Integral and ideals in Riesz spaces

A. Boccuto – D. Candeloro *

Abstract

A convergence in Riesz spaces is given axiomatically. A Bochner-type integral for Riesz space-valued functions is introduced and some Vitali and Lebesgue dominated convergence theorems are proved. Some properties and examples are investigated.


KEY WORDS: Riesz spaces, ideals, Bochner integral, Vitali convergence theorems, Lebesgue dominated convergence theorem.

1 Introduction.

The theory of limits and convergence with respect to a fixed admissible ideal of the set of all positive integers is quite recent: for the real case, it has been focused and investigated in several papers. In particular, in [20] the main properties of this convergence and the affinities and differences with the usual one are explained. The related Cauchy property is studied in [14], while in [26, 27] some Cauchy conditions are presented, related with sequences of functions. Further results are proved in [3, 4, 10, 13, 15, 19, 21]. An example of this convergence is the statistical convergence, that is the convergence with respect to the ideal consisting of all subsets of $\mathbb{N}$ with zero asymptotic density (see [20] and its bibliography), which was introduced in [17]. Clearly, this notion of convergence includes the classical case: it suffices to take the ideal $\mathcal{I}_\text{fin}$ consisting of all finite subsets of $\mathbb{N}$. Note that convergence with respect to ideals is, in the context of Riesz spaces, a particular case of an abstract model, presented axiomatically in [8, 9], where some topological properties

*Authors’ Address: Dipartimento di Matematica e Informatica, via Vanvitelli, 1 I-06123 PERUGIA (Italy).
E-mail: boccuto@yahoo.it, candelor@dipmat.unipg.it
of the involved convergence and limits were investigated. Recent applications of statistical convergence to fuzzy numbers can be found in [1, 2, 23].

In this paper we continue the study of this type of abstract convergence and in particular we introduce an integral, proving absolute continuity and some extensions of the Lebesgue dominated convergence theorem and of the Scheffé theorem. Moreover, for the case of real- valued functions and $\sigma$ -additive Riesz space-valued measures, further results are demonstrated, among which the monotone convergence theorem and completeness of $L^1$. For the usual convergences existing in the context of Riesz spaces (that is w. r. to the ideal $I_{\text{fin}}$), the corresponding results were proved in [5, 6, 7].

2 Preliminaries.

A Riesz space $R$ is said to be Dedekind complete if every nonempty subset $H \subset R$, bounded from above, has supremum in $R$. From now on we suppose that $R$ is a Dedekind complete Riesz space, $R \neq \{0\}$, and denote by $R^+$ the set of all positive nonzero elements of $R$. A unit (of $R$) is any element $u \in R^+$.

We now give an axiomatic approach to convergence in Riesz spaces (see also [8, 9]). Let $T$ be the set of all sequences $(x_n)_n$ in $R$.

**Definition 2.1** A convergence is a pair $(S, L)$, where $S$ is a linear subspace of $T$ and $L$ is a map $L : S \rightarrow R$, satisfying the following axioms:

a) $L((\zeta_1 x_n + \zeta_2 y_n)_n) = \zeta_1 L((x_n)_n) + \zeta_2 L((y_n)_n)$ for every pair of sequences $(x_n)_n$, $(y_n)_n \in S$ and for each $\zeta_1, \zeta_2 \in \mathbb{R}$; if $(x_n)_n$, $(y_n)_n \in S$ and $x_n \leq y_n$ definitely, then $L((x_n)_n) \leq L((y_n)_n)$.

b) If $(x_n)_n$ satisfies $x_n = l$ definitely, then $(x_n)_n \in S$ and $L((x_n)_n) = l$; if $(x_n)_n$, $(y_n)_n$ are such that the set $\{n \in \mathbb{N} : x_n \neq y_n\}$ is at most finite and $(x_n)_n \in S$, then $(y_n)_n \in S$ too and $L((y_n)_n) = L((x_n)_n)$.

c) If $(x_n)_n \in S$, then $(|x_n|)_n \in S$ and $L((|x_n|)_n) = |L((x_n)_n)|$.

d) Given three sequences, $(x_n)_n$, $(y_n)_n$, $(z_n)_n$, satisfying $(x_n)_n$, $(z_n)_n \in S$, $L((x_n)_n) = L((z_n)_n)$, and $x_n \leq y_n \leq z_n$ definitely, then $(y_n)_n \in S$.

e) If $u$ is a unit, then the sequence $\left(\frac{1}{n} u\right)_n$ belongs to $S$ and $L\left(\left(\frac{1}{n} u\right)_n\right) = 0$. 
We now introduce some structural assumptions, since they will be useful in the construction of our integral. Let $R_1$, $R_2$, $R$ be three Dedekind complete Riesz spaces. From now on, we endow them with three respective convergence structures, obeying axioms a), ..., e) and "compatible" with each other, and that we will always denote by the same symbol $(\mathcal{S}, \mathcal{L})$, for the sake of simplicity.

In what follows, fixed convergence structures $(\mathcal{S}, \mathcal{L})$ in Riesz spaces $R_1$, $R_2$ and $R$ are not mentioned explicitly.

**Assumptions 2.2** $(R_1, R_2, R)$ is a product triple if a map $\cdot : R_1 \times R_2 \to R$ (product) is given, such that

i) $(r_1 + s_1) \cdot r_2 = r_1 \cdot r_2 + s_1 \cdot r_2$,

ii) $r_1 \cdot (r_2 + s_2) = r_1 \cdot r_2 + r_1 \cdot s_2$,

iii) $[r_1 \geq s_1, r_2 \geq 0] \Rightarrow [r_1 \cdot r_2 \geq s_1 \cdot r_2]$,

iv) $[r_1 \geq 0, r_2 \geq s_2] \Rightarrow [r_1 \cdot r_2 \geq r_1 \cdot s_2]$ for all $r_j, s_j \in R_j, j = 1, 2$;

v) if $(a_n)_n$ is any sequence in $R_2$ and $b \in R_1$, then $[\mathcal{L}((a_n)_n) = 0] \Rightarrow [\mathcal{L}((b \cdot a_n)_n) = 0]$;

vi) if $(a_n)_n$ is any sequence in $R_1$ and $b \in R_2$, then $[\mathcal{L}((a_n)_n) = 0] \Rightarrow [\mathcal{L}((a_n \cdot b)_n) = 0]$.

**3 The integral.**

**Assumptions 3.1** We will construct an integral for Riesz space-valued functions defined on an arbitrary nonempty set $G$, with respect to a positive finitely additive measure $\mu : A \to R_2$, where $A \subset \mathcal{P}(G)$ is a fixed $\sigma$-algebra.

We define our integral for the "basic" space $L$ of all simple measurable functions, in the usual way, and then extend it to a larger class. More precisely, we shall assume the existence of a linear monotone mapping from $L$ to $R$ (which will be denoted by $\int_G (\cdot) d\mu$) such that $\int_G f(g) d\mu(g) = c \mu(A)$ whenever $f(g) = c$ for $g \in A$, with $c \in R_1$ and $A \in \mathcal{A}$, and $f(g) = 0$ if $g \notin A$. Moreover, we put

$$\int_A f \ d\mu = \int_G f \ 1_A \ d\mu$$

for all $A \in \mathcal{A}$, where $1_A$ is the function which associates the real number 1 to every $x \in A$ and 0 to all $x \in G \setminus A$.

