A note on the improper Kurzweil-Henstock integral

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ABSTRACT. A connection is studied between the improper Kurzweil-Henstock integral on the real line and the integral over a compact space.


KEY WORDS: Kurzweil-Henstock construction, improper integral, compact spaces.

1 Introduction.

In [5] two possibilities are mentioned of defining the improper Kurzweil-Henstock integral on the real line (see also [2] for a more general range). In [1] and [6] the Kurzweil-Henstock construction has been examined for a general compact range. It is natural to consider one-point compactification of the real line. Therefore we work with the compactification and we prove a convergence theorem in compact spaces describing the situation from the real case.

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2 Kurzweil-Henstock integral in compact topological spaces.

Let \( \mathbb{N} \) be the set of all strictly positive integers, \( \mathbb{R} \) the set of the real numbers, \( \mathbb{R}^+ \) be the set of all strictly positive real numbers. Let \( X \) be a Hausdorff compact topological space. If \( A \subseteq X \), then the interior of the set \( A \) is denoted by \( \text{int} \, A \).

We shall work with a family \( \mathcal{F} \) of compact subsets of \( X \) such that \( X \in \mathcal{F} \) and closed under the intersection and finite union, and a monotone and additive mapping \( \lambda : \mathcal{F} \to [0, +\infty] \). The additivity means that
\[
\lambda(A \cup B) + \lambda(A \cap B) = \lambda(A) + \lambda(B)
\]
whenever \( A, B, A \cup B \in \mathcal{F} \).

By a partition (detailed, \((\mathcal{F}, \lambda)\)-partition ) of a set \( A \in \mathcal{F} \) we mean a finite collection \( \{ (\mathcal{U}_1, t_1), \ldots, (\mathcal{U}_k, t_k) \} \) such that

(i) \( \mathcal{U}_1, \ldots, \mathcal{U}_k \in \mathcal{F} \),

(ii) \( \bigcup_{i=1}^{k} \mathcal{U}_i = A \),

(iii) \( \lambda(\mathcal{U}_i \cap \mathcal{U}_j) = 0 \) whenever \( i \neq j \),

(iv) \( t_i \in \mathcal{U}_i \) (\( i = 1, \ldots, k \)).

A finite collection \( \{ (\mathcal{U}_1, t_1), \ldots, (\mathcal{U}_k, t_k) \} \) of subsets of \( A \in \mathcal{F} \), satisfying conditions (i), (iii) and (iv), but not necessarily (ii), is said to be decomposition of \( A \). We shall assume that \( \mathcal{F} \) separates points in the following way: to any \( A \in \mathcal{F} \) there exists a sequence \( (\mathcal{A}_n) \) of partitions of \( A \) such that

(i) \( \mathcal{A}_{n+1} \) is a refinement of \( \mathcal{A}_n \),

(ii) to any \( x, y \in A, x \neq y \), there exist \( n \in \mathbb{N} \) and \( B \in \mathcal{A}_n \) such that \( x \in B \) and \( y \notin B \).

We note that this assumption is fulfilled if the topological space \( X \) is metrizable or it satisfies the second axiom of countability (see [6]).
A gauge on a set $A \subset X$ is a mapping $\delta$ assigning to every point $x \in A$ a neighborhood $\delta(x)$ of $x$. If $D = \{(U_1, t_1), \ldots, (U_k, t_k)\}$ is a decomposition of $A$ and $\delta$ is a gauge on $A$, then we say that $D$ is $\delta$-fine if $U_i \subset \delta(t_i)$ for any $i \in \{1, 2, \ldots, k\}$.

We obtain a simple example putting $X = [a, b] \subset \mathbb{R}$ with the usual topology, $\mathcal{F} =$ the family of all finite unions of closed subintervals of $X$, $\lambda([\alpha, \beta]) = \beta - \alpha$, $a \leq \alpha < \beta \leq b$. Any gauge can be represented by a real function $d : [a, b] \rightarrow \mathbb{R}^+$, if we put $\delta(x) = (x - d(x), x + d(x))$.

