The Kurzweil-Henstock Integral for Riesz Space-Valued Maps Defined in Abstract Topological Spaces and Convergence Theorems

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Abstract

We prove some convergence theorems for the Kurzweil-Henstock integral in Riesz spaces, for functions defined in a compact topological space with respect to Riesz space-valued measures, which can assume even the value ”$+\infty$”.

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1 Introduction

In this paper we deal with convergence theorems (monotone convergence theorem and Lebesgue dominated convergence theorem) for the Kurzweil-Henstock integral in an abstract setting, for functions defined in a compact topological space $T$, satisfying suitable properties, and with values in a Dedekind complete Riesz space, with respect to a positive Riesz space-valued measure $\mu$, which can assume even the value ”$+\infty$”. A particular case is $T = [A,B]$ an interval (possibly unbounded) of the extended real line, or the whole of $\mathbb{R}$. 
and \( \mu = \) the Lebesgue measure (this case was investigated in [2]). We continue the investigation started in [4], in which \( \mu \) is \( \mathbb{R} \)-valued, and in which some other kinds of convergence theorems were demonstrated. Our results given here extend the ones in [4], Chapter 5, which were proved in the case in which \( \mu(T) \) is finite, and the ones of [9], which were proved in the case \( T = [A, B] \), where \([A, B]\) is as above, and all the involved Riesz spaces coincide with the (eventually extended) real line. A similar Kurzweil-Henstock type integral was investigated in [5] for Banach space-valued maps.

This approach can be used in several contexts: for example, in the study of states and observables (see [14]), which can be viewed as functions or probability measures with values in special Riesz spaces, and in the theory of stochastic processes, which can be considered as \( L^0 \)-valued maps, where the involved convergence is the almost everywhere convergence, which does not have a "topological" nature.

2 Preliminaries

Assumption 2.1. We assume that \((T, T)\) is a Hausdorff compact topological space, and that there are given a family \( \mathcal{B} \) of Borel subsets of \( T \) such that \( T \in \mathcal{B} \) and closed under intersections, and a monotone positive additive mapping \( \mu : \mathcal{B} \to \mathbb{R} \) (here, additive means

\[
\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F)
\]

whenever \( E, F, E \cup F \in \mathcal{B} \).

Definition 2.2. A gauge on \( T \) is a mapping \( \gamma : T \to T \) assigning to each \( t \in T \) its neighborhood \( \gamma(t) \).

Example 2.3. Let \( T = [a, b] \subset \mathbb{R} \), \( \delta : T \to \mathbb{R}^+ \) be a map. Let \( T \) be the usual topology on the real line. Put \( \gamma(t) = (t - \delta(t), t + \delta(t)) \). Then \( \gamma \) is a gauge in the sense of Definition 2.2.

Definition 2.4. A partition of \( T \) is a finite collection \( \Pi = \{(E_i, t_i) : i = 1, \ldots, k\} \) of couples such that

(i) \( \bigcup_{i=1}^{k} E_i = T \);

(ii) \( t_i \in E_i, E_i \in \mathcal{B} \);

(iii) \( \mu(E_i \cap E_j) = 0 \) (\( i \neq j \)).

A collection \( \Pi \) satisfying axioms (ii) and (iii), but not necessarily (i), is called decomposition of \( T \). The partition or decomposition \( \Pi \) is \( \gamma \)-fine (\( \Pi \prec \gamma \)), if \( E_i \subseteq \gamma(t_i) \) (\( i = 1, 2, \ldots, k \)).

Definition 2.5. We say that \( \mathcal{B} \) is separating, if there exists a sequence \( (\Pi_n)_n \) of partitions such that \( \Pi_{n+1} \) is a refinement of \( \Pi_n \) (\( n \in \mathbb{N} \)) and, for any \( x, y \in T \), \( x \neq y \), there exist \( n \) and \( E \in \Pi_n \) such that, if \( x \in E \) for some \( E \in \Pi_n \), then \( y \in T \setminus E \).

Let \( R \) be a Dedekind complete Riesz space. We now give the following preliminary definitions.
Definition 2.6. A bounded double sequence \((a_{i,j})_{i,j}\) in \(R\) is called a \((D)\)-sequence if \(a_{i,j} \geq a_{i,j+1} \forall i, j \in \mathbb{N}\) and \(\bigwedge_{j=1}^{\infty} a_{i,j} = 0 \forall i \in \mathbb{N}\).

Definition 2.7. A sequence \((p_n)_{n}\) is called an \((O)\)-sequence if \(p_n \geq p_{n+1} \forall n \in \mathbb{N}\) and \(\bigwedge_{n=1}^{\infty} p_n = 0\). In this case, we write also \(p_n \downarrow 0\).

Definition 2.8. Given a sequence \((r_n)_{n}\) in \(R\), we say that \((r_n)_{n}\) \((D)\)-converges to an element \(r \in R\) if there exists a \((D)\)-sequence \((a_{i,j})_{i,j}\) in \(R\), satisfying the following condition:

\[
\forall \varphi \in \mathbb{N}^N, \text{there exists an integer } n_0 \text{ such that } |r_n - r| \leq \bigwedge_{i=1}^{\infty} a_{i,\varphi(i)},
\]

for all \(n \geq n_0\). In this case, we write \(\lim_{n} r_n = r\).

Definition 2.9. Given a sequence \((r_n)_{n}\) in \(R\), we say that \((r_n)_{n}\) \((O)\)-converges to an element \(r \in R\) if there exists an \((O)\)-sequence \((p_n)_{n}\) in \(R\), such that

\[
|r_n - r| \leq p_n \quad \forall n \in \mathbb{N}.
\]

In this case, we write \(\lim_{n} r_n = r\).

