Comparison between some Kurzweil-Henstock type integrals for Riesz-space-valued functions

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ABSTRACT. Some kinds of integral, like variational, Kurzweil-Henstock and (SL)-integral are introduced and investigated in the context of Riesz-space-valued functions and with respect to abstract interval bases.


KEY WORDS: Riesz spaces, derivation basis, Kurzweil-Henstock integral, Variational integral, Lusin property, interval functions.

1 Introduction

In this note we compare three kinds of definitions of Kurzweil-Henstock type integrals with respect to an abstract derivation bases: the original definition of the Kurzweil-Henstock integral based on generalized Riemann sums, variational integral and the so-called SL-integral, introduced in [7]. The relation between those integrals depends on whether we consider them in application to the real-valued functions or to the Banach-space-valued functions or at last to the functions with values in Riesz spaces (vector lattices). In the case of real-valued functions all the three integrals are equivalent (see [7]). In the Banach case the SL-integral is equivalent to the variational integral, but both are strictly included into the Kurzweil-Henstock integral. As for the

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Riesz-space-valued case, we shall check that the Kurzweil-Henstock integral is equivalent to the variational integral, and the $SL$-integral is not equivalent to them. Only if we impose some additional assumption on the Riesz space all three integrals will be equivalent.

2 Preliminaries

We introduce some definitions and notations. For the sake of simplicity we consider here integrals for the functions defined on an interval of the real line $\mathbb{R}$, although the results are true for much more general setting. A derivation basis (or simply a basis) $\mathcal{B}$ on an interval $[a, b] \subset \mathbb{R}$ is a filter base on the product space $\mathcal{I} \times [a, b]$ where $\mathcal{I}$ is a family of intervals which we shall call $\mathcal{B}$-intervals. That is $\mathcal{B}$ is a nonempty collection of subsets of $\mathcal{I} \times [a, b]$ so that each $\beta \in \mathcal{B}$ is a set of pairs $(I, x)$, where $I \in \mathcal{I}$, $x \in [a, b]$, and $\mathcal{B}$ has the filter base property: $\emptyset \notin \mathcal{B}$ and for every $\beta_1, \beta_2 \in \mathcal{B}$ there exists $\beta \in \mathcal{B}$ such that $\beta \subset \beta_1 \cap \beta_2$. So each basis is an ordered directed set and the order is given by the "reversed" inclusion. We shall refer to the elements $\beta$ of $\mathcal{B}$ as basis sets. In this paper we shall always suppose that $(I, x) \in \beta$ implies $x \in I$, although it is not the case in the general theory (see [11]). For a set $E \subset [a, b]$ and $\beta \in \mathcal{B}$ we write

$$\beta(E) = \{(I, x) \in \beta : I \subset E\} \quad \text{and} \quad \beta[E] = \{(I, x) \in \beta : x \in E\}.$$  

We say that a basis $\mathcal{B}$ is a Vitali basis, if for any $x$, for each neighborhood $U(x)$ of $x$ and for every $\beta \in \mathcal{B}$ the set $\{(I, x) \in \beta[x] : I \subset U(x)\}$ is nonempty. The simplest derivation basis on $\mathbb{R}$ is the full interval basis. In this case, $\mathcal{I}$ is the set of all intervals in $\mathbb{R}$ and each basis set is defined by a positive function $\delta$ on $\mathbb{R}$ called gage as

$$\beta_\delta = \{(I, x) : I \in \mathcal{I}, \ x \in I \subset (x - \delta(x), x + \delta(x))\}.$$  

So the full interval basis is the family $(\beta_\delta)_\delta$ where $\delta$ runs over the set of all possible gages.