Another example is the unbounded interval $[a, +\infty] = [a, +\infty) \cup \{+\infty\}$ considered as the one-point compactification of the locally compact space $[a, +\infty)$. The base of open sets consists on open subsets of $[a, +\infty)$ and the sets of the type $(b, +\infty) \cup \{+\infty\}$, $a \leq b < +\infty$. Any gauge in $[a, +\infty]$ has the form $\delta(x) = (x - d(x), x + d(x))$, if $x \in [a, +\infty] \cap \mathbb{R}$, and $\delta(+\infty) = (b, +\infty] = (b, +\infty) \cup \{+\infty\}$, where $d$ denotes a positive real-valued function defined on $[a, +\infty)$, and $b$ denotes a real number.

Let us return to the definition of Kurzweil-Henstock integral (KH-integral) on $X$. If $D = \{(U_1, t_1), \ldots, (U_k, t_k)\}$ is a decomposition of a set $A$, and $f : X \rightarrow \mathbb{R}$, then we define the Riemann sum as follows:

$$S(f, D) = \sum_{i=1}^{k} f(t_i) \lambda(U_i),$$

if the sum exists in $\mathbb{R}$, with the convention $0 \cdot (+\infty) = 0 \cdot (-\infty) = 0$.

We note that the fact that $\mathcal{F}$ separates points guarantees the existence of at least one $\delta$-fine partition $D$ such that $S(f, D)$ is well-defined for any gauge $\delta$ (see [6], [8]).

**Definition 2.1** A function $f : X \rightarrow \mathbb{R}$ is integrable on a set $A$ if there exists $I \in \mathbb{R}$ such that $\forall \varepsilon > 0$ there exists a gauge $\delta$ on $A$ such that

$$|S(f, D) - I| \leq \varepsilon$$

whenever $D$ is a $\delta$-fine partition of $A$ such that $S(f, D)$ exists in $\mathbb{R}$. We denote

$$I = \int_A f$$

(see also [6], Definition 1.8., p. 154).
3 The convergence theorem.

We now prove the following:

**Theorem 3.1** Let \( X = X_0 \cup \{x_0\} \) be the one-point compactification of a locally compact space \( X_0 \). Let \( f : X \to \mathbb{R} \) be a function such that \( f(x_0) = 0 \). Let \((A_n)_n\) be a sequence of sets, such that \( A_n \in \mathcal{F}, A_n \subset \text{int} A_{n+1}, A_{n+1} \setminus \text{int} A_n \in \mathcal{F}, \) \( \lambda(A_n \setminus \text{int} A_n) = 0 \) \((n = 1, 2, \ldots)\), \( \bigcup_{n=1}^{\infty} A_n = X_0 \). Let \( f \) be integrable on every \( A \in \mathcal{F}, \) with \( A \subset X_0 \), and let there exist in \( \mathbb{R} \) an element \( I \) such that, \( \forall \varepsilon > 0 \), there exists an integer \( n_0 \) such that

\[
\left| \int_A f - I \right| \leq \varepsilon \quad \forall A \in \mathcal{F}, X_0 \supset A \supset A_{n_0}.
\]

Then \( f \) is integrable on \( X \) and \( \int_X f = I \).

**Proof:** Let \( \varepsilon \) be an arbitrary positive real number, and \( n_0 \in \mathbb{N} \) be as in the hypotheses of the theorem. Put \( A_0 = \emptyset, B_n = A_{n+1} \setminus \text{int} A_n \) \((n = 1, 2, \ldots)\). For all \( n \in \mathbb{N} \) there exists a gauge \( \delta_n \) on \( B_n \) such that

\[
\left| \int_{B_n} f - S(f, D_n) \right| \leq \frac{\varepsilon}{2^{n+3}}
\]

for any \( \delta_n \)-fine partition \( D_n \) of \( B_n \). From (3) and Henstock’s Lemma (see also [6], Lemma 2.1., pp. 158-159; [5], Theorem 3.2.1., pp. 81-83), it follows that

\[
\left| \int_{\bigcup_{i=1}^{h} V_i} f - S(f, E_n) \right| \leq \frac{\varepsilon}{2^{n+2}}
\]

for each \( \delta_n \)-fine decomposition \( E_n = \{(V_1, t_1), \ldots, (V_h, t_h)\} \) of \( B_n \). Evidently

\[
B_n \cap B_{n-1} = A_n \setminus \text{int} A_n \quad \forall n \in \mathbb{N}.
\]