We now recall the very famous Fremlin Lemma (see also [6, 14]):

Lemma 2.10. Let \((a_{i,j}^{(k)})_{i,j}, k \in \mathbb{N}\), be any countable family of \((D)\)-sequences. Then for each fixed element \(u \in R, u \geq 0\), there exists a \((D)\)-sequence \((a_{i,j})_{i,j}\) such that, for every \(\varphi \in \mathbb{N}^N\), one has

\[
u \wedge \sum_{k=1}^{s} \left( \bigvee_{i=1}^{\infty} a_{i,\varphi(i+s+k)}^{(k)} \right) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \quad \forall s \in \mathbb{N}.
\]

Definition 2.11. A Riesz space \(R\) is said to be weakly \(\sigma\)-distributive if for every bounded double sequence \((b_{i,j})_{i,j}\) with \(b_{i,j} \geq b_{i,j+1} \forall i, j\) one has:

\[
\bigvee_{i=1}^{\infty} \left( \bigwedge_{j=1}^{\infty} b_{i,j} \right) = \bigwedge_{\varphi \in \mathbb{N}^N} \left( \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \right) \tag{1}
\]

(see [19]). The following characterization holds (see also [7], Proposition 2.1, pp. 156-157).

Proposition 2.12. A Dedekind complete Riesz space \(R\) is weakly \(\sigma\)-distributive if and only if for every \((D)\)-sequence \((a_{i,j})_{i,j}\) in \(R\) one has:

\[
\bigwedge_{\varphi \in \mathbb{N}^N} \left( \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) = 0. \tag{2}
\]
(This implies that in weakly σ-distributive spaces (D)-limits are unique)

From now on, we shall always suppose that $R$ is a Dedekind complete weakly σ-distributive Riesz space.

**Definition 2.13.** Let $S$ be the σ-algebra of all Borel subsets of $T$. We say that $µ : S \rightarrow R$ is *regular*, if to any $E \in S$ there exists a (D)-sequence $(a_{i,j})_{i,j}$ such that for every $ϕ : \mathbb{N} \rightarrow \mathbb{N}$ there exist $C$ compact and $U$ open such that $C \subseteq E \subseteq U$ and $\sup_{B \in S, B \subseteq U \setminus C} |µ(B)| \leq \bigvee_{i=1}^{\infty} a_{i,ϕ(i)}$.

**Remark 2.14.** If $(T, d)$ is a compact metric space, $B$ is a semiring of subsets of $T$ such that to any $x \in T$ and every neighborhood $U$ of $x$ there exists $E \in B$ such that $x \in \text{int} E \subseteq E \subseteq U$ (where, given any subset $E$ of any topological space, we denote by $\text{int} E$ its interior in the topological sense) and $B$ is separating, then to every gauge $γ : T \rightarrow T$ there exists a $γ$-fine partition and in correspondence with any set $B \in B$ there exists a $γ|_{B}$-fine partition (see also [11], Lemma 1.2, p. 154 and Proposition 1.7., p. 156).

From now on, we shall always suppose that $B$ satisfies conditions in 2.14.

We now introduce some structural assumptions, which will be needed later.

**Assumptions 2.15.** Let $R_1$, $R_2$, $R$ be three Dedekind complete Riesz spaces. We say that $(R_1, R_2, R)$ is a *product triple* if there exists a map $\cdot : R_1 \times R_2 \rightarrow R$, which we will call *product*, such that

1. $r_1 + s_1 \cdot r_2 = r_1 \cdot r_2 + s_1 \cdot r_2,$
2. $r_1 \cdot (r_2 + s_2) = r_1 \cdot r_2 + r_1 \cdot s_2,$
3. $|r_1 \geq s_1, r_2 \geq 0 \Rightarrow |r_1 \cdot r_2 \geq s_1 \cdot r_2|,$
4. $|r_1 \geq 0, r_2 \geq s_2 \Rightarrow |r_1 \cdot r_2 \geq r_1 \cdot s_2|$ for all $r_j, s_j \in R_j, j = 1, 2$;
5. if $r_j \in R_j$, $j = 1, 2$ and $α \in \mathbb{R}$, then $(αr_1) \cdot r_2 = r_1 \cdot (αr_2) = α(r_1 \cdot r_2);$  
6. if $(a_{λ})_{λ \in Λ}$ is any net in $R_2$ and $b \in R_1$, then $[a_{λ} \downarrow 0, b \geq 0] \Rightarrow [b \cdot a_{λ} \downarrow 0];$
7. if $(a_{λ})_{λ \in Λ}$ is any net in $R_1$ and $b \in R_2$, then $[a_{λ} \downarrow 0, b \geq 0] \Rightarrow [a_{λ} \cdot b \downarrow 0].$

We will write often $a \cdot b$ instead of $a \cdot b$. A Dedekind complete Riesz space $R$ is called an *algebra* if $(R, R, R)$ is a product triple.

We always assume that $(R_1, R_2, R)$ is a product triple and that $R$ is weakly σ-distributive. Furthermore, we add to $R_2$ two extra elements, $+\infty$ and $-\infty$, extending ordering and operations in a natural way, and denote $R = R_2 \cup \{+\infty, -\infty\}$. 