A finite collection $\pi \subset \beta$ is called a $\beta$-partition if for any distinct elements $(I', x')$ and $(I'', x'')$ in $\pi$, the $\mathcal{B}$-intervals $I'$ and $I''$ are non-overlapping. If a partition $\pi = \{(I_i, x_i)\} \subset \beta(I)$ for some $I \in \mathcal{I}$ is such that $\bigcup I_i = I$, then we say that $\pi$ is a $\beta$-partition of $I$. We say that a basis $\mathcal{B}$ has the partitioning
property if for each \( \mathcal{B} \)-interval \( I \) and for any \( \beta \in \mathcal{B} \) there exists a \( \beta \)-partition of \( I \). In the particular case of the full interval basis on \( \mathbb{R} \), this property has long been known as the Cousin lemma. But for some bases this property was proved only recently (see [4]), and there are bases for which it is not valid at all or holds true only in some weaker sense as it is in the case of the symmetric approximate basis (see [10]). We say that a basis \( \mathcal{B} \) ignors a set \( E \subset [a, b] \) if there exists a basis set \( \beta \in \mathcal{B} \) such that \( \beta[E] \) is empty. We say that \( \mathcal{B} \) is a complete basis if it ignores no point. We say that \( \mathcal{B} \) is an almost complete basis if it ignores no set of Lebesgue non-zero measure \( \mu \). Having a complete basis \( \mathcal{B} \) on \( [a, b] \) we can always extend it to the almost complete basis by including into \( \mathcal{B} \) together with each basis set \( \beta \) all the sets of the form \( \beta \setminus \beta[K] \), where \( K \) is any set of Lebesgue measure \( \mu \) zero. We shall denote this almost complete extension of basis \( \mathcal{B} \) by \( \mathcal{B}_0 \). Furthermore, we shall assume that, for each \( \theta \in \mathcal{B}_0 \), with \( N = \{ x : \theta[\{x\}] = \emptyset \} \), for each \( \beta \in \mathcal{B} \), there exists \( \gamma \in \mathcal{B} \) with the following property:

\[
\gamma[\{x\}] = \begin{cases} 
\beta[\{x\}] & \text{if } x \in N,
\theta[\{x\}] & \text{if } x \notin N.
\end{cases}
\]

Note that the partitioning property of a basis \( \mathcal{B} \) does not guarantee that \( \mathcal{B}_0 \) has the same property.

We denote by \( R \) a Dedekind complete Riesz space (see [8]). We add to \( R \) two extra elements, \(+\infty\) and \(-\infty\), extending to \( R \cup \{+\infty, -\infty\} \) ordering and operations in a natural way.

A Riesz space \( R \) satisfies property \( \sigma \) if, given any sequence \((u_n)_n\) in \( R \) with \( u_n \geq 0 \forall n \in \mathbb{N} \), there exists a sequence \((\lambda_n)_n\) of positive real numbers, such that the sequence \((\lambda_n u_n)_n\) is bounded in \( R \). A Riesz space \( R \) satisfies the Swartz property if there exists a sequence \((h_n)_n\) in \( R \) such that, for each \( x \in R \), \( \exists k, n \in \mathbb{N} \) such that \( |x| \leq k h_n \) (see [12]).

3 The Kurzweil-Henstock integral

We now introduce a Kurzweil-Henstock type integral related to the basis \( \mathcal{B} \) (for the real case, see [3]). If \( E \) is a fixed \( \mathcal{B} \)-set, \( f : E \to R \) and \( \pi = \{(J_i, \xi_i) : i = 1, \ldots, q\} \) is a partition of \( E \), we will call Riemann sum associated with \( \pi \)
and we will write it by the symbol $S(f, \pi)$ the quantity $\sum_{i=1}^{q} \mu(J_i) f(\xi_i)$.

**Definition 3.1** Let $R$ be a Dedekind complete Riesz space, $B$ be a fixed basis having the partitioning property and $E \subset X$ be a $B$-set. We say that $f : E \rightarrow R$ is Kurzweil-Henstock integrable (in brief, $H_B$-integrable) on $E$ (with respect to $B$) if there exists an element $Y \in R$ such that

$$\inf_{\beta \in B} (\sup \{|S(f, \pi) - Y| : \pi \text{ is a } \beta-\text{partition of } E\}) = 0.$$  

(1)

In this case we write $(H_B) \int_E f = Y$.

It is easy to see that the element $Y$ in (1) is uniquely determined.

The following two propositions can be proved repeating the arguments of the corresponding propositions in [2].

**Proposition 3.2** If $E = I \cup J$ where $E$, $I$, $J$ are $B$-intervals, $I$ and $J$ are non-overlapping and $f$ is $(H_B)$-integrable on $I$ and on $J$, then $f$ is $(H_B)$-integrable on $E$ and $(H_B) \int_E f = (H_B) \int_I f + (H_B) \int_J f$.

**Proposition 3.3** If $f$ is $(H_B)$-integrable on a $B$-interval $I$ and $J \subset I$ is a $B$-interval, then $f$ is $(H_B)$-integrable on $J$ too.

