Kondurar Theorem in Riesz spaces

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Abstract

In this paper we give a version of the Kondurar theorem for functions taking values in Riesz spaces with respect to a general notion of convergence. If the Riesz space involved is a space of random variables, the most common types of convergence are included. Some comments and possible applications conclude the paper.

Key words: Riesz spaces, convergence, Riemann-Stieltjes integration, Kondurar theorem, Itô formula.

1 Introduction

In this paper we investigate the Kondurar theorem in Riesz spaces. The classical (scalar) version of the Kondurar theorem asserts that, whenever \( f \) and \( g \) are two real-valued functions, defined on a compact interval \( [a, b] \subset \mathbb{R} \), the Riemann-Stieltjes integral \( \int_{a}^{b} f \, dg \) exists, as soon as \( f \) and \( g \) are Hölder-continuous, with orders \( \alpha \) and \( \beta \) respectively, with \( \alpha + \beta > 1 \). Here we have chosen to adopt the harmless restriction that \( [a, b] = [0, 1] \), and to fix the Riesz space setting as a product triple \((R_1, R_2, R)\) such that \( f \) is \( R_1 \)-valued, \( g \) is \( R_2 \)-valued, and the integral takes values in \( R \). We formulated Riemann-Stieltjes integration with respect to an interval function \( q \), because this is a more general approach, and in our opinion is more suitable for applications.

Beyond the differences deriving from the Riesz space setting, and from the general notion of convergence here introduced \(((L)\)-convergence), our approach differs from the classical one (see [5]) because we chose to split the main proof into a number of partial results, whose purpose is to reduce the proof to considering very special decompositions (decompositions with rational points, decompositions with intervals of the same length, dyadic decompositions). We think that this approach better clarifies the importance of continuity properties, and explains the precise moment in which the crucial property \( \alpha + \beta > 1 \) plays its role (mainly, in dealing with dyadic decompositions). Moreover, the choice of a general interval function \( q \) makes the results more interesting even in the real-valued case.

In a previous paper (see [3]), a different (less general) version of Kondurar’s theorem has been found and applied in order to obtain some kinds of Itô formula; other consequences will be deduced in a forthcoming paper, so we don’t present applications here. We just observe that the classical stochastic integrals (Itô, Stratonovich, Backward) and the classical Itô formula are included. In Section 2 we introduce some preliminary definitions and notations and prove some results, needed for the subsequent investigations. In Section 3 we present our version of the Kondurar theorem, concluding with some comments.

2 Preliminaries

In this section we introduce the definitions used in proving the Kondurar theorem in
Riesz spaces. As we noticed in the Introduction, we shall consider only functions defined in the interval $[0, 1]$. A Riesz space $R$ is said to be **Dedekind complete** if every nonempty subset $A \subset R$, bounded from above, has supremum in $R$. A unit of a Riesz space $R$ is an element $u \in R$ such that $u \geq 0$, $u \neq 0$. From now on, we always suppose that $R$ is a Dedekind complete Riesz space. In such a Riesz space $R$, let $T$ be the set of all bounded sequences $(r_n)_n$ in $R$. In the literature, there exist several types of convergence in Riesz spaces (see [6], [9]). In this paper we give an axiomatic approach to convergence.

**Definition 2.1** A convergence is a pair $(\mathcal{S}, \mathcal{L})$, where $\mathcal{S}$ is a subspace of $T$ and $\mathcal{L}$ is a map $\mathcal{L} : \mathcal{S} \to R$, satisfying the following axioms:

a) $\mathcal{L}((\zeta_1 r_n + \zeta_2 r_n)_n) = \zeta_1 \mathcal{L}((r_n)_n) + \zeta_2 \mathcal{L}((s_n)_n)$ for every pair of sequences $(r_n)_n, (s_n)_n \in \mathcal{S}$ and for each $\zeta_1, \zeta_2 \in R$.

b) If $(r_n)_n$ satisfies $r_n = I$ definitely, then $(r_n)_n \in \mathcal{S}$ and $\mathcal{L}((r_n)_n) = I$.

c) Given three sequences, $(r_n)_n, (s_n)_n, (t_n)_n$, satisfying $(r_n)_n, (t_n)_n \in \mathcal{S}$, $\mathcal{L}((r_n)_n) = \mathcal{L}((t_n)_n) = r$, if $r_n \leq s_n \leq t_n$ definitely, then $(s_n)_n \in \mathcal{S}$ and $\mathcal{L}((s_n)_n) = r$.

d) If $u \in R$, $u \geq 0$, then the sequence $(\frac{1}{n}u)_n$ belongs to $\mathcal{S}$ and $\mathcal{L}((\frac{1}{n}u)_n) = 0$.