3 The Kurzweil-Henstock integral

We now give our definition of Kurzweil-Henstock integrability. We suppose that \( \mu : S \rightarrow R \) is an additive positive regular measure on the Borel \( \sigma \)-algebra \( S \) of \( T \). If \( \Pi = \{(E_i, t_i) : i = 1, \ldots, q\} \) is a partition or a decomposition of a set \( A \in \mathcal{B} \), and \( f : T \rightarrow R \), then we define the Riemann sum as follows:

\[
\sum_{\Pi} f = \sum_{i=1}^{q} f(t_i) \mu(E_i)
\]

if the sum exists in \( R \), with the conventions that \( 0 \cdot (+\infty) = 0 \) and it is not considered any contribution in which \( \mu(E_i) = +\infty \). The points \( t_i, i = 1, \ldots, q \), are called tags.

**Definition 3.1.** A function \( f : T \rightarrow R \) is \((KH)\)-integrable (or, in short, integrable) if there exist \( I \in R \) and a \((D)\)-sequence \((a_{i,j})_{i,j}\) such that \( \forall \varphi \in N^N \) there exists a gauge \( \gamma \) such that

\[
\left| \sum_{\Pi} f - I \right| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}
\]

whenever \( \Pi = \{(E_i, t_i), i = 1, \ldots, q\} \) is a \( \gamma \)-fine partition of \( T \).

Evidently the number \( I \) is determined uniquely. It will be denoted by

\[
(KH) \int_{T} f \, d\mu, \int_{T} f \, d\mu \text{ or } \int_{T} f \, d\mu.
\]

**Proposition 3.2.** The integral is a linear positive functional.

**Proof:** The linearity follows by the identity

\[
\sum_{\Pi} (\alpha f + \beta g) = \alpha \sum_{\Pi} f + \beta \sum_{\Pi} g.
\]

The positivity follows by the implication

\[
f \geq 0 \Rightarrow \sum_{\Pi} f \geq 0
\]

and weak \( \sigma \)-distributivity of \( R \). \( \square \)

**Definition 3.3.** A function \( f : T \rightarrow R_1 \) is \((KH)\)-integrable (or, in short, integrable) on a set \( E \in \mathcal{B} \) if there exist \( I_E \in R \) and a \((D)\)-sequence \((a_{i,j}^{(E)})_{i,j}\) such that \( \forall \varphi \in N^N \) there exists a gauge \( \gamma \) such that

\[
\left| \sum_{\Pi} f - I_E \right| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}^{(E)}
\]

whenever \( \Pi = \{(E_i, t_i), i = 1, \ldots, q\} \) is a \( \gamma \)-fine partition of \( E \).
The element $I_E$ is denoted by $\int_E f \, d\mu$. When we say simply "integrable", we will intend "integrable on $T$".

We now state the Bolzano-Cauchy condition.

**Theorem 3.4.** A map $f : T \to R_1$ is $(KH)$-integrable on $E \in \mathcal{B}$ if and only if there exists a $(D)$-sequence $(b_{i,j}^{(E)})_{i,j}$ such that, $\forall \varphi \in \mathbb{N}^\mathbb{N}$, $\exists$ a gauge $\gamma$ such that for all $\gamma$-fine partitions $\Pi_1, \Pi_2$ of $E$ we have

$$\left|\sum_{\Pi_1} f - \sum_{\Pi_2} f\right| \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}^{(E)}.$$ 

The proof is similar to the one of [14], Proposition 5.2.9, pp. 77-79.

**Proposition 3.5.** If $E,F,G \in \mathcal{B}$, $E = F \cup G$, $\mu(F \cap G) = 0$ and $f$ is integrable on $E$, then $f$ is integrable on $F$ and on $G$ too, and

$$\int_E f \, d\mu = \int_F f \, d\mu + \int_G f \, d\mu.$$ 

The proof is similar to the one of [14], Proposition 5.2.10, pp. 79-80.

By induction, it is possible to prove the following:

**Proposition 3.6.** If $E \in \mathcal{B}$, $E_i \in \mathcal{B}$, $i = 1, \ldots, n$, and $E = \bigcup_{i=1}^{n} E_i$, $\mu(E_i \cap E_j) = 0$ whenever $i \neq j$ and $f$ is integrable on $E$, then $f$ is integrable on $E_i$ for every $i$, and

$$\int_E f \, d\mu = \sum_{i=1}^{n} \int_{E_i} f \, d\mu.$$ 

Of course, if we want to get an integral corresponding with our intuition, the value of the integral of a simple function of the type $\sum_{i=1}^{n} \alpha_i \chi_{G_i}$ should be $\sum_{i=1}^{n} \alpha_i \mu(G_i)$, if $\mu(G_i) \in R_2$ for every $i = 1, \ldots, n$. In order to do this, it is enough to prove the following theorem (see also [14], Proposition 5.2.11, pp. 80-81):

**Theorem 3.7.** Let $\mathcal{S}$ be the class of all Borel sets of $T$, $\mu : \mathcal{S} \to R_2$ be positive, additive and regular, $E \in \mathcal{S}$, with $\mu(E) \in R_2$. Let $r \in R_1$, and $g = \chi_{E \cap r}$ (defined by the relation $g(t) = r$, if $t \in E$, and $g(t) = 0$, if $t \notin E$). Then $g$ is integrable, and

$$\int_T g \, d\mu = r \mu(E).$$ 

**Proof:** First of all, assume $r \geq 0$. By regularity of $\mu$ on $\mathcal{S}$ there exists a $(D)$-sequence $(a_{i,j})_{i,j}$ such that for every $\varphi \in \mathbb{N}^\mathbb{N}$ there exist an open set $U$ and a compact set $C$, $C \subset E \subset U$, such that