It follows from the last two propositions that for any $H_B$-integrable function $f : E \rightarrow R$ the indefinite $H_B$-integral is defined as an additive $R$-valued $B$-interval function on the family of all $B$-intervals in $E$. We shall denote it by

$$F(I) = (H_B) \int_I f.$$  

(2)

The following version of the so called Saks-Henstock Lemma can be proved using argument similar to the one used in the proof of a little bit less general version in [2] (see also [9], Lemma 12, pp. 353-354):

**Lemma 3.4** If $f$ is $(H_B)$-integrable on $E$ and $F$ is as in (2), then

$$\inf_{\beta \in B} \left\{ \sup_{(I,x) \in \pi} \left| \mu(I) f(x) - F(I) \right| : \pi \text{ is a } \beta-\text{partition of } E \right\} = 0.$$  

(3)
Having fixed a set \( W \subset X \), a complete basis \( B \) and a point-set function \( F : \mathcal{I} \times X \to R \) we define, for each \( E \subset W \),
\[
Var(\beta, F, E) = \sup_{\pi} \sum_{(I,x) \in \pi} |F(I,x)|,
\]
where the involved supremum is taken over the totality of all \( \beta \)-partitions \( \pi \) on \( E \). We also define
\[
V(B, F, E) = \inf_{\beta} Var(\beta, F, E).
\]

Being considered as a set function on the family of all the subsets \( E \subset W \), we call \( Var(\beta, F, \cdot) \) the \( \beta \)-variation and \( V(B, F, \cdot) \) the variational measure on \( W \), generated by \( F \), with respect to the basis \( B \). (We are using the term ”measure” here because in the real-valued case the variational measure is in fact a metric outer measure.) If in the above definitions we replace a complete basis \( B \) by the almost complete basis \( B_0 \) generated by \( B \), we get the essential \( \beta \)-variation and the essential variational measure, respectively. That is
\[
Var_{ess}(\beta, F, E) = \sup_{\pi} \sum_{(I,x) \in \pi} |F(I,x)|,
\]
where the involved supremum is taken over the totality of all \( \beta \)-partitions \( \pi \) on \( E \), \( \beta \in B_0 \) and
\[
V_{ess}(B_0, F, E) = \inf_{\beta} Var_{ess}(\beta, F, E).
\]

It is clear that
\[
V_{ess}(B_0, F, E) \leq V(B, F, E) \tag{4}
\]

**Definition 3.5** We say that two interval-point functions \( F \) and \( G \) are variationally equivalent if \( V(B, F - G, E) = 0 \).

**Definition 3.6** We say that a function \( f : E \to R \) defined on a \( B \)-interval \( E \) is variationally integrable (\( V_HB \)-integrable) if there exists an additive \( B \)-interval function \( \tau \) which is variationally equivalent to the interval-point function \( \mu(I)f(x) \). In this case we write \( (V_HB) \int_E f = \tau(E) \).
As an immediate consequence of the Saks-Henstock lemma we get

**Theorem 3.7** The variational integral \( V_{H_B} \) is equivalent to the \( H_B \)-integral.

**Definition 3.8** We say that two interval-point functions \( F \) and \( G \) are *almost variationally equivalent* if \( V_{\text{ess}}(B, F - G, E) = 0 \).

It follows from (4) that variational equivalence implies almost variational equivalence. Now we introduce the concept of \((SL)\)-integral with respect to a basis \( B \) (in the case of the full interval basis this integral was considered in [6] and [7] for real-valued functions and in [1] for Riesz-space-valued functions).

**Definition 3.9** We say that the variational measure \( V(B, F, \cdot) \) is *absolutely continuous with respect to a measure \( \mu \) or \( \mu \)-absolutely continuous on \( E \), if \( V(B, F, N) = 0 \) whenever \( \mu(N) = 0 \) for \( N \subset E \).

**Definition 3.10** We say that a \( B \)-interval \( R \)-valued function \( \tau \) is of class \((SL)\) or has property \((SL)\) on a \( B \)-interval \( E \) if the variational measure generated by this function is \( \mu \)-absolutely continuous on \( E \).

**Definition 3.11** Let \( R \) be any Dedekind complete Riesz space, We say that \( f : E \to R \) is \((SL)\)-integrable on a \( B \)-interval \( E \) if there exists a \( B \)-interval \( R \)-valued additive function \( \tau \) of class \((SL)\) which is almost variationally equivalent to the interval-point function \( \mu(I) f(x) \). The function \( \tau \) is called the *indefinite SL-integral* of \( f \). In this case we put (by definition ) \((SL) \int_E f = \tau(E)\).