**Definitions 3.2** A sequence of functions $(f_n)_n$ in $R_1^G$ converges uniformly to $f \in R_1^G$ if $\mathcal{L}((\sup_{g \in G} |f_n(g) - f(g)|)_n) = 0$. 


A sequence \((f_n)_n\) in \(R_1^G\) converges in measure to \(f \in R_1^G\) if there is a sequence \((A_n)_n\) in \(A\), with \(\mathcal{L}((\mu(A_n))_n) = 0\) and \(\mathcal{L}((\sup_{g \in A_n}|f_n(g) - f(g)|)_n) = 0\), where the symbol \(\complement\) denotes the complement with respect to \(G\).

Obviously, every uniformly convergent sequence is convergent in measure.

**Remark 3.3** Observe that, when \(R_1 = R_2 = \mathbb{R}\) and the involved convergence is the usual one, the given definition of convergence in measure coincides with the classical one: indeed, in this case the sequence \((f_n)_n\) of measurable functions converges in measure to \(f\) if and only if there exist two sequences \((p_n)_n\), \((q_n)_n\) of positive real numbers such that \(p_n \downarrow 0\), \(q_n \downarrow 0\) and

\[\mu(\{g \in G : |f_n(g) - f(g)| > q_n\}) \leq p_n \quad \text{for all } n \in \mathbb{N}\]

(see also [12], Theorem 4.4).

The following results hold.

**Proposition 3.4** Let \((f_n)_n, (h_n)_n \in R_1^G\) converge in measure to \(f, h\) respectively, and \(\alpha \in \mathbb{R}\). Then \((f_n + h_n)_n, (f_n \vee h_n)_n, (f_n \wedge h_n)_n, (\alpha f_n)_n, (\|f_n\|)_n\) converge in measure to \(f + h, f \vee h, f \wedge h\) respectively, and \((\alpha f_n)_n, (\|f_n\|)_n\) converge in measure to \(\alpha f, \|f\|\) respectively.

**Proof:** We report it only for the case \((f_n \vee h_n)_n\). Observe that, by virtue of the Birkhoff inequality (see [22], p. 64), we get

\[|f_n(g) \vee h_n(g) - f(g) \vee h(g)| \leq |f_n(g) \vee h_n(g) - f_n(g) \vee h(g)| + |f_n(g) \vee h(g) - f(g) \vee h(g)|\]
\[\leq |h_n(g) - h(g)| + |f_n(g) - f(g)|, \quad n \in \mathbb{N}, g \in G.\]

The assertion follows from convergence in measure of \((f_n)_n\) and \((h_n)_n\) to \(f\) and \(h\) respectively and axioms a), c), d) of Definition 2.1.

The proofs of the other cases are similar. \(\square\)

In the following two definitions, the involved integral is the linear functional introduced in 3.1.

**Definition 3.5** A sequence \((f_n)_n\) in \(L\) converges in \(L^1\) to \(f \in L\) if

\[\mathcal{L}\left(\left(\int_G |f_n(g) - f(g)| \, d\mu(g)\right)_n\right) = 0.\]
Definition 3.6 A sequence \((f_n)_n\) in \(L\) is said to be **uniformly integrable** if

\[ \mathcal{L}\left(\left(\int_{A_n} |f_n(g)| \, d\mu(g)\right)_n\right) = 0 \]

whenever \(\mathcal{L}((\mu(A_n))_n) = 0\).

If \(R_1 = R_2 = \mathbb{R}\) with the usual convergence, thanks to [12], Theorem 4.16, this notion coincides with the classical one (see also [18]).

We now give a first version of a Vitali-type theorem.

**Theorem 3.7** Let \(f_n, f \in L, n \in \mathbb{N}\). If \((f_n)_n\) converges in measure to \(f\) and is uniformly integrable, then \((f_n)_n\) converges in \(L^1\) to \(f\).

**Proof:** By construction, \(f\) is bounded, and hence there is a unit \(v \in R_1\), such that \(|f(g)| \leq v\) for all \(g \in G\). Then,

\[ \int_A |f(g)| \, d\mu(g) \leq \int_A v \, d\mu(g) = v \cdot \mu(A) \]

whenever \(A \in \mathcal{A}\). Let \((A_n)_n\) be related with convergence in measure. For all \(n \in \mathbb{N}\) we get:

\[ \int_G |f_n(g) - f(g)| \, d\mu(g) = \int_{A_n^c} |f_n(g) - f(g)| \, d\mu(g) + \int_{A_n} |f_n(g) - f(g)| \, d\mu(g) \]

\[ \leq \left(\sup_{g \in A_n^c} |f_n(g) - f(g)|\right) \cdot \mu(G) + \int_{A_n} |f_n(g) - f(g)| \, d\mu(g) \]

\[ \leq \left(\sup_{g \in A_n^c} |f_n(g) - f(g)|\right) \cdot \mu(G) + \int_{A_n} |f_n(g)| \, d\mu(g) \]

\[ + \int_{A_n} |f(g)| \, d\mu(g). \]

Taking the limits with respect to the involved convergence, from the assumptions of convergence in measure and uniform integrability we get the assertion, also thanks to axioms a), c) and d) and property vi) of the product. \(\square\)

We now turn to the concept of integrability. We begin with the following:

Definition 3.8 A map \(f \in R^G_1\) is called **measurable** if there is a sequence \((f_n)_n\) in \(L\), convergent in measure to \(f\).

Definition 3.9 Given a map \(f \in R^G_1\), we say that a sequence \((f_n)_n\) in \(L\) is **defining** for \(f\) if it is uniformly integrable and converges in measure to \(f\).
Definition 3.10 A positive map $f \in R^G_1$ is integrable if it admits a defining sequence $(f_n)_n$ and there is a map $l : \mathcal{A} \to \mathbb{R}$, such that

$$\mathcal{L}\left(\sup_{A \in \mathcal{A}} \left| \int_A f_n(g) \, d\mu(g) - l(A) \right| \right)_n = 0.$$  \hspace{1cm} (1)

Proposition 3.11 For any $A \in \mathcal{A}$ we get:

$$l(A) = \mathcal{L}\left(\left(\int_A f_n(g) \, d\mu(g)\right)_n\right);$$  \hspace{1cm} (2)

moreover $l(A)$ is independent on the choice of the defining sequence.

Proof: The first part follows easily from (1) and axiom d).

We now check that the entity in (2) does not depend on the chosen defining sequence.