$$\mu(U \setminus C) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$
Since $C$ is compact and $U$ is open, there exists a gauge $\eta$ such that $\eta(t) \subset U \forall t \in C$, $\eta(t) \subset U \setminus C \forall t \in U \setminus C$, $\eta(t) \cap C = \emptyset \forall t \notin U$. Take any partition $\Pi \prec \eta, \Pi = \{(E_i, t_i) : i = 1, \ldots, n\}$. Then

$$r \mu(C) \leq r \mu(E) \leq r \mu(U),$$

$$r \mu(U \setminus C) = r \mu(U) - r \mu(C),$$

and

$$r \mu(E) - r \sum_{i=1}^{\infty} a_{i, \varphi(i)} \leq r \mu(U) - r \sum_{i=1}^{\infty} a_{i, \varphi(i)} \leq r \mu(C)$$

$$\leq r \mu \left( \bigcup_{t_i \in C} E_i \right) \leq \sum_{t_i \in C} r \mu(E_i)$$

$$= \sum_{i=1}^{n} \chi_C(t_i) r \mu(E_i) \leq \sum_{i=1}^{n} \chi_C(t_i) r \mu(E_i) \sum_{i} g$$

$$\leq \sum_{t_i \in U} r \mu(E_i) \leq r \mu(U) \leq r \mu(E) + r \sum_{i=1}^{\infty} a_{i, \varphi(i)},$$

and hence

$$\left| \sum_{i} g - r \mu(E) \right| \leq \sum_{i=1}^{\infty} a_{i, \varphi(i)}$$

for any partition $\Pi \prec \eta$. From the properties of the product it follows that the double sequence $(r \cdot a_{i,j})_{i,j}$ is a $(D)$-sequence. Thus, we get the assertion, at least when $r \geq 0$.

In the general case $(r \in R_1)$ we get:

$$\int_T \chi_E r \, d\mu = \int_T \chi_E (r^+ - r^-) \, d\mu$$

$$= \int_T \chi_E r^+ \, d\mu - \int_T \chi_E r^- \, d\mu$$

$$= r^+ \mu(E) - r^- \mu(E) = r \mu(E).$$

4 Convergence theorems

We begin with proving a theorem, which will be used in the sequel, in order to demonstrate our versions of convergence theorems.

**Theorem 4.1.** Let $(f_n : T \rightarrow R_1)_n$ be a sequence of integrable functions. Suppose that:

4.1.1) there is a $(D)$-sequence $(b_{i,j})_{i,j}$ such that to every $\varphi : N \rightarrow N$ there exist a gauge $\zeta$ and $n_0 \in N$ such that

$$\left| \int_T f_n \, d\mu - \sum_{i} f_n \right| \leq \sum_{i=1}^{\infty} b_{i, \varphi(i)}$$

for every partition $\Pi \prec \zeta$ and $n \geq n_0$:
4.1.2) There exist a function $f : T \rightarrow R_1$, a $(KH)$-integrable map $h^* : T \rightarrow R^+$ (with respect to $\mu$) and a $(D)$-sequence $(a_{i,j}^*)_{i,j}$ such that, $\forall \varphi \in \mathbb{N}^N$, $\forall t \in T$, $\exists p(t) \in \mathbb{N}$: $\forall n \geq p(t)$,

$$|f_n(t) - f(t)| \leq h^*(t) \left( \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}^* \right).$$

(5)

Then $f$ is integrable and

$$(D) \lim_{n} \int_T f_n d\mu = \int_T f d\mu.$$

**Proof:** We shall use the Bolzano-Cauchy condition. Let $(b_{i,j})_{i,j}$, $\zeta$ and $n_0$ be as in 4.1.1). By 4.1.2) we get the existence of an element $0 \leq w \in R_1$ such that $\forall \varphi \in \mathbb{N}^N$ there exists a gauge $\eta \subset \zeta$ (without loss of generality) such that, for every $\eta$-fine partition $\Pi$ of $T$, $\Pi = \{(E_i, t_i) : i = 1, \ldots, q\}$, we have:

$$\left| \sum_{\Pi} f - \sum_{\Pi} f_n \right| \leq \sum_{\Pi} |f(t_i) - f_n(t_i)| \mu(E_i)$$

$$\leq \sum_{\Pi} h^*(t_i) \left( \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}^* \right) \mu(E_i)$$

$$\leq \left( \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}^* \right) w$$

$\forall n \geq \max\{p(t_i) : i = 1, \ldots, q\}$. Put $a_{i,j} = a_{i,j}^* w$, $\forall i, j \in \mathbb{N}$.

Without loss of generality, we can suppose that $p(t_i) \geq n_0 \forall i = 1, \ldots, n$. Choose now a $(D)$-sequence $(c_{i,j})_{i,j}$ such that

$$2 \left( \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}^* + \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}^* \right) \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}.$$

Then for all partitions $\Pi_1, \Pi_2 \prec \eta$, we have (using sufficiently large $n$, depending on the involved partitions $\Pi_1$ and $\Pi_2$):

$$\left| \sum_{\Pi_1} f - \sum_{\Pi_2} f \right| \leq \left| \sum_{\Pi_1} f - \sum_{\Pi_1} f_n \right|$$

$$+ \left| \sum_{\Pi_1} f_n - \int_T f_n d\mu \right| + \left| \int_T f_n d\mu - \sum_{\Pi_2} f_n \right|$$

$$+ \left| \sum_{\Pi_2} f_n - \sum_{\Pi_2} f \right| \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}.$$