By means of this concept, we define also convergence for nets, as follows.

Given any bounded net $\Psi : \to \to R$, we say that it is $\mathcal{L}$-**convergent** to the limit $\lambda$, if there exists a positive sequence $(r_n)_n$ in $R$, belonging to $\mathcal{S}$ and satisfying $\mathcal{L}((r_n)_n) = 0$, such that for every $n$ it is possible to find an element $\xi \in \to$, $\xi = \xi(n)$, for which one has $|\Psi(\xi') - \lambda| \leq r_n$ for every $\xi' > \xi$.

It is easy to check that $(O)$-, $(r)$- and $(*)$-convergence satisfy axioms a), . . . , d) in 2.1 (see also [6]). Moreover, if $(X, \mathcal{B}, P)$ is a probability space, and $R = L^0(X, \mathcal{B}, P)$ is the Riesz space of all measurable functions, with the identification up to sets of probability zero, then $(O)$- and $(r)$-convergence coincide with almost everywhere convergence, and $(*)$-convergence is equivalent to convergence in probability (see [9]). We now introduce some structural assumptions, which will be needed in the construction of the Riemann-Stieltjes integral.

**Assumptions 2.2** Let $R_1$, $R_2$, $R$ be three Dedekind complete Riesz spaces. We say that $(R_1, R_2, R)$ is a product triple if there exists a map $\cdot : R_1 \times R_2 \to R$, which we will call product, such that

2.2.1) $(r_1 + s_1) \cdot r_2 = r_1 \cdot r_2 + s_1 \cdot r_2$,

2.2.2) $r_1 \cdot (r_2 + s_2) = r_1 \cdot r_2 + r_1 \cdot s_2$,

2.2.3) $[r_1 \geq s_1, r_2 \geq 0] \Rightarrow [r_1 \cdot r_2 \geq s_1 \cdot r_2]$,

2.2.4) $[r_1 \geq 0, r_2 \geq s_2] \Rightarrow [r_1 \cdot r_2 \geq r_1 \cdot s_2]$ for all $r_j, s_j \in R_j, j = 1, 2$.

2.2.5) if $\Lambda$ is any nonempty directed set, $(a_\lambda)_{\lambda \in \Lambda}$ is any net in $R_2$ and $b \in R_1$, then $[a_\lambda \downarrow 0, b \geq 0] \Rightarrow [b \cdot a_\lambda \downarrow 0]$;

2.2.6) if $(a_\lambda)_{\lambda \in \Lambda}$ is any net in $R_1$, then $b, b \geq 0] \Rightarrow [a_\lambda \cdot b \downarrow 0]$.

We now give some examples of product triples:

1) $R_1 = R_3 = R$ is a Dedekind complete Riesz space, $R_2 = R$, and the product involved is the "scalar product";

2) $R_3 = R_2 = R = \mathcal{B}(\Omega)$, where $\mathcal{B}(\Omega)$ is the space of all real-valued bounded functions defined on $\Omega$, $\Omega$ is any nonempty set (clearly, this is representative of many function spaces, such as $L_p$ spaces and so on);

3) $R_3 = X, R_2 = L(X,Y), R_3 = Y$, where $X$ and $Y$ are two Dedekind complete Riesz spaces, and $L(X,Y)$ is the space of all (order) continuous linear operators, defined on $X$ and taking values in $Y$. In particular, this includes the case where $R_1$ is a Banach lattice, $R_2$ is the dual space of $R_1$, and $R_3$ is $\mathcal{B}R$.

Now, let us denote by $\{I\}$ the family of all (nontrivial) closed subintervals of the interval $[0, 1]$, and by $\mathcal{D}$ the family of all finite decompositions of $[0, 1]$ into pairwise nonoverlapping elements from $\{I\}$. Usually, any such decomposition is denoted by a symbol like $D$, and we shall write $D = \{0 \leq t_0 < t_1 < \ldots < t_{n-1} < t_n = 1\}$ or also $D = \{I_1, \ldots, I_n\}$,
where $I_i = [t_{i-1}, t_i]$ for each $i = 1, \ldots, n$.

Given any decomposition $D = \{I_1, \ldots, I_n\}$, the mesh of $D$ is the number $\delta(D) := \max\{|I_i| : i = 1, \ldots, n\}$, where $|I|$ as usual denotes the length of the interval $I$. Clearly the mesh defines a natural filtering order relation among all decompositions: we shall say that a decomposition $D_1$ precedes a decomposition $D_2$ if $\delta(D_2) \leq \delta(D_1)$.