Let $(f_n^j)_n$, $j = 1, 2$, be two defining sequences for $f$, set

$$l_j(A) := \mathcal{L}\left(\left(\int_A f_n^j(g) \, d\mu(g)\right)_n\right), \quad A \in \mathcal{A},$$

and put $p_n(g) = |f_n^1(g) - f_n^2(g)|$, $g \in G$, $n \in \mathbb{N}$. Since $(f_n^j)_n$, $j = 1, 2$, are uniformly integrable, then $(p_n)_n$ is uniformly integrable too. Moreover, $(f_n^1)_n$ and $(f_n^2)_n$ converge in measure to $f$, and so $(p_n)_n$ converges in measure to 0. Thus, by Theorem 3.7, $(p_n)_n$ converges to zero in $L^1$. Hence,

$$|l_1(A) - l_2(A)| \leq \left|\int_A f_n^1(g) \, d\mu(g) - l_1(A)\right| + \left|l_2(A) - \int_A f_n^2(g) \, d\mu(g)\right| + \left|\int_A f_n^1(g) \, d\mu(g) - \int_A f_n^2(g) \, d\mu(g)\right|$$

$$\leq \left|\int_A f_n^1(g) \, d\mu(g) - l_1(A)\right| + \left|l_2(A) - \int_A f_n^2(g) \, d\mu(g)\right| + \int_G p_n(g) \, d\mu(g).$$

So, by virtue of the axioms of the convergence, we get:

$$|l_1(A) - l_2(A)| = 0,$$

i.e. $l_1(A) = l_2(A)$ for all $A \in \mathcal{A}$. \hfill \Box

From now on, we write $\int_A f(g) \, d\mu(g) := l(A)$, $A \in \mathcal{A}$.

Definition 3.12 We say that a function $f \in R^G_1$ is integrable if $f^+$ and $f^-$ are integrable, and in this case we set

$$\int_A f(g) \, d\mu(g) = \int_A f^+(g) \, d\mu(g) - \int_A f^-(g) \, d\mu(g), \quad A \in \mathcal{A}.$$
**Remark 3.13** It is easy to check that, if \( f \) is integrable, then \(|f|\) is too, and
\[
\int_A |f(g)| \, d\mu(g) = \int_A f^+(g) \, d\mu(g) + \int_A f^-(g) \, d\mu(g), \quad A \in \mathcal{A}:
\]
this is a consequence of the fact that, if \((h_n)_n\) and \((k_n)_n\) are two defining sequences for \(f^+\) and \(f^-\) respectively, then \((h_n + k_n)_n\) is a defining sequence for \(|f|\). Furthermore, note that our integral turns out to be a linear functional.

We now prove absolute continuity of the integral.

**Theorem 3.14** Let \( f \in R_1^G \) be an integrable map. Then
\[
\mathcal{L}\left(\left(\int_{A_n} |f(g)| \, d\mu(g)\right)_n\right) = 0
\]
whenever \( \mathcal{L}(\mu(A_n))_n = 0 \).

**Proof:** Without loss of generality, we can assume that \( f \geq 0 \).

Let \((A_n)_n\) be in \( \mathcal{A} \), \( \mathcal{L}(\mu(A_n))_n = 0 \), and fix \( n \in \mathbb{N} \). We get:
\[
\int_{A_n} f(g) \, d\mu(g) \leq \left| \int_{A_n} f(g) \, d\mu(g) - \int_{A_n} f_n(g) \, d\mu(g) \right| + \int_{A_n} f_n(g) \, d\mu(g) \\
\leq \sup_{A \in \mathcal{A}} \left| \int_A f(g) \, d\mu(g) - \int_A f_n(g) \, d\mu(g) \right| + \int_{A_n} f_n(g) \, d\mu(g).
\]

From this and uniform integrability it follows that
\[
\mathcal{L}\left(\left(\int_{A_n} f(g) \, d\mu(g)\right)_n\right) = 0.
\]
This completes the proof. \( \square \)

Exactly like in the previous definitions, we can re-formulate the notions of convergence in \( L^1 \) and uniform integrability, by simply requiring that the maps \( f_n, n \in \mathbb{N} \), and \( f \) are integrable. By using Theorem 3.14 and proceeding as in the proof of Theorem 3.7, one can prove the following Vitali-type theorem:

**Theorem 3.15** Let \((f_n)_n\) be a sequence of integrable maps in \( R_1^G \), convergent in measure to an integrable function \( f \) and uniformly integrable. Then \((f_n)_n\) converges in \( L^1 \) to \( f \).

Taking the limits, from Theorem 3.14 and thanks to the assumptions, we obtain easily the assertion. \( \square \)
Remark 3.16 Conversely, it is easy to check that convergence in $L^1$ implies uniform integrability.

Before introducing the Lebesgue dominated convergence theorem, we state some further properties of our integral.

Lemma 3.17 Let $f \in R^G_1$ be integrable, and let $h \in R^G_1$ coincide with $f$ in the complement of a set $N$ with $\mu(N) = 0$. Then $h$ is integrable and

$$\int_A h(g) \, d\mu(g) = \int_A f(g) \, d\mu(g) \quad \text{for all } A \in \mathcal{A}.$$

We now turn to the Lebesgue dominated convergence theorem.

Theorem 3.18 Let $(f_n)_n$ be a sequence of integrable functions, $f_n \in R^G_1$, and suppose that there exists an integrable map $h \in R^G_1$ such that $|f_n(g)| \leq |h(g)|$ for all $n \in \mathbb{N}$ and $\mu$-almost everywhere. Assume that $(f_n)_n$ converges in measure to an integrable function $f$. Then $(f_n)_n$ converges in $L^1$ to $f$.

Proof: Thanks to the domination hypothesis, the functions $f_n$, $n \in \mathbb{N}$, are uniformly integrable. So Theorem 3.18 is a consequence of Theorem 3.15.

We now turn to a Scheffé-type theorem.

Theorem 3.19 Let $(f_n)_n$, $0 \leq f_n$, $n \in \mathbb{N}$, be a sequence of integrable functions, converging in measure to an integrable map $f \geq 0$. Assume that

$$\mathcal{L} \left( \left( \int_G f_n(g) \, d\mu(g) \right)_n \right) = \int_G f(g) \, d\mu(g).$$

Then, $(f_n)_n$ converges in $L^1$ to $f$.

Proof: Let $h_n(g) = f_n(g) - f(g)$, $n \in \mathbb{N}$, $g \in G$. As $f_n(g) \geq 0$, $f(g) \geq 0$ for all $g$ and $n$, then $|h_n(g)| \leq f_n(g) + f(g)$ for each $g \in G$ and $n \in \mathbb{N}$. Thus,

$$0 \leq (h_n(g))^+ = \frac{|h_n(g)| - h_n(g)}{2} \leq \frac{f_n(g) + f(g)}{2} - f(g) = f(g), \quad \forall g, \forall n.$$

Let $H_n(g) = (h_n(g))^-$, $g \in G$, $n \in \mathbb{N}$. Then $f$ and $H_n$, $n \in \mathbb{N}$, are integrable and $(H_n)_n$ converges in measure to zero. By Theorem 3.18, we have:

$$0 = \mathcal{L} \left( \left( \int_G (h_n(g))^- \, d\mu(g) \right)_n \right)$$

and hence

$$\mathcal{L} \left( \left( \int_G (h_n(g))^+ \, d\mu(g) \right)_n \right) = \mathcal{L} \left( \left( \int_G h_n(g) \, d\mu(g) \right)_n \right) = 0.$$
Finally, we get:

\[
\mathcal{L}\left(\left(\int_G |h_n(g)| d\mu(g)\right)_n\right) = \mathcal{L}\left(\left(\int_G (h_n(g))^+ d\mu(g)\right)_n\right) + \mathcal{L}\left(\left(\int_G (h_n(g))^- d\mu(g)\right)_n\right) = 0.
\]

Thus the assertion follows. □

4 Some particular cases.

We now give some examples, and begin with the definition of ideal (see also [20], Definition A, pp. 670-671).