Therefore

\[
B_n = (B_n \cap B_{n-1}) \cup (\text{int} B_n) \cup (B_n \cap B_{n+1}) \quad \forall n.
\]

Moreover, it is easy to check that

\[
B_j \cap B_l = \emptyset \text{ whenever } |j - l| \geq 2
\]
and that
\[(\text{int } B_n) \cap (\text{int } B_{n+1}) = \emptyset \quad \forall n \in \mathbb{N}. \tag{6}\]

Now define a gauge \(\delta\) on \(X\) by the following formula:
\[
\delta(x) = \begin{cases}
\delta_n(x) \cap (\text{int } B_n) & \text{if } x \in \text{int } B_n, \\
\delta_n(x) \cap \delta_{n+1}(x) \cap (\text{int } A_{n+1}) & \text{if } x \in B_n \cap B_{n+1}, \quad (n = 1, 2, \ldots) \\
(X_0 \setminus A_{n_0}) \cup \{x_0\} & \text{if } x = x_0.
\end{cases} \tag{7}\]

Let \(D = \{(U_1, t_1), \ldots, (U_k, t_k)\}\) be a \(\delta\)-fine partition of \(X\). There exists \((U_{i_0}, t_{i_0}) \in D\), with \(i_0 \in \{1, 2, \ldots, k\}\), such that \(x_0 \in U_{i_0}\). We shall prove that \(t_{i_0} = x_0\). Namely, in the opposite case,

\[x_0 \in U_{i_0} \subset \delta(t_{i_0}) \subset \delta_n(t_{i_0}) \quad \text{for some } n. \]

But \(\delta_n(t) \subset X_0\) for \(t \neq x_0\). We have obtained \(x_0 \in X_0\), that is a contradiction.

Since \(f(x_0) = 0\), the Riemann sum \(S(f, D)\) has the form
\[
\sum_{i=1,\ldots,k,i \neq i_0} f(t_i) \lambda(U_i),
\]
and \(t_i \in X_0 \quad (i = 1, \ldots, k, i \neq i_0).\) Let
\[
A = \bigcup_{i=1,\ldots,k,i \neq i_0} U_i,
\]
and
\[
T = \{n \in \mathbb{N} : \exists i \in \{1,\ldots,k\}, i \neq i_0 : B_n \cap U_i \neq \emptyset\}. \tag{8}\]

By (7), and since \(D\) is a \(\delta\)-fine partition of \(X\), we get that
\[
X_0 \supset A \supset A_{n_0}. \tag{9}\]

By hypothesis we have
\[
\left| \int_A f - I \right| \leq \varepsilon. \tag{10}\]
We claim that, if $U_i$, $i \neq i_0$, has nonempty intersection with at least two of the $int B_n$'s, then necessarily there exists $n \in \mathbb{N}$ such that the point $t_i$ corresponding to $U_i$ belongs to $B_n \cap B_{n+1}$. Indeed, if $t_i \in int B_n$ for some $n$, then, from (7) and the fact that $D$ is a $\delta$-fine partition of $X$, we'd have

$$U_i \subset \delta(t_i) \subset int B_n,$$

this is impossible, by virtue of (5) and (6). From this and since

$$(B_{n-1} \cap B_n) \cap (B_n \cap B_{n+1}) = \emptyset \quad \forall n,$$

it follows that, for every $i = 1, 2, \ldots, k$, $i \neq i_0$, the $B_n$'s having nonempty intersection with $U_i$ are at most two, while the $B_n$'s which have nonempty intersection with $U_{i_0}$ can be infinitely many (even all the $B_n$'s). Thus we proved that the set $T$ in (8) is finite.