**Proposition 3.12** If a function \( f \) is \( H_B \)-integrable and its indefinite \( H_B \)-integral \( F \) is of class \((SL)\), then \( f \) is \( SL \)-integrable with \( F \) being its indefinite \( SL \)-integral.

**Proof:** It follows from the fact that variational equivalence implies almost variational equivalence. \( \square \)

**Proposition 3.13** Let \( R \) be any Dedekind complete Riesz space, \( N \subset X \) be a set with \( \mu(N) = 0 \), and \( f_0 : X \to R \) be such that \( f_0(x) = 0 \) for all \( x \notin N \). Then \( f_0 \) is \( SL \)-integrable and the identically zero function is its indefinite \( SL \)-integral.
Proof: Straightforward. □

As a corollary we get:

**Proposition 3.14** Let $R$ be as in Proposition 3.13, and $f, g : X \to R$ be two functions, which differ only on a set of measure $\mu$ zero. Then $f$ is $(SL)$-integrable if and only if $g$ does, and in this case

\[
(SL) \int_X f = (SL) \int_X g.
\]

The $SL$-integral in general does not coincide with the $HB$-integral. An example of a function which is $SL$-integrable but is not $HB$-integrable can be obtained from Example 4.21 given in [1]. In this example the space $R$ does not satisfy property $\sigma$. So this property is a necessary condition for the equivalence of $SL$- and $HB$-integrals. An example in the opposite direction also can be constructed.

**Example 3.15** Consider the dyadic intervals $\Delta_j^k = \left[ \frac{j}{2^k}, \frac{j+1}{2^k} \right)$, $0 \leq j \leq 2^k - 1$, $k \in \mathbb{N} \cup \{0\}$. Note that $(0,1) = \bigcup_{k=1}^{\infty} \Delta_1^k$ and $\Delta_1^k = \Delta_2^{k+1} \cup \Delta_3^{k+1}$. We define a basis $\mathcal{B}$ on $[0,1)$ as follows. Let $\mathcal{I}$ consist of right-open intervals $I = [u, v)$ such that: if $u = 0$, then $I = \Delta_0^k$, $k \in \mathbb{N} \cup \{0\}$; if $u \in \Delta_j^{k+1}$, $j = 2$, or 3, then $I \subset \Delta_j^{k+1}$ with $j = 2$ or 3, respectively. We consider a family of gages $\delta$, each of which is constant on every interval $\Delta_j^{k+1}$, $j = 2, 3$, $k \in \mathbb{N}$. Now every basis set of $\mathcal{B}$ is defined by a gage from the above family as

\[
\beta_\delta = \{(I, x) : I = [u, v) \in \mathcal{I}, x = u, \mu(I) \leq \delta(x)\},
\]

where $\mu$ denotes the Lebesgue measure. It is easy to check that this basis has the partitioning property and so the $HB$-integral is defined for it. Now let $R$ be a Riesz space which does not satisfy property $\sigma$ and let a sequence $(p_k)_k$ in $R$ with $p_k \geq 0 \ \forall k \in \mathbb{N}$ be such that for any sequence $(\lambda_k)_k$ of positive real numbers the sequence $(\lambda_k p_k)_k$ is not bounded in $R$. We define a function $f : [0,1) \to R$ as follows: $f(x) = 0$ if $x = 0$; $f(x) = (-1)^j p_k$ if $x \in \Delta_j^{k+1}$, $j = 2, 3$, $k \in \mathbb{N}$. It is easy to check that with the considered basis the function $f$ is $HB$-integrable. Moreover,

\[
(HB) \int_{\Delta_k^0} f = 0 \text{ for any } k \in \mathbb{N} \cup \{0\}
\]
and

\[ (H_B) \int_I f = \mu(I) f(x) \] for any \( I = [u, v) \in \mathcal{I} \) with \( x = u \neq 0 \).

In particular, denoting by \( F \) the indefinite integral of \( f \), we have

\[ F(I) = (H_B) \int_I f = \mu(I) p_k \]

if \( I \subset \Delta_2^{k+1} \). Now take a countable set \( E = \{x_k : k \in \mathbb{N}\} \) with \( x_k \in \Delta_2^{k+1} \). Note that for any \( \beta_\delta \)-partition \( \pi = \{(I_k, x_k)\}_k \) on \( E \) we have \( I_k \subset \Delta_2^{k+1} \) and so

\[ \sum_{(I,x) \in \pi} |F(I, x)| = \sum_{(I,x) \in \pi} \mu(I) p_k. \]

Because of the choice of the sequence \((p_k)_k\) the set of the values of these sums is not bounded. Hence \( Var(\beta_\delta, F, E) = V(\mathcal{B}, F, E) = +\infty \) and so \( F \) is not of class \((SL)\).