**Definition 4.1** Let \( X \) be any nonempty set. A family of sets \( \mathcal{I} \subset \mathcal{P}(X) \) is called an *ideal* of \( X \) if \( A \cup B \in \mathcal{I} \) whenever \( A, B \in \mathcal{I} \) and for each \( A \in \mathcal{I} \) and \( B \subset A \) we get \( B \in \mathcal{I} \). An ideal is said to be *non-trivial* if \( \mathcal{I} \neq \emptyset \) and \( X \notin \mathcal{I} \). A non-trivial ideal \( \mathcal{I} \) is said to be *admissible* if it contains all singletons.

We obtain the classical convergences of sequences if we take \( X = \mathbb{N} \) and \( \mathcal{I} := \mathcal{I}_{\text{fin}} \), the class of all finite subsets of \( \mathbb{N} \). Another example of ideals, which gives rise to the *statistical convergence* (see [20]), is the class of all subsets of \( \mathbb{N} \) with zero asymptotic density.

Fix now an admissible ideal \( \mathcal{I} \) of \( \mathbb{N} \).

**Definitions 4.2** Let \( R \) be a Dedekind complete Riesz space, and fix a unit \( u \). The sequence \((x_n)_n\) in \( R \) is said to be *\( u \)-convergent* to \( x \in R \) (\( u \)-Cauchy) if \( \forall \varepsilon > 0, \exists D(\varepsilon) \in \mathcal{I}: |x_n - x| \leq \varepsilon u \) for all \( n \notin D \) (resp. \( |x_n - x_m| \leq \varepsilon u \) for each \( n, m \notin D \)). We say that \((x_n)_n\) *\( (r) \)-converges* to \( x \) (is \( (r) \)-Cauchy) if it \( u \)-converges (is \( u \)-Cauchy) for some unit \( u \).

If \((x_n)_n\) is \( (r) \)-convergent to \( x \), we write \( (r) \lim_n x_n = x \).

It is easy to prove that the introduced convergence satisfies the axioms a), \ldots, e) of Definition 2.1.

A Riesz space \( R \) is said to be \( (r) \)-*complete* if every \( (r) \)-Cauchy sequence is \( (r) \)-convergent.

We now prove the following

**Proposition 4.3** Every Dedekind complete Riesz space is \( (r) \)-complete.
Proof: Let \((x_n)_n\) be any \((r)\)-Cauchy sequence. By hypothesis there is a unit \(u\) with the following property: \(\forall \varepsilon > 0 \ \exists D = D(\varepsilon) \in I\) with \(|x_n - x_m| \leq \varepsilon u\) whenever \(n, m \notin D\). Put

\[ K := \{D \in I : \exists \varepsilon > 0 : D = D(\varepsilon) \}. \]

Fix \(D \in K\), and let \(m_0 := \min(\mathbb{N} \setminus D)\). Then, for every \(n \notin D\),

\[ |x_n| \leq |x_n - x_{m_0}| + |x_{m_0}| \leq \varepsilon u + |x_{m_0}|. \]

For each \(D \in I\), set \(D_D := \{x_n : n \notin D\}\). Then \(D_D\) is bounded for any \(D \in K\), and thus the entities

\[ a_D := \land D_D, \quad b_D := \lor D_D \]

belong to \(R\), thanks to Dedekind completeness of \(R\).

Fix now arbitrarily \(D_1, D_2 \in K\), and pick \(D_0 := D_1 \cup D_2\). Then \(D_{D_0} \subset D_{D_1} \cap D_{D_2}\), and hence \(D_{D_0}\) is bounded. So, \(a_{D_1} \leq a_{D_0} \leq b_{D_0} \leq b_{D_2}\), and thus, by Dedekind completeness again, an element \(x \in R\) can be found, with \(\sup_{D_1 \in K} a_{D_1} \leq x \leq \inf_{D_2 \in K} b_{D_2}\).

To every \(\varepsilon > 0\) there corresponds \(D = D(\varepsilon) \in K\) such that

\[ x_n \leq x_m + \varepsilon u \quad \forall m, n \notin D(\varepsilon). \] \hspace{1cm} (3)

By fixing \(m\) and taking the supremum with respect to \(n\), we get

\[ b_{D(\varepsilon)} \leq x_m + \varepsilon u; \]

taking the infimum as \(m\) varies in \(\mathbb{N} \setminus D(\varepsilon)\), we obtain

\[ b_{D(\varepsilon)} - a_{D(\varepsilon)} \leq \varepsilon u. \]

Thus,

\[ 0 \leq \inf_{D_2 \in K} b_{D_2} - \sup_{D_1 \in K} a_{D_1} = \inf_{\varepsilon > 0} b_{D(\varepsilon)} - \sup_{\varepsilon > 0} a_{D(\varepsilon)} \leq 0, \]

and hence

\[ \inf_{\varepsilon > 0} b_{D(\varepsilon)} = \sup_{\varepsilon > 0} a_{D(\varepsilon)} = x. \]

So, if \(D = D(\varepsilon)\) is chosen as in (3), then for all \(n \notin D(\varepsilon)\) we get

\[ x - x_n \leq b_{D(\varepsilon)} - x_n \leq \varepsilon u; \]

similarly,

\[ x_n - x \leq x_n - a_{D(\varepsilon)} \leq \varepsilon u. \]
Thus the assertion follows. □

In a similar way one can prove that, in any Dedekind complete Riesz space $R$, we get completeness of $(r)$-convergence also for indexed families of sequences, in the uniform sense.

We now define $(D)$-convergence in Riesz spaces with respect to a fixed admissible ideal $I$. We begin with the following

**Definitions 4.4** A bounded double sequence $(a_{i,j})_{i,j}$ in $R$ is called a regulator or a $(D)$-sequence if, for each $i \in \mathbb{N}$, $a_{i,j} \downarrow 0$, that is $a_{i,j} \geq a_{i,j+1}$ $\forall j \in \mathbb{N}$ and $\bigwedge_{j \in \mathbb{N}} a_{i,j} = 0$.

Given a sequence $(x_n)_n$ in $R$, we say that $(x_n)_n$ $(D)$-converges to an element $x \in R$ (is (D)-Cauchy) if there is a regulator $(a_{i,j})_{i,j}$ with the following property:

for all maps $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exists $D = D(\varphi) \in I$ with

$$|x_n - x| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

for any $n \notin D$

$$(|x_n - x_m| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

whenever $n, m \notin D$). If $(x_n)_n$ $(D)$-converges to $x$, we write $(D)\lim_n x_n = x$.