The integrability of $f$ follows from this and the Bolzano-Cauchy condition.
By integrability of $f$ we obtain the existence of a $(D)$-sequence $(\pi_{i,j})_{i,j}$ such that, for every $\varphi \in \mathbb{N}^\mathbb{N}$, there exists a gauge $\eta_1$, depending on $\varphi$, such that

$$\left| \int_T f \, d\mu - \sum_{\Pi} f \right| \leq \sqrt[\infty]{\pi_{i,\varphi(i+1)}}$$

for every partition $\Pi \prec \eta_1$. By 4.1.1) there is a $(D)$-sequence $(b_{i,j})_{i,j}$ such that

$$\left| \sum_{\Pi} f_k - \int_T f_k \, d\mu \right| \leq \sqrt[\infty]{b_{i,\varphi(i+2)}}$$

for every $k$ greater than a suitable integer $k_0$ (depending on the involved $\varphi$) and for each partition $\Pi \prec \eta_2$. By 4.1.2), proceeding as in (6), we get the existence of a $(D)$-sequence $(c_{i,j})_{i,j}$ such that

$$\left| \sum_{\Pi} f - \sum_{\Pi} f_k \right| \leq \sqrt[\infty]{c_{i,\varphi(i+3)}}$$

for every $k \geq k'$, where $k'$ is a positive integer depending on the involved partition $\Pi$. Without loss of generality, we can assume $k' \geq k_0$. Choose a $(D)$-sequence $(d_{i,j})_{i,j}$ such that

$$\sqrt[\infty]{\pi_{i,\varphi(i+1)}} + \sqrt[\infty]{b_{i,\varphi(i+2)}} + \sqrt[\infty]{c_{i,\varphi(i+3)}} \leq \sqrt[\infty]{d_{i,\varphi(i)}}.$$

Then (II being a partition chosen arbitrarily, $\Pi \prec \eta_1 \cap \eta_2$)

$$\left| \int_T f \, d\mu - \int_T f_k \, d\mu \right| \leq \left| \int_T f \, d\mu - \sum_{\Pi} f \right| + \left| \sum_{\Pi} f - \sum_{\Pi} f_k \right| + \left| \sum_{\Pi} f_k - \int_T f_k \, d\mu \right| \leq \sqrt[\infty]{d_{i,\varphi(i)}}$$

for every $k \geq k'$. We have proved that

$$(D) \lim \int_T f_k \, d\mu = \int_T f \, d\mu$$

with respect to the $(D)$-sequence $(d_{i,j})_{i,j}$. This concludes the proof. □

We now prove a version of the Henstock lemma.

**Theorem 4.2.** Let $g : T \to R_1$ be an integrable function. Let $(a_{i,j})_{i,j}$ be a $(D)$-sequence such that to every $\varphi : \mathbb{N} \to \mathbb{N}$ there exists a gauge $\eta$ such that

$$\left| \int_T g \, d\mu - \sum_{\Pi} g \right| \leq \sqrt[\infty]{a_{i,\varphi(i)}}$$
for every partition $\Pi \prec \eta$. If $\Pi = \{(E_i, t_i) : i = 1, \ldots, n\} \prec \eta$, then for every $L \neq \emptyset$, $L \subseteq \{1, \ldots, n\}$, we have:

$$\left| \sum_{i \in L} \int_{E_i} g \, d\mu - \sum_{i \in L} g(t_i) \mu(E_i) \right| \leq \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}.$$

**Proof:** First of all, we note that integrability of $g$ on all $E_i$'s follows from Proposition 3.6. So there exists a $(D)$-sequence $(b_{i,j})_{i,j}$ such that to every $\psi : \mathbb{N} \to \mathbb{N}$ there is a gauge $\eta'$ such that

$$\left| \sum_{i \notin L} \int_{E_i} g \, d\mu - \sum_{i \notin L} \sum_{\Pi'_i} g \right| \leq \bigvee_{i=1}^{\infty} b_{i, \psi(i)}$$

whenever $\Pi_i \prec \eta'_{|E_i} \setminus L$. Put $\eta_i = (\eta'_E)|\cap \eta'$, take $\Pi'_i \prec \eta_i$, $i \notin L$, and set $\Pi' = \{(E_i, t_i) : i \in L\} \cup (\cup_{i \notin L} \Pi'_i)$. Then $\Pi' \prec \eta$, hence

$$\left| \int_{T} g \, d\mu - \sum_{\Pi'} g \right| \leq \bigvee_{i=1}^{\infty} a_{i, \varphi(i)},$$

and

$$\left| \sum_{i \notin L} \int_{E_i} g \, d\mu - \sum_{i \notin L} \sum_{\Pi'_i} g \right| \leq \bigvee_{k=1}^{\infty} b_{k, \psi(k)}.$$

Now

$$\left| \sum_{i \in L} \int_{E_i} g \, d\mu - \sum_{i \in L} g(t_i) \mu(E_i) \right|$$

$$= \left| \int_{T} g \, d\mu - \sum_{i \notin L} \sum_{\Pi'_i} \int_{E_i} g \, d\mu - \sum_{i \notin L} \sum_{\Pi'_i} g \right| + \sum_{i \in L} \sum_{\Pi'_i} g - \sum_{i \notin L} \sum_{\Pi'_i} \int_{E_i} g \, d\mu$$

$$\leq \bigvee_{i=1}^{\infty} a_{i, \varphi(i)} + \bigvee_{k=1}^{\infty} b_{k, \psi(k)}.$$

Since

$$\left| \sum_{i \in L} \int_{E_i} g \, d\mu - \sum_{i \in L} g(t_i) \mu(E_i) \right| - \bigvee_{i=1}^{\infty} a_{i, \varphi(i)} \leq \bigvee_{k=1}^{\infty} b_{k, \psi(k)}$$

for every $\psi : \mathbb{N} \to \mathbb{N}$, by weak $\sigma$-distributivity of $R$ we obtain:

$$\left| \sum_{i \in L} \int_{E_i} g \, d\mu - \sum_{i \in L} g(t_i) \mu(E_i) \right| - \bigvee_{i=1}^{\infty} a_{i, \varphi(i)} \leq 0. \quad \Box$$