Another different ordering will be considered: for any two decompositions, $D_1, D_2$, we say that $D_2$ is finer than $D_1$ if every interval from $D_2$ is entirely contained in some interval from $D_1$. We shall denote this fact by writing $D_2 \text{refines} D_1$ ($D_2$ refines $D_1$). Moreover, we shall say that a decomposition $D$ is rational if all its endpoints are rational, and that a decomposition $D$ is equidistributed if all intervals in $D$ have the same length (clearly, if $D$ is equidistributed, then it is also rational). In case $D$ is equidistributed, and consists of $2^n$ intervals, we shall say that $D$ is dyadic of order $n$. Of course, equidistributed and dyadic decompositions of any interval $[a, b]$ are similarly defined. In general, the family of all decompositions of an interval $[a, b]$ will be denoted by $D_{[a,b]}$.

We now introduce the concept of integral of an interval function $q$, according with the definition given in [3]. We shall skip the proof of the main properties of such integral, because they can be found in [3].

**Definition 2.3** Let $L$ be a convergence as in 2.1. Assume that $q : \{I\} \rightarrow R$ is any interval function. We say that $q$ is $(L)$-integrable, if there exists an element $Y \in R$ such that $L(\sup\{|Y - \sum_{I \in D} q(I)} : D \in D, \delta(D) \leq \frac{1}{n}\}) = 0$. The element $Y$ is called the $(L)$-integral of $q$, and, when no confusion can arise, we will write $Y := \int q$. Recalling our definition of $L$-convergence for nets, the integral $Y$ is nothing but the $L$-limit of the net $S(D) = \sum_{I \in D} q(I)$ where the decompositions are ordered according to the mesh: $D_1 > D_2$ iff $\delta(D_1) \leq \delta(D_2)$.

**Definition 2.4** We say that an interval function $q$ is Hölder-continuous if there exist a unit $u \in R$ and a positive real constant $\gamma$, such that $|q(I)| \leq |I|^\gamma u$ for all subintervals $I \subset [0, 1]$. If this is the case, we say also that $q$ is Hölder-continuous of order $\gamma$.

A very useful tool to prove $(L)$-integrability is the Cauchy property, according with the next theorem: a more complete formulation (and a proof) is given in [3] (see also [4]).

**Theorem 2.5** Assume that $q : \{I\} \rightarrow R$ is any interval function. The following conditions are equivalent:

(i) $q$ is $(L)$-integrable;

(ii) the sequence $(r_n)_n := (\sup\{|S(D) - S(D_0) : D, D_0 \in D, \delta(D_0) \leq \frac{1}{n}, D \text{refines} D_0\})_n$ is $(L)$-convergent to $0$.

Other important facts are the following: if an interval function is integrable in $[a, b]$, then it is integrable in any subinterval, and the integral is an additive function. These results are summarized in the following theorem.

**Theorem 2.6** Let $q : \{I\} \rightarrow R$ be an integrable function. Then, for every subinterval $J \subset [0, 1]$, the function $q_J$ is integrable, where $q_J$ is defined as $q_J(I) = q(I \cap J)$, as soon as $I \cap J$ is nondegenerate, and 0 otherwise. Moreover, if $\{J_1, J_2\}$ is any decomposition of some interval $J \subset [0, 1]$, we have $\int_J q = \int_{J_1} q + \int_{J_2} q$ (Here, as usual, $\int_J q$ means the $(L)$-integral of $q_J$). Finally, denoting $Z(I) := q(I) - \int_J q$ for every interval $I$, it turns out that $Z$ is integrable, and it has null integral.

Let now $(R_1, R_2, R)$ be a product triple of Riesz spaces. Usually, given a bounded function $q : \{I\} \rightarrow R_2$, for every subinterval $[a, b] \subset [0, 1]$, we set: $\omega(q)([a, b]) := \sup\{|q([u, v])| : a \leq u < v \leq b\}$. The interval function $\omega(q)$ is called the oscillation of $q$. Moreover, given a bounded function $g : [0, 1] \rightarrow R_2$, we can associate with $g$ a bounded interval function $\Delta(g)$, as follows: $\Delta(g)([a, b]) := g(b) - g(a)$, and so we can also define $\omega(g)([a, b]) := \omega(\Delta(g))(\{a, b\}) = \sup\{|\Delta(g)([u, v])| : a \leq u < v \leq b\}$. $\Delta(g)([a, b])$ is also called the jump of $g$ in
If $[a, b]$, while $\omega(g)([a, b])$ is called the oscillation of $g$ in $[a, b]$. As usual, $g$ is said to be Hölder-continuous of order $\alpha$ (with $0 < \alpha \in \mathbb{R}$) if the function $\Delta(g)$ is.