We say that $R$ is weakly $\sigma$-distributive if, for every $(D)$-sequence $(a_{i,j})_{i,j}$,

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left( \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) = 0. \tag{4}$$

From now on, when we deal with $(D)$-convergence, we always assume that $R$ is a Dedekind complete weakly $\sigma$-distributive Riesz space.

It is easy to prove the axioms a), ..., e) of Definition 2.1.

In any Dedekind complete Riesz space, $(r)$-convergence (w. r. to any fixed admissible ideal $I$ of $\mathbb{N}$) always implies $(D)$-convergence, w.r. to the same ideal (see [16, 22]).

The converse implication is not always true, even if $I = I_{\text{fin}}$. For example it will be enough to take $R = l_\infty$, the space of all bounded sequences of real numbers: in this space, $(r)$-convergence means uniform convergence, while $(D)$-convergence is pointwise convergence (see also [22, 25]).
However we now check that, if \( R = \mathbb{R} \), then \((D)\)-convergence implies \((r)\)-convergence w. r. to any fixed admissible ideal \( \mathcal{I} \) of \( \mathbb{N} \).

Fix arbitrarily \( \varepsilon > 0 \), and pick a \((D)\)-sequence \((a_{i,j})_{i,j}\) associated with \((D)\)-convergence. Then to every \( i \in \mathbb{N} \) there correspond countably many integers \( \mathcal{J} \) with \( |a_{i,j}| \leq \varepsilon \), and so in particular there is \( \varphi \in \mathbb{N}^{\mathbb{N}} \), \( \varphi = \varphi(\varepsilon) \), with \( a_{i,\varphi(i)} \leq \varepsilon \), and thus
\[
\limsup_{i=1}^{\infty} a_{i,\varphi(i)} \leq \varepsilon, \tag{5}
\]
thanks to arbitrariness of \( i \in \mathbb{N} \). By hypothesis, to \( \varphi \) there corresponds an element \( D \in \mathcal{I} \) with
\[
|x_n - x| \leq \limsup_{i=1}^{\infty} a_{i,\varphi(i)} \quad \text{whenever } n \notin D.
\]
Hence for \( n \notin D \) from (5) we get: \( |x_n - x| \leq \varepsilon \), that is the assertion. \( \square \)

A Riesz space \( R \) is said to be \((D)\)-complete if every \((D)\)-Cauchy sequence is \((D)\)-convergent.

Analogously as in the case of \((r)\)-completeness, it is possible to prove that

**Proposition 4.5** Every Dedekind complete Riesz space is \((D)\)-complete (w. r. to every fixed admissible ideal \( \mathcal{I} \)).

Similarly as in Proposition 4.5, we can check that, in all Dedekind complete weakly \( \sigma \)-distributive Riesz spaces, \((D)\)-convergence is complete also for indexed families of sequences, in the uniform sense.

We now turn to the \((r)\)-convergence with respect to a fixed admissible ideal \( \mathcal{I} \). We begin with the following

**Definition 4.6** A Riesz space \( R \) satisfies property \( \sigma \) if to every sequence \((u_n)\) of positive elements of \( R \) there correspond a unit \( u \) in \( R \) and a sequence \((\lambda_n)\) of positive real numbers, with
\[
\lambda_n u_n \leq u \quad \text{for all } n \in \mathbb{N}.
\]

It is well-known that there exist Dedekind complete Riesz spaces, having property \( \sigma \) but in which \((r)\)- and order convergence do not coincide, for example the space \( l_\infty \) of all bounded sequences of real numbers (see [22], p. 479).

From now on we assume that \((R_1, R_2, R)\) is a product triple, and that, when we deal with \((r)\)-convergence, the involved Riesz spaces satisfy property \( \sigma \).
In this context we formulate the concepts of convergence [Cauchy] in measure, in $L^1$ and [uniform] integrability.

So, let $G$ be any arbitrary nonempty set, and assume that a $\sigma$-algebra $\mathcal{A}$ and a positive finitely additive measure $\mu : \mathcal{A} \to \mathbb{R}$ are fixed.

**Definition 4.7** We say that a sequence $(f_n)_n$ in $R^G_1$ converges in measure to $f \in R^G_1$ is [Cauchy in measure] if, for a suitable [double] sequence $(A_n)_n$ in $\mathcal{A}$ and suitable units $v$, $w$, to any $\varepsilon$ and $\sigma > 0$ there corresponds an element $D \in I$ with $\mu(A_n) \leq \sigma v$ and

$$A_n \supset \{g \in G : |f_n - f(g)| \leq \varepsilon w\}$$

whenever $n \not\in D$ [\(\mu(A_{n,m}) \leq \sigma v\) and $A_{n,m} \supset \{g \in G : |f_n - f_m(g)| \leq \varepsilon w\}$] for all $n, m \not\in D$.

**Definition 4.8** We say that the sequence $(f_n)_n$ in $L$ is uniformly integrable if

$$(r) \lim_n \int_{A_n} |f_n(g)| d\mu(g) = 0$$

whenever $(A_n)_n$ is any sequence in $\mathcal{A}$ with $(r) \lim_n \mu(A_n) = 0$. (Here the $(r)$-limit is intended as in 4.2, that is with respect to the fixed ideal $I$.)

**Definition 4.9** A positive function $f \in R^G_1$ is integrable if there exist a uniformly integrable sequence of functions $(f_n)_n$ in $L$ convergent in measure to $f$ (defining sequence), and a mapping $l : \mathcal{A} \to \mathbb{R}$, such that

$$(r) \lim_n \left( \sup_{A \in \mathcal{A}} \left| \int_A f_n(g) d\mu(g) - l(A) \right| \right) = 0.$$ 

We set

$$\int_A f(g) d\mu(g) = l(A), \quad A \in \mathcal{A}.$$ 

A map $f \in R^G_1$ is integrable if $f^+$ and $f^-$ are integrable, and in this case we put

$$\int_A f(g) d\mu(g) = \int_A f^+(g) d\mu(g) - \int_A f^-(g) d\mu(g), \quad A \in \mathcal{A}.$$ 

Of course the concept of uniform integrability is now extended to all integrable functions, analogously as above. Similarly, we have:
Definition 4.10 A sequence of integrable maps \((f_n)_n\) converges in \(L^1\) to an integrable function \(f\) if \(Cauchy in L^1\) if, for some unit \(u\), for every \(\varepsilon > 0\) there is \(D \in \mathcal{I}\) satisfying
\[
\int_G |f_n(g) - f(g)| \, d\mu(g) \leq \varepsilon u
\]
for all \(n \notin D\)
\[
\int_G |f_n(g) - f_m(g)| \, d\mu(g) \leq \varepsilon u
\]
whenever \(n, m \notin D\).

In this particular context, the concept of integrability can be formulated in a more classical way, thanks to the following characterization of integrability, which will be useful to demonstrate completeness of the space \(L^1\) and the monotone convergence theorem. Without loss of generality, we deal with \((r)-convergence and an admissible ideal \(\mathcal{I}\). Similar results were proved in [12] in the context of order convergence in the classical case of the theory of Riesz spaces.