We give now some kind of indirect characterization of the space \( R \) for which \( H_B \)-integrability implies \( SL \)-integrability.

**Proposition 3.16** The indefinite \( H_B \)-integral of a \( R \)-valued function \( f \) which is \( H_B \)-integrable on a \( \mathcal{B} \)-interval \( E \) is of class \( SL \) if and only if for any set \( N \subset E \) with \( \mu(N) = 0 \) the function \( f\chi_N \) is \( H_B \)-integrable with integral equal to zero.

We omit the easy proof, which is based on using lemma 3.4.

**Theorem 3.17** An \( R \)-valued function \( f \), \( H_B \)-integrable on \( E \), is \( SL \)-integrable with the same integral value if and only if for any set \( N \subset E \) with measure zero the function \( f\chi_N \) is \( H_B \)-integrable with integral equal to zero.

**Proof:** It is enough to apply Propositions 3.16 and 3.12. \( \square \)

As a corollary of the above theorem we get the following

**Theorem 3.18** If a space \( R \) is such that any \( R \)-valued function, which is equal to zero almost everywhere, is \( H_B \)-integrable with integral equal to zero, then any \( H_B \)-integrable \( R \)-valued function is \( SL \)-integrable with the same integral value.
As a very strong assumption imposed on the space $R$ which gives a sufficient condition for any $R$-valued function, which is equal to zero almost everywhere, to be $H_B$-integrable with integral equal to zero, is the assumption that $R$ has both property $\sigma$ and the Swartz property.

We consider now conditions under which $SL$-integrability implies $H_B$-integrability.

**Theorem 3.19** If a $R$-valued function $f$ is $SL$-integrable on a $B$-interval $E$ with $SL$-integral $F$ and if for any set $N \subset E$ with $\mu(N) = 0$ the function $f\chi_N$ is $H_B$-integrable with integral equal to zero, then $f$ is $H_B$-integrable on $E$ and $F$ is its indefinite $H_B$-integral.

**Proof:** Let $(r_\theta)_{\theta \in B_0}$ be an $(o)$-net given by $SL$-integrability of $f$, such that, for any partition $\pi \subset \theta$,

$$\sum_{(I,x) \in \pi} |\mu(I) f(x) - F(I)| \leq r_\theta.$$

Each fixed $\theta \in B_0$ defines a set $N$ with $\mu(N) = 0$, such that $\theta[N]$ is empty. By assumption the function $f\chi_N$ is $H_B$-integrable to zero and this together with Lemma 3.4 implies the existence of an $(o)$-net $(p_{\theta,\beta})_{\beta \in B}$ such that, for any partition $\pi \subset \beta[N]$,

$$\sum_{(I,x) \in \pi} |\mu(I) f(x) \chi_N| \leq p_{\theta,\beta}.$$

Another $(o)$-net $(s_{\theta,\beta})_{\beta \in B}$ is defined by the fact that $F$ is of class $(SL)$. With this net, for any partition $\pi \subset \beta[N]$, we get

$$\sum_{(I,x) \in \pi} |F(I)| \leq s_{\theta,\beta}.$$

Now take a basis set $\gamma \in B$ defined by setting

$$\gamma([x]) = \begin{cases} 
\beta([x]) & \text{if } x \in N, \\
\theta([x]) & \text{if } x \not\in N.
\end{cases}$$

(This basis set of $B$ exists by assumption imposed on the basis).
Now we can sum up the above estimations getting, for any partition \( \pi \subset \gamma \),
\[
\sum_{(I,x) \in \pi} |\mu(I) f(x) - F(I)| \leq \sum_{(I,x) \in \pi, x \in N^c} |\mu(I) f(x) - F(I)| + \\
+ \sum_{(I,x) \in \pi, x \in N} |\mu(I) f(x)| + \sum_{(I,x) \in \pi, x \in N} |F(I)| \leq r_\theta + p_{\theta, \beta} + s_{\theta, \beta}.
\]
It is clear that
\[
\inf \{(r_\theta + p_{\theta, \beta} + s_{\theta, \beta}) : \theta \in B_0, \beta \in B\} = 0,
\]
and consequently \( F \) is \( H_B \)-integrable with \( F \) being its \( H_B \)-integral. \( \square \)

References