Now we introduce the concept of Riemann-Stieltjes integral, with respect to an interval function $q$. Again, details and proofs can be found in [3].

**Definition 3.1** Let $f : [0, 1] \to R_1$ and $q : \{I\} \to R_2$ be two bounded functions. We say that $f$ is Hölder-continuous of order $\gamma$, with $\gamma > 0$, and $L$ is a convergence as in 2.1, then $f$ is Riemann-integrable.

3 The Kondurar theorem in Riesz spaces

We begin with the following condition;

**Definition 3.1** Let $f : [0, 1] \to R_1$ and $q : \{I\} \to R_2$ be two fixed functions. We say that $f$ and $q$ satisfy assumption (C) if the interval function $\phi : \{I\} \to R$, defined as $\phi(I) = \omega(f(I))\|q(I)|$, is $(L)$-integrable, and its integral is 0.

The following result is easy (see also [3]).

**Proposition 3.2** Let $f : [0, 1] \to R_1$ and $q : \{I\} \to R_2$ be two bounded functions, satisfying assumption (C). Then the $(L)$-Riemann-Stieltjes integral $(L)(RS) \int_0^1 f dq$ exists in $R$ if and only if the interval function $Q(I) = f(a_1)q(I)$ is $(L)$-integrable, where $a_1$ denotes the left endpoint of $I$.

In the next theorems, we shall assume in addition that the interval function $q$ is $(L)$-integrable, according with Definition 2.3. These theorems differ from similar results obtained in [3], because there $q$ is required to be additive (of course, if $q$ is additive, it is trivially integrable, but the converse in general is not true).

According with the previous theorem 2.6, the integral function $\psi(I) = \int_I q$ turns out to be additive, and the interval function $|Z| = |q - \psi|$ has null integral.

The next step will be proving that, provided $q$ is integrable, $f$ and $q$ are Hölder-continuous and assumption (C) is satisfied, existence of the $(L)$-Riemann-Stieltjes integral $(L)(RS) \int_0^1 f dq$ can be checked considering only rational decompositions.

**Proposition 3.3** Assume that $f : [0, 1] \to R_1$ and $q : \{I\} \to R_2$ are two Hölder-continuous functions, for which assumption (C) is satisfied. Suppose also that $q$ is $(L)$-integrable, and that $L((\rho_n)_n) = 0$, where $\rho_n = \sup\{\sum_{I \in D} f(a_1)q(I) - \sum_{I \in D_0} f(a_1)q(I) : D_0, D \in D, D_0, D$ rational, $\delta(D_0) \leq \frac{1}{n}, D \notin D_0 \}$ for every $n \in \mathbb{N}$. Then, there exists the $(L)(RS)$-integral of $f$ w.r.t. $q$.

**Proof.** Since assumption (C) is satisfied, from Proposition 3.2 it’s enough to prove that the $R$-valued interval function $Q(I) := f(a_1)q(I)$ (or, equivalently, $Q(I) := f(a_1)\psi(I)$) is $(L)$-integrable.

For every integer $n$, let us define $\sigma_n := \sup\{\sum_{I \in D} |Z(I)| : D \in D, \delta(D) \leq \frac{1}{n}\}$. As we already observed, $(\sigma_n)_n$ is $(L)$-convergent to 0. Moreover, by hypotheses, there exist two units $u_1 \in R_1, u_2 \in R_2$ and two positive real numbers, $\alpha$ and $\beta$, such
that \(|f(b) - f(a)| \leq u_1 |b - a|^{\alpha} \), \(|q([a, b])| \leq u_2 |b - a|^{\beta}\) for all \([a, b] \subset [0, 1]\). It is easy to deduce that \(|f|\) and \(|q|\) are bounded, so we can denote by \(F\) and \(G\) two majorants for them, respectively. As usual, for every decomposition \(D\) of \([0, 1]\), we denote \(S(D) = \sum_{I \in D} Q(I) = \sum_{I \in D} f(a_I) q(I)\), and \(\overline{S}(D) = \sum_{I \in D} \overline{Q}(I) = \sum_{I \in D} f(a_I) \psi(I)\). The main step in the proof will be the following Claim.