Theorem 4.11 A necessary and sufficient condition for integrability of a map \(f\) is that there is a sequence \((s_n)_n\) in \(L\), Cauchy in \(L^1\) and convergent in measure to \(f\). In this case the sequence \((s_n)_n\) is uniformly integrable and converges in \(L^1\) to \(f\).

Proof: We first prove the sufficient part. If \((s_n)_n\) is any sequence in \(L\), Cauchy in \(L^1\) and convergent in measure to \(f\), then from the inequality
\[
\left| \left( \int_A s_n(g) \, d\mu(g) - \int_A s_m(g) \, d\mu(g) \right) \right| \leq \int_G |s_n(g) - s_m(g)| \, d\mu(g), \quad A \in \mathcal{A},
\]
(6)
it follows that the limit
\[
(r) \lim_n \int_A s_n(g) \, d\mu(g)
\]
(w. r. to the fixed ideal \(\mathcal{I}\)) exists uniformly with respect to \(A \in \mathcal{A}\). Set
\[
I_A(f) = I_A(f; (s_n)_n) = (r) \lim_n \int_A s_n(g) \, d\mu(g), \quad A \in \mathcal{A}.
\]
Proceeding analogously as in [12], it is easy to check that the sequence \((|s_n - s_m|)_m\) converges in measure to \(|s_n - f|\) and is Cauchy in \(L^1\) for each fixed \(n \in \mathbb{N}\). Then, arguing analogously as in (6), where the role of \((s_n)_n\) is played by the sequence \((|s_n - s_m|)_m\), the entity
\[
I_A(|s_n - f|) = I_A(|s_n - f|; (|s_n - s_m|)_m) = (r) \lim_m \int_A |s_n(g) - s_m(g)| \, d\mu(g)
\]
w. r. to the involved ideal $I$ is well-defined and the limit is uniform with respect to $A \in \mathcal{A}$. Since $(s_n)_n$ is Cauchy in $L^1$, for a suitable unit $u$ and every $\varepsilon > 0$ there corresponds an element $D \in I$ with

$$\int_G |s_n(g) - s_m(g)| \, d\mu(g) \leq \varepsilon u \quad \text{for all } n, m \notin D.$$ 

Keeping fixed $n$ and taking the $(r)$-limit as $m$ tends to $+\infty$, we get:

$$I_G(|s_n - f|) = I_G(|s_n - f|; (|s_n - s_m|)_m) = \lim_{m \to +\infty} \int_G |s_n(g) - s_m(g)| \, d\mu(g) \leq \varepsilon u$$

whenever $n \notin D$.

From this it follows that $(r) \lim_n I_G(|s_n - f|) = 0$.

Moreover, observe that the sequence $(|s_n|)_n$ converges in measure to $|f|$ and is Cauchy in $L^1$. From this, arguing as above, where the roles of $s_n$, $f$ are played by $|s_n|$, $|f|$ respectively, we get also $(r) \lim_n I_G(| |s_n| - |f| |) = 0$.

Let now $(r) \lim_n \mu(A_n) = 0$. Then, for all $n, m, k \in \mathbb{N}$,

$$\int_{A_n} |s_m(g)| \, d\mu(g) \leq \int_{A_n} |s_m(g) - s_k(g)| \, d\mu(g) + \int_{A_n} |s_k(g)| \, d\mu(g) \leq \int_G |s_m(g) - s_k(g)| \, d\mu(g) + \int_{A_n} |s_k(g)| \, d\mu(g).$$

Taking the $(r)$-limit as $m$ tends to $+\infty$, we obtain:

$$I_{A_n}(|f|) = I_{A_n}(|f|; (|s_n|)_m) \leq I_G(|f - s_k|; (|s_m - s_k|)_m) + \int_{A_n} |s_k(g)| \, d\mu(g), \quad (7)$$

and hence, arguing analogously as in the classical case and taking in $(7)$ first the $(r)$-limits as $n$ tends to $+\infty$ with $k$ fixed, and afterwards the $(r)$-limit as $k$ tends to $+\infty$, we get that $(r) \lim_n I_{A_n}(|f|) = 0$.

Uniform integrability of the sequence $(s_n)_n$ can be proved by means of a technique similar to the one used in the classical context.

From the inequalities

$$\sup_{A \in \mathcal{A}} \left| \int_A s_n(g) \, d\mu(g) - I_A(f) \right| \leq I_G(|s_n - f|),$$

$$\sup_{A \in \mathcal{A}} \left| \int_A |s_n(g)| \, d\mu(g) - I_A(|f|) \right| \leq I_G(| |s_n| - |f| |), \quad n \in \mathbb{N},$$
and the axioms of convergence it follows that two entities \( l(A), \tilde{l}(A) \), \( A \in \mathcal{A} \), satisfying condition (1) with respect to \((s_n)_n, f\) and \((|s_n|)_n, |f|\) respectively do exist and coincide with \( I_A(f; (s_n)_n) \) and \( I_A(|f|; (|s_n|)_n) \) respectively for each \( A \in \mathcal{A} \).

From this it follows that the sequences \((s^+_n)_n\) and \((s^-_n)_n\) are defining for \( f^+ \) and \( f^- \) respectively, and thus we get integrability of \( f \), and

\[
\int_A f(g) \, d\mu(g) = I_A(f) \quad \text{for all} \quad A \in \mathcal{A}.
\]

Conversely, let \( f \) be an integrable function and \((s_n)_n\) be a sequence in \( L \), convergent in measure to \( f \) and uniformly integrable. By the Vitali theorem we get convergence in \( L^1 \) of \((s_n)_n\) to \( f \), and hence the sequence \((s_n)_n\) is Cauchy in \( L^1 \). \( \Box \)

From now on, when we deal with \((r)\)-convergence, we always suppose that the involved Riesz spaces satisfy property \( \sigma \). We begin with the following technical lemma.

**Lemma 4.12** Let \((f_n)_n\) be a sequence of integrable functions. Then a unit \( u \) in \( R \) and a sequence \((s_n)_n\) in \( L \) can be found, with

\[
\int_G |f_n(g) - s_n(g)| \, d\mu(g) \leq \frac{1}{n} u \quad \text{for all} \quad n \in \mathbb{N}.
\]

**Proof:** Let \( n \in \mathbb{N} \). Since \( f_n \) is integrable, thanks to Theorem 4.11 there exist a unit \( u_n \in R \) and a function \( t_n \in L \) with

\[
\int_G |f_n(g) - t_n(g)| \, d\mu(g) \leq \frac{1}{n} u_n.
\]

By virtue of property \( \sigma \), in correspondence with the sequence \((u_n)_n\), a unit \( u \) and a sequence of real positive numbers \((\lambda_n)_n\) can be found, satisfying

\[
\lambda_n u_n \leq u \quad \text{for all} \quad n \in \mathbb{N}.
\]

From (9) it follows that for any \( n \in \mathbb{N} \) there is an element \( s_n \) of \( L \), such that

\[
\int_G |f_n(g) - s_n(g)| \, d\mu(g) \leq \frac{\lambda_n}{n} u_n \leq \frac{1}{n} u.
\]

Thus the assertion follows. \( \Box \)

The following proposition will be useful to prove completeness of \( L^1 \) and the monotone convergence theorem.
Proposition 4.13 Let \((f_n)_n\) be a sequence of integrable functions, convergent in measure to \(f\) and Cauchy in \(L^1\). Then \(f\) is integrable and \((f_n)_n\) converges in \(L^1\) to \(f\).