Finally, we are ready to prove the monotone convergence theorem.
The Kurzweil-Henstock Integral for Riesz Space-Valued Maps

Theorem 4.3. Let \((f_n : T \to R_1)_n\) be a sequence of integrable functions, \(f_n \leq f_{n+1} (n \in \mathbb{N})\), and let the sequence \(\left( \int_T f_n \, d\mu \right)_n\) be bounded. Suppose that:

4.3.1) there exist a function \(f : T \to R_1\), a \((KH)\)-integrable map \(h^* : T \to \mathbb{R}^+\) (with respect to \(\mu\)) and a \((D)\)-sequence \((a_{i,j})_{i,j}\) such that, \(\forall \varphi \in \mathbb{N}^\mathbb{N}\), 
\(\forall t \in T\), \(\exists p(t) \in \mathbb{N}\): \(\forall n \geq p(t)\),
\[ |f(t) - f_n(t)| \leq h^*(t) \left( \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right). \] (7)

Furthermore, assume that

4.3.2) there exist \(a \in \mathbb{R}\), \(a \geq 0\), and a gauge \(\gamma^*\), such that, for every \(\gamma^*\)-fine partition \(\Pi\) of \(T\), we have:
\[ \left| \sum_{\Pi} f_n - \int_T f_n \, d\mu \right| \leq a \quad \forall n \in \mathbb{N}. \]

Then \(f\) is integrable on \(T\), and
\[ \int_T f \, d\mu = (D) \lim_n \int_T f_n \, d\mu. \]

Remark 4.4. We note that regularity does imply \(\sigma\)-additivity of \(\mu\), at least for \(R_2\)-valued positive finitely additive measures.

Moreover we observe that, if needed, a distinguished point \(x_0\) could be dropped from \(T\), provided it is possible to define \(f(x_0) = f_n(x_0) = 0\), \(n \in \mathbb{N}\), without affecting the other hypotheses.

Furthermore, we observe that, when \(R_1 = R_2 = R = \mathbb{R}\) and \(T = [A, +\infty]\) or \(T = [-\infty, B]\) is a halfline of the extended real line, condition 4.3.1) is equivalent to pointwise convergence of the sequence \((f_n)_n\) to \(f\): indeed, it is sufficient to take the function \(h^*\) defined by setting
\[ h^*(t) = \frac{1}{1 + t^2}, \quad t \in [A, +\infty) \text{ or } t \in (-\infty, B]. \]

Proof of Theorem 4.3: Since the sequence \(\left( \int_T f_n \, d\mu \right)_n\) is bounded and increasing, it admits the \((D)\)-limit in \(R\). Thus, there exists a \((D)\)-sequence \((c_{i,j})_{i,j}\) in \(R\) such that, for every \(\varphi \in \mathbb{N}^\mathbb{N}\), there exists \(k_0 \in \mathbb{N}\) such that, \(\forall k, l \geq k_0\),
\[ \left| \int_T f_k \, d\mu - \int_T f_l \, d\mu \right| \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}. \] (8)

Furthermore, from 4.3.1) we get the existence of an element \(0 \leq w \in R_2\) such that \(\forall \varphi \in \mathbb{N}^\mathbb{N}\) there exists a gauge \(\gamma^*\) such that, for every \(\gamma^*\)-fine partition \(\Pi\) of \(T\), \(\Pi = \{(E_i, t_i) : i =

A. Boccuto and B. Riečan

1, \ldots, q}, we have:
\[ \sum_{i} \left[ f(t_i) - f_{p(t_i)}(t_i) \right] \mu(E_i) \leq \sum_{i} h^*(t_i) \left( \bigvee_{i_1, \varphi(i_1)} a_{i, \varphi(i)} \right) \mu(E_i) \]
\[ \leq \left( \bigvee_{i=1}^{\infty} a_{i, \varphi(i)} \right) w. \]  
(9)

Note that in (9) the natural numbers \( p(t_i) \) can be chosen greater than \( k_0 \). Since \( f_k \) is integrable \( \forall k \in \mathbb{N} \), then for each \( k \in \mathbb{N} \) there exists a \((D)\)-sequence \((a_{i,j}^{(k)})_{i,j}\) such that, for every \( \varphi \in \mathbb{N}^\mathbb{N} \), there exists a gauge \( \gamma_k \) such that for every partition \( \Pi \prec \gamma_k \) we have
\[ \left| \sum_{\Pi} f_k - \int f_k d\mu \right| \leq \left( \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}^{(k)} \right). \]  
(10)

For each \( i,j \in \mathbb{N} \), put \( b_{1}^{(i,j)} = 2a_{i,j} w \), and \( b_{m}^{(i,j)} = a_{i,j}^{(m-1)} \) \((m = 2, 3, \ldots)\). Moreover, let \( a \) be as in 4.3.2). By virtue of the Fremlin lemma there exists a \((D)\)-sequence \((b_{i,j})_{i,j}\) such that, \( \forall \varphi \in \mathbb{N}^\mathbb{N} \) and \( \forall s \in \mathbb{N} \),
\[ a \wedge \left( \sum_{m=1}^{s} \left( \bigvee_{i=1}^{\infty} a_{i, \varphi(i+m)}^{(m)} \right) \right) \leq \left( \bigvee_{i=1}^{\infty} b_{i, \varphi(i)} \right). \]  
(11)

