega),$$

for all  $\omega \in S$ . As  $S$  is an open dense subset of  $\Omega$ , this concludes the proof.

### 3.6. PROPOSITION.

Let  $b \in B$ ,  $b \equiv (k, z, \mathbf{t}, \mathbf{g})$ , and fix any point  $x_0 \in X \setminus Z$ . Let  $a$  be any element of  $R$ , such that

$$\beta_*(x_0) \leq a \leq \beta^*(x_0).$$

If it holds

$$z + \sum_{i=0}^k t_i g_i x_0 \geq 0 \quad (\text{resp. } \leq 0)$$

then we have

$$\mu(z) + a S(\mathbf{t}) \geq 0 \quad (\text{resp. } \leq 0).$$

PROOF. The proof is essentially the same as in Theorem 2.5. of [4], for the case  $S(\mathbf{t}) \neq 0$ , and simply relies on the definition of  $\beta^*$ ,  $\beta_*$  and monotonicity of  $p$ .

In case  $S(\mathbf{t}) = 0$ , assume for instance that  $z + \sum_{i=0}^k t_i g_i x_0 \leq 0$ . For all  $g \in G$  we have

$$0 \geq gz + \sum_{i=0}^k t_i g g_i x_0 .$$

Fix now  $\varphi$  and  $S$  as in the proof of 3.4. For all  $\omega \in S$  we have

$$0 \geq \mu(z)(\omega) + \sum_{i=0}^k t_i \varphi(g g_i x_0)(\omega).$$

By integrating in  $g$ , we get

In case  $S(\mathbf{t}) = 0$ , assume for instance that  $z + \sum_{i=0}^k t_i g_i x_0 \leq 0$ . For all  $g \in G$  we have

$$0 \geq gz + \sum_{i=0}^k t_i g g_i x_0 .$$

Fix now  $\varphi$  and  $S$  as in the proof of 3.4. For all  $\omega \in S$  we have

$$0 \geq \mu(z)(\omega) + \sum_{i=0}^k t_i \varphi(g g_i x_0)(\omega).$$

By integrating in  $g$ , we get

$$0 \geq \mu(z)(\omega) + \left(\sum_{i=0}^k t_i\right) \int_G \varphi(g x_0)(\omega) dP(g) = \mu(z)(\omega), \text{ for all } \omega \in S.$$

This shows that  $\mu(z) \leq 0$ .

A similar argument shows that  $\mu(z) \geq 0$  in case  $S(\mathbf{t}) = 0$  and  $z + \sum_{i=0}^k t_i g_i x_0 \geq 0$ .

It's now easy to deduce the following

### 3.7. COROLLARY.

Let  $b \in B$ ,  $b \equiv (k, z, \mathbf{t}, \mathbf{g})$ , and assume that  $z = \sum_{i=0}^k t_i g_i x_0$  for some  $x_0 \in X$ . Then we have

$$(3.7.1.) \quad \beta^*(x_0) = \beta_*(x_0) \text{ and } \mu(z) = S(\mathbf{t}) \beta^*(x_0), \text{ whenever } S(\mathbf{t}) \neq 0$$

$$(3.7.2.) \quad \mu(z) = 0, \text{ if } S(\mathbf{t}) = 0$$

### 3.8. THEOREM.

In the same situation as above , let  $x_0 \notin Z$  be fixed, and let  $Y \subset X$  be the least invariant variety , containing  $Z$  and  $x_0$ . Then , for any element  $a \in R$  , such that  $\beta_*(x_0) \leq a \leq \beta^*(x_0)$ , there exists at least one  $G$ -functional  $\tilde{\mu} : X \rightarrow R$  , such that  $\tilde{\mu}|_Z \equiv \mu$  ,  $\tilde{\mu} \leq p$ , and  $\tilde{\mu}(x_0) = a$ . Furthermore,  $\tilde{\mu}|_Y$  is unique.