**Claim:** For every \(n \in \mathbb{N}\) and every \(D \in \mathcal{D}\) with \(\delta(D) < \frac{1}{n}\), there exists a rational decomposition \(D_0\) such that \(\delta(D_0) < \frac{1}{n}\) and

\[
|\overline{S}(D) - \overline{S}(D_0)| \leq \frac{F u_2 + G u_1}{n} + 2F \sigma_n. \tag{1}
\]

To this aim, fix \(n \in \mathbb{N}\), and take a decomposition \(D\), with \(\delta(D) < \frac{1}{n}\). Set \(D := \{0 = t_0, t_1, \ldots, t_N, 1 = t_{N+1}\}\), and let \(\theta = \frac{1}{n} - \delta(D)\). Choose now any positive number \(\varepsilon\) such that \(\varepsilon < \theta\), \(\varepsilon^\alpha < \frac{1}{n^\alpha}\), \(\varepsilon^\beta < \frac{1}{n^\beta}\). Now, for each \(i = 1, 2, \ldots, N\) choose a rational point \(\tau_i\) such that \(t_{i-1} < \tau_i < t_i\) and \(t_i - \tau_i < \varepsilon\). If we define \(D_0 := \{0 \in \tau_0, \tau_1, \ldots, \tau_N, 1\}\), it is clear that \(D_0\) is a rational decomposition, with \(\delta(D_0) < \frac{1}{n}\).

Now we compute \(|\overline{S}(D) - \overline{S}(D_0)|\), as follows:

\[
\begin{align*}
|\overline{S}(D) - \overline{S}(D_0)| &= |(f(0) - f(t_1))\psi([t_1, t_2]) + (f(t_1) - f(t_2))\psi([t_2, t_3]) + \ldots + (f(t_{N-1}) - f(t_N))\psi([t_{N-1}, 1])| \\
&\leq 2F \sigma_n + 2FN u_2 \varepsilon^{\beta} + NG u_1 \varepsilon^\alpha.
\end{align*}
\]

Now we deduce that

\[
|\overline{S}(D) - \overline{S}(D_0)| \leq 2F \sigma_n + \frac{F u_2 + G u_1}{n},
\]

i.e. the assertion of the Claim. From this, we get

\[
|S(D) - S(D_0)| \leq |S(D) - \overline{S}(D)| + |\overline{S}(D) - \overline{S}(D_0)| \leq 2F \sigma_n + \frac{F u_2 + G u_1}{n}.
\]

Now we shall prove that \(Q\) is \((\mathcal{L})\)-integrable. According to the hypotheses, one easily sees that \(\sup\{||S(D_1) - S(D_2)|: D_1 \text{ and } D_2 \text{ rational}, \delta(D_1) < \frac{1}{n}, \delta(D_2) < \frac{1}{n}\} \leq 2 \rho_n\) for every \(n\); hence, using the Claim above, we see also that \(\sup\{||S(D) - S(D')|: D \in \mathcal{D}, D' \in \mathcal{D}, \delta(D) < \frac{1}{n}, \delta(D') < \frac{1}{n}\} \leq 2 \rho_n + 8F \sigma_n + \frac{F u_2 + G u_1}{n}\) for every \(n\). Now, the assertion follows from Theorem 2.5.

Up to the end of this section, we shall always assume that \(q\) is integrable, that \(f\) and \(q\) are Hölder-continuous of order \(\alpha\) and \(\beta\) respectively, with relative units \(u_1\) and \(u_2\) respectively. Moreover, we shall set \(\gamma = \alpha + \beta\) and assume that \(\gamma > 1\). As above, we shall define \(Q(I) := f(a_I) q(I)\), \(\overline{Q}(I) := f(a_I) \psi(I)\), \(Z(I) := \psi(I) - q(I)\), for all intervals \(I \subset [0, 1]\), where \(a_I\) denotes the left endpoint of \(I\), and \(\psi\) is the integral function of \(q\). Finally, we shall define \(\sigma(J) := \sup\{\sum_{I \in D} |Z(I)|: D \in \mathcal{D}_J\}\) for every sub-interval \(J\).