Proof: By virtue of Theorem 4.11 and Lemma 4.12, for any \(n \in \mathbb{N}\) there is a function \(s_n\) belonging to \(L^1\), satisfying (8) for a suitable unit \(u\) and such that

\[
\mu(\{g \in G : |f_n(g) - s_n(g)| \leq \frac{1}{n} w\}) \leq \frac{1}{nv} \tag{11}
\]

for two suitable units \(v, w\). We now check that the sequence \((s_n)_n\) converges in measure to \(f\). Let \(z\) be a unit, associated with convergence in measure of \((f_n)_n\) to \(f\). Observe that

\[
\{g \in G : |s_n(g) - f(g)| \leq \varepsilon (z + w)\} \tag{12}
\]

\[
\subset \{g \in G : |f_n(g) - f(g)| \leq \varepsilon z\} \cup \{g \in G : |f_n(g) - s_n(g)| \leq \varepsilon w\}
\]

for all \(\varepsilon > 0\) and \(n \in \mathbb{N}\). Thus convergence in measure of \((s_n)_n\) to \(f\) follows from the one of \((f_n)_n\) to \(f\), (11) and (12).

We now prove that the sequence \((s_n)_n\) is Cauchy in \(L^1\). For every \(n, m \in \mathbb{N}\) we get:

\[
\int_G |s_n(g) - s_m(g)| \, d\mu(g) \leq \int_G |s_n(g) - f_n(g)| \, d\mu(g) + \int_G |f_n(g) - f_m(g)| \, d\mu(g) + \int_G |f_m(g) - s_m(g)| \, d\mu(g)
\]

\[
\leq \int_G |f_n(g) - f_m(g)| \, d\mu(g) + \frac{1}{n} u + \frac{1}{m} u \tag{13}
\]

for a suitable unit \(u \in R\), thanks to (8). From (13) and Theorem 4.11 we obtain integrability of \(f\) and convergence in \(L^1\) of \((s_n)_n\) to \(f\). Finally, since

\[
\int_G |f_n(g) - f(g)| \, d\mu(g) \leq \int_G |f_n(g) - s_n(g)| \, d\mu(g) + \int_G |s_n(g) - f(g)| \, d\mu(g) \tag{14}
\]

\[
\leq \frac{1}{n} u + \int_G |s_n(g) - f(g)| \, d\mu(g), \ n \in \mathbb{N},
\]

then convergence in \(L^1\) of \((f_n)_n\) to \(f\) follows from the one of \((s_n)_n\) to \(f\) and (14). \(\square\)

Let now \(\mathcal{I}_{\text{fin}}\) be the ideal of all finite subsets of \(\mathbb{N}\). Observe that \((r)\)- and \((D)\)-convergence in Riesz spaces satisfy the property that ordinary convergence (i.e. with respect to \(\mathcal{I}_{\text{fin}}\)) implies convergence w. r. to any fixed admissible ideal \(\mathcal{I}\): indeed, \(\mathcal{I}_{\text{fin}}\) is contained in every admissible ideal \(\mathcal{I}\).

The following result holds for both \((r)\)- and \((D)\)-convergence.
**Proposition 4.14** Let \( I \) be any admissible ideal. If \( f \) is integrable with respect to \( I_{\text{fin}} \), then \( f \) is integrable with respect to \( I \).

Again concerning both \((r)\)- and \((D)\)-convergence with respect to an admissible ideal \( I \), for the sake of simplicity, we denote by the symbol "lim" the involved limit in any case, since no confusion can arise.

We observe that, in general, there are Riesz spaces \( R \) satisfying property \( \sigma \) and increasing sequences \((a_n)\) in \( R \), admitting supremum, but not \((r)\)-limit: for example, let us choose \( A = \{ 1 - \frac{1}{m} : m \in \mathbb{N} \} \cup \{ 1 \} \), and, in the space \( R = \{ f : A \to \mathbb{R}, f \text{ bounded} \} \), the monotone sequence \( a_n = x \mapsto 1 - x^n, x \in A, n \in \mathbb{N} \). The particular choice of \( A \) ensures property \( \sigma \) of \( R \), but \((r)\)-convergence of the sequence \((a_n)\) would imply uniform convergence, which does not occur here (see also [22, 25]).

The following result holds.

**Proposition 4.15** Let \((a_n)\) be an increasing sequence in \( R \), and \( l \in R \). Then \((r)\lim_n a_n = l \) with respect to an admissible ideal \( I \) if and only if \((r)\lim_n a_n = l \) with respect to \( I_{\text{fin}} \).

**Proof:** It is enough to prove the "only if" part. By hypothesis, thanks to admissibility of \( I \), a unit \( u \) can be found, with the following property: to any \( \varepsilon > 0 \) there corresponds a positive integer \( n_0 \) with \( 0 \leq l - a_{n_0} \leq \varepsilon u \). From this and monotonicity of the sequence \((a_n)\) we obtain:

\[
0 \leq l - a_n \leq l - a_{n_0} \leq \varepsilon u
\]

whenever \( n \geq n_0 \), and thus the assertion. \( \square \)

We now consider the particular case when \( R_1 = \mathbb{R} \) and \( R = R_2 \) is any Dedekind complete Riesz space, endowed with \((r)\)- or \((D)\)-convergence with respect to any fixed admissible ideal \( I \).

Let now \( A \subset \mathcal{P}(G) \) be a \( \sigma \)-algebra, \( L \) be the subspace of all simple measurable functions (w. r. to \( A \)), and assume that \( \mu : A \to R \) is finitely additive and continuous (this last condition means that

\[
\mu(A) = \lim_n \mu(A_n)
\]

whenever \((A_n)\) is an increasing \([\text{decreasing}] \) sequence in \( A \) with

\[
A = \bigcup_n A_n \quad [A = \bigcap_n A_n].
\]

Observe that, in this setting, the concepts of integrability and related topics are intended with respect to the considered \( \sigma \)-algebra \( A \).
We note that, since every \((r)\)-convergent sequence is order convergent too and \((D)\)-convergence is equivalent to order convergence in weakly \(\sigma\)-distributive Riesz spaces (in the usual sense, that is when we consider \(I_{\text{fin}}\), see also [16, 22]), then \(\mu\) turns out to be \(\sigma\)-additive.

We now prove the monotone convergence theorem.

**Theorem 4.16** Let \((f_n)_n\) be a monotone sequence of integrable non-negative real-valued functions, convergent pointwise to \(f\), and assume that \(\mu\) is continuous.