PROOF. We can proceed exactly as in [4] for the extension of  $\mu$  to the variety  $Y$  : in simple words, one can easily see that

$$Y = \{z + \sum_{i=0}^k t_i g_i x_0 : (k, z, \mathbf{t}, \mathbf{g}) \in B\}.$$

For all  $y \in Y$ ,  $y = z + \sum_{i=0}^k t_i g_i x_0$ , we set

$$\tilde{\mu}(y) = \mu(z) + a S(\mathbf{t}).$$

In view of 3.7.,  $\tilde{\mu}$  is well-defined on  $Y$ . It's clear that  $\tilde{\mu}$  is a  $G$ -functional, extending  $\mu$ , the property of monotonicity being ensured by 3.6. Moreover, for  $y \in Y$ ,  $y = z + \sum_{i=0}^k t_i g_i x_0$ , and  $\sum_{i=0}^k t_i > 0$ , it holds

$$\begin{aligned} \tilde{\mu}(y) &= \mu(z) + a S(\mathbf{t}) \leq \mu(z) + \beta^*(x_0) S(\mathbf{t}) \leq \\ &\leq \mu(z) + \frac{p(z + \sum_{i=0}^k t_i g_i x_0) - \mu(z)}{S(\mathbf{t})} S(\mathbf{t}) = p(y). \end{aligned}$$

So  $\tilde{\mu}(y) \leq p(y)$  at least when  $y = z + \sum_{i=0}^k t_i g_i x_0$  and  $\sum_{i=0}^k t_i > 0$ . Still assuming  $\sum_{i=0}^k t_i > 0$ , we have also

$$\tilde{\mu}(y) \geq \mu(z) + \beta_*(x_0) S(\mathbf{t}) \geq \mu(z) + S(\mathbf{t}) \frac{\mu(-z) - p(-z - \sum_{i=0}^k t_i g_i x_0)}{S(\mathbf{t})}$$

by definition of  $\beta_*(x_0)$ , hence

$$(*) \quad \tilde{\mu}(y) \geq -p(-y)$$

at least when  $y = z + \sum_{i=0}^k t_i g_i x_0$  and  $\sum_{i=0}^k t_i > 0$ .

Now , if  $\sum_{i=0}^k t_i < 0$ , replacing  $y$  by  $-y$  we can use (\*) and get  $\tilde{\mu}(-y) \geq -p(y)$ , which means  $\tilde{\mu}(y) \leq p(y)$ .

Thus,  $\tilde{\mu}(y) \leq p(y)$  at least when  $y = z + \sum_{i=0}^k t_i g_i x_0$  and  $\sum_{i=0}^k t_i \neq 0$ . In case  $\sum_{i=0}^k t_i = 0$ , we have  $\tilde{\mu}(y) = \mu(z)$  and then the domination follows from 3.5.

So,  $\tilde{\mu}$  is a  $G$ -functional extending  $\mu$  to  $Y$ , and it is dominated by  $p$ . Of course, the definition of  $\tilde{\mu}$  in  $Y$  is only depending on  $a$ , so uniqueness is obvious.

The possibility of further extending  $\mu$  follows from Zorn's Lemma, by iterating the previous process.

So the theorem is completely proved.

### 3.9. COROLLARY.

*Amenability implies property (H-B).*

We can also deduce some complementary results collected in the following theorem, whose proof is easy (see also [4]).

### 3.10. THEOREM.

*Under the same hypothesis as above, one has*

$$(3.10.1.) \quad \beta_*(x) \leq \tilde{\mu}(x) \leq \beta^*(x)$$

*for all  $x \in X$ , and every  $G$ -functional  $\tilde{\mu}$  extending  $\mu$ ,  $\tilde{\mu} \leq p$ .*

$$(3.10.2.) \quad \beta^*(x) = \max \{ \tilde{\mu}(x) : \tilde{\mu} \text{ is an extension of } \mu \},$$

$$(3.10.3.) \quad \beta_*(x) = \min \{ \tilde{\mu}(x) : \tilde{\mu} \text{ is an extension of } \mu \},$$

*for all  $x \in X$ .*

## 4. The sandwich theorem and consequences.

In n.3 we saw that amenability implies (H-B). Our next goal is to prove that (H-B) implies (S): this can be done in essentially the same lines as in [18], however we shall outline the main steps.