**Lemma 3.4** Let \([a, b]\) be any sub-interval of \([0, 1]\), and let us denote by \(c\) its midpoint. Then

\[
|\mathcal{Q}([a, b]) - (\mathcal{Q}([a, c]) + \mathcal{Q}([c, b]))| \leq \left(\frac{b - a}{2}\right)^\alpha u_1 |Z([c, b])| + u_1 u_2 \left(\frac{b - a}{2}\right)^\gamma.
\]

**Proof.** We have

\[
|\mathcal{Q}([a, b]) - (\mathcal{Q}([a, c]) + \mathcal{Q}([c, b]))| = |f(a)\psi([a, b]) - f(a)\psi([a, c]) - f(c)\psi([c, b])| =
\]

\[
|f(a)\psi([a, b]) - f(a)\psi([a, c])| + |f(c)\psi([c, b]) - f(c)\psi([c, b])|.
\]
\[ |f(a)\psi([c, b]) - f(c)\psi([c, b])| \leq |f(c) - f(a)|\psi([c, b]) | \leq |f(c) - f(a)||Z([c, b])| + |q([c, b])| \leq \left( \frac{b - a}{2} \right)^\alpha u_1 |Z([c, b])| + u_1 u_2 \left( \frac{b - a}{2} \right)^{\alpha + \beta}. \]

**Proposition 3.5** Let \([a, b]\) be any subinterval of \([0, 1]\) and assume that \(D\) is a dyadic decomposition of \([a, b]\), of order \(n\). Then there exists a positive element \(v \in R\), independent of \(a\) and \(n\), such that

\[
|\mathcal{Q}([a, b]) - \sum_{I \in D} \mathcal{Q}(I)| \leq v \left( (b - a)^\gamma + (b - a)^\alpha\sigma([a, b]) \right).
\]

**Proof.** In this proof, for brevity we shall write \(\sigma\) in the place of \(\sigma([a, b])\). Let us consider the dyadic decompositions of \([a, b]\) of order \(1, 2, \ldots, n\) and denote them by \(D_1, D_2, \ldots, D_n = D\) respectively. Then we have

\[
|\mathcal{Q}([a, b]) - \sum_{I \in D} \mathcal{Q}(I)| \leq |\mathcal{Q}([a, b])| - \sum_{I \in D_1} \mathcal{Q}(I)| + \sum_{i=2}^n |\mathcal{Q}(J) - \sum_{I \in D_{i-1}} \mathcal{Q}(I)|.
\]

Thanks to Lemma 3.4, the first summand in the right-hand side is less than

\[
\left( \frac{b - a}{2} \right)^\alpha u_1 |Z([c, b])| + u_1 u_2 \left( \frac{b - a}{2} \right)^\gamma,
\]

where \(c\) is the midpoint of \([a, b]\). Since \(|Z([c, b])| \leq \sigma\), then we can write

\[
|\mathcal{Q}([a, b]) - \sum_{I \in D_1} \mathcal{Q}(I)| \leq \left( \frac{b - a}{2} \right)^\alpha u_1 \sigma + u_1 u_2 \left( \frac{b - a}{2} \right)^\gamma.
\]

Similarly, we obtain

\[
|\sum_{J \in D_2} \mathcal{Q}(J) - \sum_{I \in D_1} \mathcal{Q}(I)| \leq 2 u_1 u_2 \left( \frac{b - a}{4} \right)^\gamma + u_1 \left( \frac{b - a}{2} \right)^\alpha\sigma.
\]

where \(c_1\) is the midpoint of \([a, c]\) and \(c_2\) is the midpoint of \([c, b]\). Again, we have \(|Z([c_1, c])| + |Z([c_2, b])| \leq \sigma\), so

\[
|\sum_{J \in D_2} \mathcal{Q}(J) - \sum_{I \in D_1} \mathcal{Q}(I)| \leq 2 u_1 u_2 \left( \frac{b - a}{4} \right)^\gamma + u_1 \left( \frac{b - a}{2} \right)^\alpha\sigma.
\]

In the same fashion, for every index \(i\) from 3 to \(n\), we have

\[
|\sum_{J \in D_i} \mathcal{Q}(J) - \sum_{I \in D_{i-1}} \mathcal{Q}(I)| \leq 2 u_1 u_2 \left( \frac{b - a}{4} \right)^\gamma + u_1 \left( \frac{b - a}{2} \right)^\alpha\sigma,
\]

hence

\[
|\mathcal{Q}([a, b]) - \sum_{I \in D} \mathcal{Q}(I)| \leq u_1 u_2 \left( \frac{b - a}{2} \right)^\gamma + 2 \left( \frac{b - a}{4} \right)^\gamma + \ldots + 2^n \left( \frac{b - a}{2^n} \right)^\gamma + u_1 \sigma \left( \frac{b - a}{2} \right)^\alpha + \ldots + \left( \frac{b - a}{2^n} \right)^\alpha.
\]