Let \( \varphi \in \mathbb{N}^\mathbb{N} \) and \( k_0 = k_0(\varphi) \) be as in (8). Put
\[ \gamma_0(t) = \gamma^*(t) \cap \gamma(t) \cap \gamma_1(t) \cap \gamma_2(t) \cap \ldots \cap \gamma_{p(t)}(t), \]
where the involved gauges are the ones associated with \( \varphi \), as above. Choose a partition \( \Pi \prec \gamma_0 \), \( \Pi = \{(E_i, t_i) : i = 1, \ldots, q\} \). Fix arbitrarily \( k > k_0 \), where \( k_0 \) is as in (8). We have:
\[ \left| \sum_{\Pi} f_k - \int f_k d\mu \right| \leq \left| \sum_{p(t_i) \geq k} f_k(t_i) \mu(E_i) - \sum_{p(t_i) \geq k} \int_{E_i} f_k d\mu \right| + \left| \sum_{p(t_i) < k} f_k(t_i) \mu(E_i) - \sum_{p(t_i) < k} \int_{E_i} f_k d\mu \right|. \]  
(12)

Construct \( \bar{\Pi} \) similarly as in the proof of the Henstock lemma, i.e.
\[ \bar{\Pi} = \{(E_i, t_i) : k \leq p(t_i)\} \cup \left( \bigcup_{p(t_i) < k} \Pi_i \right), \]
where \( \Pi_i \) is a sufficiently fine partition of \( E_i \), in such a way that \( \bar{\Pi} \prec \gamma_k \). Then
\[ \left| \sum_{\Pi} f_k - \int f_k d\mu \right| \leq \left( \bigvee_{i=1}^{\infty} a_{i, \varphi(i+k+1)}^{(k)} \right). \]
Hence, by the Henstock lemma \((L = \{i : p(t_i) \geq k\} \text{ and } L = \{i : p(t_i) = k\})\), we obtain

\[
\left| \sum_{k} f_k(t_i) \mu(E_i) - \sum_{i \in L} \int_{E_i} f_k d\mu \right| \leq \sum_{i=1}^{\infty} a_{i,\varphi(i+k+1)}^{(i+k+1)}
\]

and

\[
\left| \sum_{k} f_k(t_i) \mu(E_i) - \sum_{k} \int_{E_i} f_k d\mu \right| \leq \sum_{i=1}^{\infty} a_{i,\varphi(i+k+1)}^{(i+k+1)}
\]

respectively. We now estimate the second part of the right side of (12). We have:

\[
\left| \sum_{k} f_k(t_i) \mu(E_i) - \sum_{k} \int_{E_i} f_k d\mu \right|
\]

\[
\leq \sum_{m=k_0}^{k-1} \sum_{m=k_0}^{k} f_k(t_i) \mu(E_i) - \sum_{m=k_0}^{k-1} \sum_{m=k_0}^{k} f_p(t_i) \mu(E_i) + \] \[
+ \sum_{m=k_0}^{k-1} \sum_{m=k_0}^{k} f_p(t_i) \mu(E_i) - \sum_{m=k_0}^{k-1} \sum_{m=k_0}^{k} \int_{E_i} f_p(t_i) d\mu
\]

\[
+ \sum_{m=k_0}^{k-1} \sum_{m=k_0}^{k} \int_{E_i} (f_k - f_m) d\mu
\]

\[
\leq \sum_{m=k_0}^{k-1} \sum_{m=k_0}^{k} (f_k(t_i) - f_p(t_i)) \mu(E_i)
\]

\[
+ \sum_{m=k_0}^{k-1} \sum_{m=k_0}^{k} f_m(t_i) \mu(E_i) - \sum_{m=k_0}^{k} \int_{E_i} f_m d\mu
\]

\[
+ \sum_{m=k_0}^{k-1} \sum_{m=k_0}^{k} \int_{E_i} (f_k - f_m) d\mu
\]

\[
\leq \sum_{i=1}^{k} b_{i,\varphi(i+1)}^{(i+1)} + \sum_{m=k_0}^{k-1} \sum_{i=1}^{\infty} a_{i,\varphi(i+m+1)}^{(i+m+1)} + \int_{T} (f_k - f_m) d\mu
\]

\[
\leq \sum_{i=1}^{k} b_{i,\varphi(i+1)}^{(i+1)} + \sum_{m=2}^{k} \sum_{i=1}^{\infty} b_{i,\varphi(i+m)}^{(i+m)} + \int_{T} (f_k - f_m) d\mu
\]

\[
= \sum_{m=1}^{k} \left( \sum_{i=1}^{\infty} b_{i,\varphi(i+m)}^{(i+m)} \right) + \int_{T} (f_k - f_m) d\mu.
\]

Thus, from 4.3.2), (8), (11) and (15) we get the existence of a \((D)\)-sequence \((d_{i,j})_{i,j}\) such that, for every \(\varphi \in \mathbb{N}^{\mathbb{N}}\), there exist a gauge \(\gamma_0\) and \(k_0 \in \mathbb{N}\) such that, for each \(\gamma_0\)-fine
partition Π and ∀k > k₀, we have:

$$\left|\sum_{\Pi} f_k - \int_T f_k d\mu\right| \leq \bigvee_{i=1}^{\infty} d_{i,\varphi(i)}.$$  \hspace{1cm} (16)

The assertion follows from Lemma 4.1. □

We now state and prove a version of the Lebesgue dominated convergence theorem.