4.1. LEMMA.

Let  $f_1, f_2$  be two  $G$ -subfunctionals on  $X$ ,  $f_1 \geq -f_2$ .

Set  $f_2^*(x) = f_2(-x)$  and

$$f(x) = \inf \{ f_1(x_1) + f_2^*(x_2) : x_1, x_2 \in X, x_1 + x_2 = x \},$$

for all  $x \in X$ . Then we have:

- a)  $f$  is a well-defined subfunctional,  $f \leq f_1$ ,  $f \leq f_2^*$ .
- b) If  $\psi : X \rightarrow R$  is any functional, then  $\psi \leq f$  if and only if  $\psi \leq f_1$  and  $\psi \leq f_2^*$ .
- c)  $f$  is a  $G$ -subfunctional, i.e.  $f$  is sub-invariant.

PROOF. For a) and b), see [18].

c) Fix  $g \in G$  and  $x \in X$ . Then

$$f(x) = \inf \{ f_1(x_1) + f_2^*(x_2) : x_1, x_2 \in X, x_1 + x_2 = x \} \geq \inf \{ f_1(gx_1) + f_2^*(gx_2) :$$

$$x_1, x_2 \in X, gx_1 + gx_2 = gx \} \geq \inf \{ f_1(s_1) + f_2^*(s_2) : s_1, s_2 \in X, s_1 + s_2 = gx \} = f(gx).$$

Thus  $f$  is subinvariant and c) is proved.

4.2. THEOREM.

Assume that  $G$  satisfies (H-B), and let  $X$  be any ordered abelian group, or vector space, on which  $G$  acts. Fix any DRS  $R$ , and any pair  $(p, q)$  of applications from  $X$  to  $R$ , such that  $p$  and  $-q$  are  $G$ -subfunctionals,  $p \geq q$ . Then there exists a  $G$ -functional  $\mu : X \rightarrow R$  satisfying  $q \leq \mu \leq p$ . Thus, (H-B) implies (S).

PROOF. Let  $Z$  be any variety, such that  $p(z) = q(z)$  for each  $z \in Z$ , and set  $\mu(z) = p(z) = q(z)$  for all  $z \in Z$ .

It's easy to see that  $\mu : Z \rightarrow R$  is a  $G$ -functional.

If we put  $f_1 = p, f_2 = -q$  in Lemma 4.1., then the subfunctional  $f$  described there satisfies

$$f(x) \leq p(x), \quad f(x) \leq -q(-x) \text{ for all } x \in X, \text{ and}$$

$$\mu(z) \leq f(z) \text{ for all } z \in Z .$$

So, by (H-B), there exists an extension  $\tilde{\mu} : X \rightarrow R$  of  $\mu$ , that is a  $G$ -functional  $\tilde{\mu}$  such that  $\tilde{\mu}|_Z \equiv \mu$  and  $\tilde{\mu} \leq f$ . As  $f \leq p$ , we have  $\tilde{\mu} \leq p$ . As  $f(x) \leq -q(-x)$ , we have  $\tilde{\mu}(-x) \geq q(-x)$  for all  $x \in X$ , i.e.  $\tilde{\mu} \geq q$ . So  $\tilde{\mu}$  is the required functional.

So far, we have proved that amenability implies (H-B), and that (H-B) implies (S). It's possible to state an analogous result of 3.8. and 3.10. for property (S), as follows.

#### 4.3. THEOREM.

Let's assume that  $G$  is amenable, and let  $X, Z, R, p$  and  $q$  be as in 4.2. For all  $x \in X$ , set

$$\begin{aligned} f(x) &= \inf \{ p(x_1) - q(-x_2) : x_1 + x_2 = x, x_1, x_2 \in X \}, \\ \gamma^*(x) &= \inf \left\{ \frac{f(z + \sum_{i=0}^k t_i g_i x) - p(z)}{S(\mathbf{t})} : (k, z, \mathbf{t}, \mathbf{g}) \in B^+ \right\}, \\ \gamma_*(x) &= \sup \left\{ \frac{p(z) - f(z - \sum_{i=0}^k t_i g_i x)}{S(\mathbf{t})} : (k, z, \mathbf{t}, \mathbf{g}) \in B^+ \right\}. \end{aligned}$$