The first summation is bounded by the quantity

\[
\frac{1}{2} u_1 u_2 (b - a)^\gamma \sum_{i=1}^n \left( \frac{1}{2^i} \right)^i \leq \frac{1}{2} u_1 u_2 (b - a)^\gamma \sum_{i=1}^\infty \left( \frac{1}{2^i} \right)^i = \frac{1}{2(2^n - 1)} (b - a)^\gamma u_1 u_2.
\]

Let us set \(v_1 := \frac{1}{2(2^n - 1)} u_1 u_2\), and turn to the second summation:

\[
u_1 \sigma \left( \frac{b - a}{2} \right)^\alpha + \left( \frac{b - a}{4} \right)^\alpha + \ldots + \left( \frac{b - a}{2^n} \right)^\alpha \leq u_1 \sigma \frac{1}{2^\alpha - 1} (b - a)^\alpha.
\]

Now, if we put: \(v = \sup\{v_1, \frac{u_1}{2^{\alpha - 1}}\}\), the assertion follows immediately.

In the next proofs, we shall again use the notation: \(S(D) := \sum_{I \in D} Q(I)\), \(S(D) := \sum_{I \in D} \mathcal{Q}(I)\) for every decomposition \(D\).
Proposition 3.6 Let \([a, b]\) be any subinterval of \([0, 1]\), and assume that \(D\) is an equidistributed decomposition of \([a, b]\), consisting of \(N\) elements. Then
\[
|Q([a, b]) - \sum_{I \in D} Q(I)| \\
\leq r((b - a)^\gamma + \sigma([a, b]))
\]
for some unit \(r \in R\), independent of \(D\) and of \([a, b]\).

Proof. Of course, if \(N = 2^h\) for some integer \(h\), the result follows from Proposition 3.5. So, let us assume that \(N\) is not a power of 2, and write \(N\) in dyadic expansion: \(N = e_02^0 + e_12^1 + \ldots + e_h2^h\), where \((e_0, e_1, \ldots, e_h)\) is a suitable element of \((0, 1)^{h+1}\), and \(h = \lfloor \log_2 N \rfloor\). Clearly then, \(e_h = 1\). Now let us define: \(t_0 := a\), \(t_{h+1} := b\), and \(t_j := a + \sum_{i=1}^{j} e_{i-1} 2^{i-1} \frac{b-a}{N}\) for \(j = 1, 2, \ldots, h\). Thus we have \(t_j - t_{j-1} = 0\) or \(2^{j-1} \frac{b-a}{N}\) according as \(e_{j-1} = 0\) or 1 for \(j = 1, 2, \ldots, h\), and in any case \(t_h = b - 2^h \frac{b-a}{N}\). Let us denote by \(D_0\) the decomposition of \([a, b]\) whose intermediate points are \(t_0, t_1, \ldots, t_{h+1}\) (dropping duplicates if there are any: however we shall keep the same notation, because this will not affect the result). We observe that, for every \(I \in D_0\), the elements from \(D\) that are contained in \(I\) form a dyadic decomposition of \(I\). So we have, from Proposition 3.5:
\[
|S(D_0) - S(D)| \\
\leq \sum_{I \in D_0} |Q(I)| - \sum_{J \in D, J \subseteq I} Q(J)| \\
\leq \sum_{I \in D_0} v(|I|) + |I|^\alpha \sigma(I) \\
\leq v \sum_{i=0}^{h} (2^i \frac{b-a}{N})^\gamma + v \sum_{i=0}^{h} (2^i \frac{b-a}{N})^\alpha \sigma([a, b]) \\
\leq v(b - a)^\gamma 2^h + v(b - a)^\alpha \frac{2^{(h+1)\alpha}}{N^\alpha(2^\alpha - 1)} \sigma([a, b]) \\
\leq v(b - a)^\gamma 2^h + v(b - a)^\alpha \frac{2^\alpha}{2^\alpha - 1} \sigma([a, b]).
\]
Thus, setting \(w := v(\max\{2^\gamma, 2^\alpha \frac{2^\alpha}{2^\alpha - 1}\})\), and recalling that \(b - a \leq 1\), we get
\[
|S(D_0) - S(D)| \leq w((b - a)^\gamma + \sigma([a, b])).
\]
Let us now evaluate \(|Q([a, b]) - S(D_0)|\). We have
\[
Q([a, b]) = f(a)\psi([a, b]) = f(a)\psi([t_1, t_2]) + f(a)\psi([t_1, t_2]) + \ldots + f(a)\psi([t_h, b]),
\]
hence
\[
|Q([a, b]) - S(D_0)| \leq |(f(a) - f(t_1))\psi([t_1, t_2]) + (f(a) - f(t_2))\psi([t_2, t_3]) + \ldots + (f(a) - f(t_h))\psi([t_h, b])| \\
\leq \sum_{j=1}^{h} |f(a) - f(t_j)||\psi([t_j, t_{j+1}])| \\
\sum_{j=1}^{h} |f(a) - f(t_j)||\psi([t_j, t_{j+1}])| \\
\sum_{j=1}^{h} u_1 u_2 (t_j - a)^\gamma (t_{j+1} - t_j)^\alpha + 2F \sum_{h=1}^{h} |Z([t_j, t_{j+1}])|,
\]
where \(F\) is any majorant for \(f\).