**Theorem 4.5.** Let \((f_n : T \rightarrow R_1)\) be a sequence of integrable functions, and suppose that \(\kappa : T \rightarrow R_1\) is an integrable map, such that \(|f_n(x)| \leq \kappa(x)\) for all \(x \in T\) and \(n \in \mathbb{N}\). Suppose that:

4.5.1) there exist a function \(f : T \rightarrow R_1\), a \((KH)\)-integrable map \(h^* : T \rightarrow \mathbb{R}^+\) (with respect to \(\mu\)) and a \((D)\)-sequence \((a_{i,j})_{i,j}\) such that, \(\forall \varphi \in \mathbb{N}^\mathbb{N}, \forall t \in T, \exists p(t) \in \mathbb{N}: \forall n \geq p(t),\)

$$|f_n(t) - f(t)| \leq h^*(t) \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right).$$  \hspace{1cm} (17)

Then \(f\) is integrable and

$$\int_T f d\mu = (D) \lim_n \int_T f_n d\mu.$$

**Proof:** For all \(s \in \mathbb{N}\) and \(k \geq s\), put

$$g_{s,k} = \bigvee_{n,m \geq s, \min(n,m) \leq k} |f_n - f_m|;$$

moreover, for each \(s \in \mathbb{N}\), set

$$g_s = \bigvee_{n,m \geq s} |f_n - f_m|.$$

We shall prove that, for each fixed \(s \in \mathbb{N}\), the sequence \((g_{s,k})_{k \geq s}\) satisfies the hypothesis of Theorem 4.3.

First of all, it is easy to check that the sequence

$$\left(\int_T g_{s,k} d\mu\right)_k$$

is well-defined and bounded in \(R\) (Indeed, it is possible to check that the \(g_{s,k}\)'s are integrable, taking into account that \(\kappa\) is integrable and proceeding analogously as in [1], Theorem 4.33 and [8], Lemma 4.2).

Let now \(h^*\) and \((a_{i,j})_{i,j}\) be as in 4.5.1). We know that, \(\forall \varphi \in \mathbb{N}^\mathbb{N}, \forall t \in T, \forall s \in \mathbb{N}, \exists p \in \mathbb{N}, \text{ with } p \geq s, \text{ such that, } \forall k \geq p,\)

$$\bigvee_{n,m \geq k} |f_n(t) - f_m(t)| \leq 2 h^*(t) \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right).$$  \hspace{1cm} (18)
Fix arbitrarily $s \in \mathbb{N}$. We have, for every $t \in T$:

$$\bigvee_{n,m \geq s} |f_n(t) - f_m(t)|$$

$$= \left( \bigvee_{n,m \geq s, \min(n,m) \leq k} |f_n(t) - f_m(t)| \right) \bigvee_{n,m \geq s, \min(n,m) \geq k} |f_n(t) - f_m(t)|$$

$$\leq \left( \bigvee_{n,m \geq s, \min(n,m) \leq k} |f_n(t) - f_m(t)| \right) + \left( \bigvee_{n,m \geq k} |f_n(t) - f_m(t)| \right),$$

and hence

$$0 \leq g_s(t) - g_{s,k}(t) \leq \bigvee_{n,m \geq k} |f_n(t) - f_m(t)| \quad \forall k \geq s, \forall t \in T.$$
that is
\[ g_s(t) = |g_s(t)| \leq h^*(t) \left( \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right). \]

So, 4.3.1) is satisfied. Concerning 4.3.2), it is enough to check that the argument in (19) works even if we replace
\[ \bigvee_{n,m \geq s, \min(n,m) \leq k} |f_n(t_i) - f_m(t_i)| \]
with
\[ \bigvee_{n,m \geq s} |f_n(t_i) - f_m(t_i)|. \]

Thus, we get that
\[
(D \lim_s \int_T g_s \, d\mu = \bigwedge_{s \in \mathbb{N}} \int_T g_s \, d\mu = 0. \tag{20}
\]

Proceeding analogously as in the proof of Theorem 4.3, it is possible to prove the existence of (D)-sequences \((e_{i,j}^{(m)}))_{i,j}, m \in \mathbb{N}\), such that, \(\forall \varphi \in \mathbb{N}^\mathbb{N}\), there exist a gauge \(\gamma'\) and \(k' \in \mathbb{N}\) such that, for each \(\gamma'\)-fine partition \(\Pi = \{ (E_i, t_i), i = 1, \ldots, q \}, \forall k > k'\), we have:
\[
\left| \sum_{\Pi} f_k - \int_T f_k \, d\mu \right| \\
\leq \sum_{m=1}^{k} \left( \bigvee_{i=1}^{\infty} e_{i,\varphi(i+m)}^{(m)} \right) + \sum_{m=k'}^{k-1} \sum_{p(t_i) = m} \left| \int_{E_i} (f_k - f_m) \, d\mu \right| \tag{21}
\]
\[
\leq \sum_{m=1}^{k} \left( \bigvee_{i=1}^{\infty} e_{i,\varphi(i+m)}^{(m)} \right) + \int_T g_{k'} \, d\mu.
\]

From (21) we get the existence of a (D)-sequence \((d_{i,j}^{(m)})_{i,j}\) such that, \(\forall \varphi \in \mathbb{N}^\mathbb{N}\), there exist a gauge \(\gamma'\) and \(k' \in \mathbb{N}\) such that, for each \(\gamma'\)-fine partition \(\Pi, \forall k > k'\), we have:
\[
\left| \sum_{\Pi} f_k - \int_T f_k \, d\mu \right| \leq \bigvee_{i=1}^{\infty} d_{i,\varphi(i)}. \tag{22}
\]

The assertion follows from (22) and Theorem 4.1. \(\square\)

References