Then, fixed any point  $x_0 \in X \setminus Z$ , for every element  $a \in R, \gamma_*(x_0) \leq a \leq \gamma^*(x_0)$ , there exists a  $G$ -functional  $\psi : X \rightarrow R$  such that  $\psi(x_0) = a$  and  $q \leq \psi \leq p$ . Furthermore, it holds

$$\gamma_*(x) \leq \tilde{\psi}(x) \leq \gamma^*(x)$$

for all  $x \in X$ , and every  $G$ -functional  $\tilde{\psi}, q \leq \tilde{\psi} \leq p$ .

PROOF. Straightforward.

We now give the announced equivalence result, which can be viewed as a characterization of amenability (For other characterizations, we refer to [12] and [20]).

#### 4.4. THEOREM.

Let  $G$  be any monoid. Then the following are equivalent:

- i)  $G$  is amenable
- ii)  $G$  satisfies (H-B)
- iii)  $G$  satisfies (S)
- iv)  $G$  satisfies (E) .

PROOF.

- i)  $\Rightarrow$  ii) : Corollary 3.9.
- ii)  $\Rightarrow$  iii) : Theorem 4.2.
- iii)  $\Rightarrow$  iv) : let  $X, Z, \mu$  be as in Definition 2.7., and put

$$\begin{aligned} p(x) &= \inf \{ \mu(z) : z \in Z, z \geq x \} \\ q(x) &= \sup \{ \mu(z) : z \in Z, z \leq x \} \end{aligned}$$

for all  $x \in X$ . As  $Z$  is cofinal in  $X$ ,  $p$  and  $q$  are well-defined mappings from  $X$  to  $R$ . It's easy to see that  $p$  and  $-q$  are  $G$ -subfunctionals, such that  $q \leq p$ .

So, by iii) there exists a  $G$ -functional  $\tilde{\mu} : X \rightarrow R$ , such that  $q \leq \tilde{\mu} \leq p$ . As  $q(z) = p(z) = \mu(z)$  for all  $z \in Z$ , it's plain that  $\tilde{\mu}|_Z \equiv \mu$ . Hence, iv) holds.

iv)  $\Rightarrow$  i). Choose  $R = \mathbf{R}$ , and  $X = \{u : G \rightarrow \mathbf{R} \mid u \text{ is bounded}\}$ . We define the action of  $G$  on  $X$  by setting

$$g u(h) = u(hg)$$

for all  $g, h \in G$  and  $u \in X$ .

For each real number  $z$ , we shall denote by  $\mathbf{z}$  the constant function on  $G$ , such that  $\mathbf{z}(g) = z$  for all  $g \in G$ .

Set  $Z = \{ \mathbf{z} : z \in \mathbf{R} \}$ . It's clear that  $Z$  is an invariant cofinal variety in  $X$ . If we define  $\mu : Z \rightarrow \mathbf{R}$  as

$$\mu(\mathbf{z}) = z \text{ for all } \mathbf{z} \in Z,$$

then we have that  $\mu$  is a  $G$ -functional on  $Z$ . By property (E),  $\mu$  can be extended to a  $G$ -functional  $\tilde{\mu}$  on  $X$ .

Now, set  $P(A) = \tilde{\mu}(1_A)$ , for all  $A \subset G$ .

It's plain that  $P$  is a finitely additive invariant measure on  $\mathcal{P}(G)$ , and  $P(G) = \tilde{\mu}(\mathbf{1}) = 1$ . Finally  $P$  is nonnegative, because of monotonicity of  $\tilde{\mu}$ .

So, the theorem is completely proved.

Before giving some complements to property (E), we need one more definition.

4.5. DEFINITION.

Let  $X, G, R, Z, \mu$  be as in Definition 2.7.