For $n \in T$ define a decomposition $E_n$ of $B_n$ in the following way:

$$E_n = \{ (U_i, t_i) : t_i \in int B_n \}$$

$$\cup \{ (U_i \cap B_n, t_i) : t_i \in B_n \cap B_{n-1} \}$$

$$\cup \{ (U_i \cap B_n, t_i) : t_i \in B_n \cap B_{n+1} \}.$$

Then, by construction, we have:

$$S(f, D) = \sum_{n \in T} S(f, E_n), \quad (11)$$

by additivity of $\lambda$ and since $A_n \setminus int A_n = B_n \cap B_{n+1} \subset int A_{n+1}$ and $\lambda(A_n \setminus int A_n) = 0 \ \forall n \in \mathbb{N}$. Similarly,

$$\sum_{n \in T} \int_{\cup_{U_i \subset int B_{n \neq i_0}} U_i} f = \int_A f. \quad (12)$$

Since $D_n$ is $\delta_n$-fine, we have (3). From (3), (10), (11) (12) and (9) we obtain:

$$|S(f, D) - I| = \left| \sum_{n \in T} S(f, E_n) - I \right|$$

$$= \left| \sum_{n \in T} \left( S(f, E_n) - \int_{\cup_{U_i \subset int B_{n \neq i_0}} U_i} f \right) \right|.$$
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\[ + \sum_{n \in T} \left| \int_{\bigcup_{i \in \mathbb{N}, i \neq i_0} U_i} f - I \right| \]

\[ \leq \sum_{n \in T} \left| S(f, \mathcal{E}_n) - \int_{\bigcup_{i \in \mathbb{N}, i \neq i_0} U_i} f \right| + \left| \int_A f - I \right| \]

\[ \leq \sum_{n \in T} \frac{\varepsilon}{2^{n+2}} + \varepsilon < 2\varepsilon. \]

From this the assertion follows. \( \square \)

4 Applications.

The following results are consequences of Theorem 3.1:

**Proposition 4.1** ([5], Theorem 2.9.3., pp. 61-63) Let \( f : [a, +\infty] \to \mathbb{R} \) be such that \( f(+\infty) = 0 \), \( f \) be integrable on \([a, b]\) for any \( b > a \), and let there exist in \( \mathbb{R} \) the limit

\[ \lim_{b \to +\infty} \int_{[a, b]} f. \]

Then \( f \) is integrable on \([a, +\infty]\), and

\[ \int_{[a, +\infty]} f = \lim_{b \to +\infty} \int_{[a, b]} f. \]

**Proposition 4.2** (see also [5], Theorem 2.8.3., pp. 57-59 and Remark 2.8.4, p. 57) Let \( a, b \in \mathbb{R}, a < b \), \( f : [a, b] \to \mathbb{R} \), \( f \) be integrable on \([a, x]\) for any \( a \leq x < b \), and let there exist in \( \mathbb{R} \) the limit

\[ \lim_{x \to b^-} \int_{[a, x]} f. \]

Then \( f \) is integrable on \([a, b]\), and

\[ \int_{[a, b]} f = \lim_{x \to b^-} \int_{[a, x]} f. \]

**Proof:** We observe that \([a, b] = [a, b) \cup \{b\}\) can be considered as the one-point compactification of \([a, b]\). The only difference is that we did not assume \( f(b) = 0 \). Of course, one can put \( g(x) = f(x) - f(b) \), and use Theorem 3.1 with respect to the function \( g \). Then we have

\[ \int_{[a, b]} g = \lim_{x \to b^-} \int_{[a, x]} g. \]
and hence

\[
\int_{[a,b]} f = f(b)(b - a) + \int_{[a,b]} g \\
= \lim_{x \to b^-} f(b)(x - a) + \lim_{x \to b^-} \int_{[a,x]} g \\
= \lim_{x \to b^-} \int_{[a,x]} (g + f(b)) = \lim_{x \to b^-} \int_{[a,x]} f.
\]

This concludes the proof. □

References