Then \(f\) is integrable if and only if \(\lim_n \int_G f_n(g) \, d\mu(g) \in \mathbb{R}\), and in this case

\[
\int_G f(g) \, d\mu(g) = \lim_n \int_G f_n(g) \, d\mu(g).
\]

**Proof:** We first deal only with \((D)\)-convergence. By Proposition 4.15 and continuity of \(\mu\) we get:

\[
\mu(A) = \lim_n \mu(A_n) = \sup_n \mu(A_n)
\]

whenever \((A_n)_n\) is an increasing sequence in \(\mathcal{A}\) with \(A = \bigcup_n A_n\) (Here, the limit is intended with respect to the ideal \(I_{\text{fin}}\)). Analogously, from hypothesis and Proposition 4.15 it follows that

\[
\lim_n f_n(g) = \sup_n f_n(g) = f(g) \quad \text{for all } g \in G.
\]

The assertion follows from Theorem 4.4 of [11] and Theorem 4.22 of [12], taking into account also of Proposition 4.14.

Now, before proving Theorem 4.16 with respect to \((r)\)-convergence, we introduce some notions and prove some basic results.

Indeed, it is not advisable to proceed as in the case of \((D)\)-convergence, because, as we pointed above, monotonic sequences may fail to have \((r)\)-limit in general, even if bounded.

So, we first state a result of completeness with respect to convergence in measure and in \(L^1\), and then we use it in order to prove the monotone convergence theorem in the case of \((r)\)-convergence. Here we consider the ideal \(I_{\text{fin}}\): this is not a substantial restriction, because of monotonicity.

We begin with the following definition and proposition, whose proof is straightforward.

**Definition 4.17** We say that the sequence \((f_n)_n\) **converges almost uniformly** to \(f\) if there is a unit \(u\) such that to any \(\varepsilon > 0\) there corresponds \(F \in \mathcal{A}\) with \(\mu(F) \leq \varepsilon \, u\) and such that \((f_n)_n\) converges uniformly on \(G \setminus F\).
Proposition 4.18 If \((f_n)_n\) converges in measure both to \(f\) and \(h\), then \(f = h\) almost everywhere; moreover convergence in \(L^1\) implies convergence in measure, and almost uniform convergence implies convergence in measure.

We now are ready to prove our result about completeness.

Theorem 4.19 If \((f_n)_n\) is a sequence of measurable real-valued functions which is Cauchy in measure, then it converges in measure to a measurable function \(f\). Moreover there is a subsequence \((f_{n_k})_k\), converging almost uniformly and almost everywhere to \(f\). Furthermore, if \((f_n)_n\) is Cauchy in \(L^1\), then it converges in \(L^1\) too, and the limit function is integrable.

Proof: By hypothesis a strictly increasing sequence \((n_k)_k\) in \(\mathbb{N}\) can be found, with

\[ \mu(E_k) < 2^{-k} u \]

for all \(k \in \mathbb{N}\), where \(u \in \mathbb{R}\) is a suitable unit, and

\[ E_k := \{ g \in G : |f_{n_k}(g) - f_{n_{k+1}}(g)| > 2^{-k} \} \], \( k \in \mathbb{N}\).

Fix now arbitrarily \(\varepsilon > 0\), pick \(k \in \mathbb{N}\) with \(2^{1-k} < \varepsilon\) and set \(G_k = \bigcup_{l=k}^{\infty} E_l\). Then, since \(\mu\) turns to be \(\sigma\)-additive, we get:

\[ \mu(G_k) \leq \sum_{l=k}^{\infty} \mu(E_l) \leq 2^{1-k} u, \]

and hence

\[ |f_{n_{l+p}}(g) - f_{n_l}(g)| \leq 2^{1-k} u \]

whenever \(g \in G \setminus G_k, \ l \geq k, \ p \in \mathbb{N}\). Thus the sequence \((f_{n_k})_k\) is Cauchy almost uniformly, and so, by completeness of \((r)\)-convergence in the uniform sense, it converges almost uniformly to a function \(f\). This implies that \((f_{n_k})_k\) converges to \(f\) even \(\mu\)-almost everywhere and in measure.

For any \(\varepsilon > 0\), \(n, k \in \mathbb{N}\) we get:

\[ \{ g \in G : |f_n(g) - f(g)| > \varepsilon \} \subset \{ g \in G : |f_n(g) - f_{n_k}(g)| > \varepsilon/2 \} \]

\[ \cup \{ g \in G : |f_{n_k}(g) - f(g)| > \varepsilon/2 \}. \] (16)

Since \((f_n)_n\) is Cauchy in measure, it is possible to find integers \(n\) and \(k\) so large that the first term in the right hand of (16) has sufficiently small measure \(\mu\).
We observe that, by Proposition 4.18, the measure of the last terms of (16) approaches 0 as \( k \) tends to \(+\infty\). From this we can conclude that for every \( \varepsilon > 0 \) the measure \( \mu \) of the first hand of (16) tends to 0 as \( n \to +\infty \), that is we get convergence in measure of the sequence \((f_n)_n\) to \( f \).

Let now \((f_n)_n\) be Cauchy in \( L^1 \). By Proposition 4.13, \( f \) is integrable and \((f_n)_n\) converges in \( L^1 \) to \( f \); hence we get completeness of the space \( L^1 \) of all integrable functions. This concludes the proof. \( \square \)

We now prove the monotone convergence theorem with respect to \((r)\)-convergence. Without loss of generality, it is enough to assume \( I = I_{\text{fin}} \). If \( f \) is integrable, then the monotone convergence theorem is an immediate consequence of the Lebesgue dominated convergence theorem. It remains to prove that, if

\[
(r) \lim_n \int_G f_n(g) \, d\mu(g) \in R,
\]

then \( f \) is integrable. By Proposition 4.14, it will be enough to prove integrability of \( f \) w.r. to \( I_{\text{fin}} \).

By hypothesis, the sequence \( \left( \int_G f_n(g) \, d\mu(g) \right)_n \) is \((r)\)-convergent, and thus \((r)\)-Cauchy. Following the lines of [18], since the differences \( f_n - f_m \) are of constant sign for every \( m, n \in \mathbb{N} \), then

\[
\left| \int_G f_m(g) \, d\mu(g) - \int_G f_n(g) \, d\mu(g) \right| = \int_G |f_m(g) - f_n(g)| \, d\mu(g),
\]

and hence \((f_n)_n\) is Cauchy in \( L^1 \). By completeness of \( L^1 \), \((f_n)_n\) converges in \( L^1 \) (and hence also in measure) to an integrable map \( h \). By Proposition 4.18, \( f = h \) \( \mu \)-almost everywhere. Thus \( f \) is integrable, thanks to integrability of \( h \) and Lemma 3.17. \( \square \)

**Conclusions:** We have introduced axiomatically the notion of convergence in the setting of Riesz spaces, and studied integration theory with respect to this convergence, extending previous results proved in [5, 6, 7]. A particular case of it is convergence with respect to a fixed admissible ideal, which was introduced in [20].

We investigated some properties of \( I \)-convergence in the context of Riesz spaces, in particular some topics on integration theory, proving some basic theorems and some further results in the particular case when we deal with real-valued functions.
References