Given any point  $x \in X$ , we say that a block  $b \equiv (k, z, \mathbf{t}, \mathbf{g})$  is *super-admissible* (resp. *sub-admissible*) for  $x$  if

$$\sum_{i=0}^k t_i g_i x \leq z \quad (\text{resp. } \sum_{i=0}^k t_i g_i x \geq z).$$

As  $Z$  is cofinal, for all  $x \in X$  there are positive super-admissible and sub-admissible blocks for  $x$ . The set of positive super-admissible (resp. sub-admissible) blocks for  $x$  is denoted by  $B^+(x)$  (resp.  $B_+(x)$ ).

Then, for all  $x \in X$ , set

$$\alpha^*(x) = \inf \left\{ \frac{\mu(z)}{S(\mathbf{t})} : (k, z, \mathbf{t}, \mathbf{g}) \in B^+(x) \right\},$$

$$\alpha_*(x) = \sup \left\{ \frac{\mu(z)}{S(\mathbf{t})} : (k, z, \mathbf{t}, \mathbf{g}) \in B_+(x) \right\}.$$

4.6. THEOREM.

*In the hypothesis of Definition 4.5., assume that  $G$  is amenable. Then, fixed any point  $x_0 \in X \setminus Z$ , and for any element  $a \in R$ ,  $\alpha_*(x_0) \leq a \leq \alpha^*(x_0)$ , there exists a  $G$ -functional  $\tilde{\mu} : X \rightarrow R$ , extending  $\mu$  and such that  $\tilde{\mu}(x_0) = a$ .*

*Furthermore, it holds*

$$\alpha_*(x) \leq \tilde{\mu}(x) \leq \alpha^*(x),$$

*for all  $x \in X$ , and for every such extension  $\tilde{\mu}$ .*

The proof is a direct consequence of 4.4. and of Theorem 2.4. in [4].

4.7. REMARKS.

- 1) Except for the ones involving property (E), our results hold true also when  $X$  is not ordered: in such case, it suffice to set  $x_1 \leq x_2 \Leftrightarrow x_1 = x_2$  for all  $x_1, x_2$  in  $X$ . Of course, monotonicity then becomes meaningless.

- 2) In general, Dedekind completeness cannot be dropped, even if  $G = \{g_0\}$ , i.e. when invariance is disregarded: indeed, if property (H-B) holds, then  $R$  must be complete, by well-known results in [9], [11], [19].

Another characterization of amenability can be given by means of coherence, as we shall now see.

4.8. DEFINITION.

Let  $W$  be any nonempty set, and let  $\mathcal{D} \subset \mathcal{P}(W)$  be any collection of subsets of  $W$ , such that  $\emptyset, W \in \mathcal{D}$ . Assume that  $G \subset W^W$  is any monoid of transformations, where the "product" is defined by setting

$$(4.8.1.) \quad g_1 g_2 (w) = g_2 (g_1(w))$$

for all  $w \in W, g_1, g_2 \in G$ . Let  $R$  be any DRS.

A map  $\varphi : \mathcal{D} \rightarrow R$ , such that  $\varphi(\emptyset) = 0$ , is said to be *coherent* if, for every  $r_1, \dots, r_n, r'_1, \dots, r'_m \in \mathbf{R}$ , and every  $D_1, \dots, D_n, D'_1, \dots, D'_m \in \mathcal{D}$ , the following implication holds:

$$\sum_{i=1}^n r_i 1_{D_i} \leq \sum_{j=1}^m r'_j 1_{D'_j} \implies \sum_{i=1}^n r_i \varphi(D_i) \leq \sum_{j=1}^m r'_j \varphi(D'_j)$$

By a similar argument as in [3] and [4] we can prove the following

4.9. THEOREM.

Let  $G, R, W, \mathcal{D}$  be as in 4.8. , and assume that  $G$  is amenable, and  $g^{-1}(D) \in \mathcal{D}$  for all  $g \in G$  and  $D \in \mathcal{D}$ . Let  $\varphi : \mathcal{D} \rightarrow R$  be any map, such that  $\varphi(\emptyset) = 0$ . Then the following are equivalent:

- i)  $\varphi$  is coherent and  $\varphi(g^{-1}(D)) = \varphi(D)$  for all  $D \in \mathcal{D}, g \in G$ .
- ii)  $\varphi$  admits a finitely additive  $G$ -invariant extension  $\tilde{\varphi} : \mathcal{P}(W) \rightarrow R$ .

As a consequence of 4.9. and 4.4., we have

4.10. THEOREM.

Let  $G$  be any monoid. Then the following are equivalent:

- 4.10.1.)  $G$  is amenable.

4.10.2.) For any choice of  $R, W, \mathcal{D}$  as in 4.9., such that  $G$  embeds in  $W^W$  as described in 4.8.1.) , and for every coherent,  $G$ -invariant map  $\varphi : \mathcal{D} \rightarrow R$  , there exists a finitely additive  $G$ -invariant extension  $\tilde{\varphi} : \mathcal{P}(W) \rightarrow R$ .

## REFERENCES

- [1] S.J.BERNAU, *Unique representation of Archimedean lattice groups and normal Archimedean lattice rings*, Proc. Lond. Math. Soc., **15** (1965), 599-631.
- [2] A.BOCCUTO - D.CANDELORO, *A sandwich-type theorem for invariant measures*, Atti Sem. Mat. Fis. Univ. Modena, **38** (1990), 415-424.
- [3] A.BOCCUTO - D.CANDELORO, *Sandwich theorems and applications to invariant measures*, Atti Sem. Mat. Fis. Univ. Modena, **38** (1990), 511-524.
- [4] A.BOCCUTO - D.CANDELORO, *Extensions and uniqueness theorems for invariant measures*, Atti Sem. Mat. Fis. Univ. Modena (to appear).
- [5] A.L. DAWIDOWICZ, *On the lifting of invariant measures*, Ann. Pol. Math., **51** (1990), 137-139.
- [6] W.FILTER, *Representations of Archimedean Riesz spaces - a survey* (1990).
- [7] B.FUCHSSTEINER - W.LUSKY, *Convex cones*, (1981), North-Holland Publishing Co.
- [8] B.GIROTTO - S.HOLZER, *Existence of invariant masses. An elementary treatment*, Rend. Ist. Mat. Univ. Trieste, **20** (1988), 24-27.
- [9] A.D.IOFFE, *A new proof of the equivalence of the Hahn-Banach extension and the least upper bound properties*, Proc. Amer. Math. Soc., **82** (1981), 385-389.
- [10] A.G.KUSRAEV - S.S.KUTATELADZE, *Local convex Analysis*, J. Sov. Math., **26** (1984), 2048-2087.
- [11] A.G.KUSRAEV - S.S.KUTATELADZE, *Subdifferential calculus*, (1987), Nauka (Russian).
- [12] A. T.-M. LAU, *Amenability of semigroups. The analytical and topological theory of semigroups*, De Gruyter Expo. Math., **1** (1990), 313-334.

- [13] F.MAEDA - T.OGASAWARA, *Representation of vector lattices*, J. Sci. Hiroshima Univ. Ser. A **12** (1942), 17-35. (Japanese)
- [14] P.PLAPPERT, *A sandwich theorem for precharges*, J. Math. Anal. Appl., **156** (1991), 395-409.
- [15] Z.SASVÁRI, *Additivity of certain functionals and the constructions of invariant integrals*, Math. Ann., **283** (1989), 573-581.
- [16] R.J.SILVERMAN, *Invariant linear functions*, Trans. Amer. Math. Soc., **81** (1956), 411-424.
- [17] R.J.SILVERMAN, *Invariant means and cones with vector interiors*, Trans. Amer. Math. Soc., **88** (1958), 75-79; R.J.SILVERMAN - T.YEN, *Addendum to "Invariant means and cones with vector interiors"*, ibidem, 327-330.
- [18] S.SIMONS, *Extended and sandwich versions of the Hahn-Banach theorem*, J. Math. Anal. Appl., **21** (1968), 112-122.
- [19] T.-O. TO, *The equivalence of the least upper bound property and the Hahn-Banach extension property in ordered vector spaces*, Proc. Amer. Math. Soc., **30** (1970), 287-296.
- [20] S.WAGON, *The Banach-Tarski Paradox*, (1985), Cambridge Univ. Press.

Dipartimento di Matematica, Università di Perugia