Recall that \(t_{j+1} - t_j \leq 2^j \frac{b-a}{N}\), and so \(t_j - a \leq \frac{b-a}{N}(1 + 2 + 4 + \ldots + 2^{j-1}) \leq 2^j \frac{b-a}{N}\) for all \(j = 1, 2, \ldots, h\). Hence
\[
|Q([a, b]) - S(D_0)| \leq \\
\sum_{j=1}^{h} u_1 u_2 (2^j \frac{b-a}{N})^\gamma (2^j \frac{b-a}{N})^\alpha + 2F \sigma([a, b]) = \\
u_1 u_2 (b - a)^\gamma \left(\frac{1}{N^\gamma} \sum_{j=1}^{h} 2^{2j}\right) \\
2F \sigma([a, b]) \leq u_1 u_2 (b - a)^\gamma \frac{2^{h\gamma}}{N^\gamma} + 2F \sigma([a, b]).
\]
Thus, setting \(v_2 := u_1 u_2 2^\gamma\), we get \(|Q([a, b]) - S(D_0)| \leq v_2 (b - a)^\gamma + 2F \sigma([a, b])\). Finally, setting \(r = w + v_2 + 2F\), we deduce
\[
|Q([a, b]) - S(D)| \leq |Q([a, b]) - S(D_0)| + \\
|S(D_0) - S(D)| \leq r((b - a)^\gamma + \sigma([a, b])).
\]
We are now ready for the main result.
Corollary 3.8 Let \( f : [0, 1] \rightarrow R_1 \) and \( g : [0, 1] \rightarrow R_2 \) be two Hölder-continuous functions, of order \( \alpha \) and \( \beta \) respectively, and assume that \( \alpha + \beta > 1 \). Then \( f \) is \((\mathcal{L})(RS)\)-integrable with respect to \( g \).

Theorem 3.7 (The Kondurar Theorem) Let \( f \) and \( q \) be as above, with \( \gamma = \alpha + \beta > 1 \). Then \( f \) is \((\mathcal{L})(RS)\)-integrable w.r.t. \( q \).

Proof. We first observe that the real valued interval function \( W(I) := |I|^{\gamma} \) is integrable, and its integral is null. By virtue of axiom d) in 2.1, this immediately implies that \( f \) and \( q \) satisfy assumption (C). Let \( r \) be the unit in \( R \) given by Proposition 3.6, and set \( b_k := \frac{2r}{\kappa^k}, \forall k \in N \). Now fix \( k \) and choose any rational decomposition \( D_0 \) such that \( \delta(D_0) \leq \frac{1}{k} \) and \( \sum_{J \in D} |J|^{\gamma} \leq \frac{1}{k} \) for every decomposition \( D \), finer than \( D_0 \). Now, if \( D \) is any rational decomposition, finer than \( D_0 \), then there exists an integer \( N \) such that the equidistributed decomposition \( D^* \), consisting of \( N \) subintervals, is finer than \( D \). Of course, \( D^* \) is also finer than \( D_0 \), so we have, by Proposition 3.6:

\[
|S(D^*) - S(D_0)| \leq |S(D^*) - S(D^*)| + \\
|S(D^*) - S(D_0)| + |S(D_0) - S(D_0)| \leq \\
\sum_{J \in D_0} r|I|^\gamma + r\sigma_k + 2F\sigma_k \leq \\
\frac{1}{k}r + (r + 2F)\sigma_k,
\]

and similarly

\[
|S(D^*) - S(D)| \leq \frac{1}{k}r + (r + 2F)\sigma_k.
\]

Therefore, by axiom d) of 2.1, we have:

\[
\mathcal{L}((\sup \{|S(D) - S(D_0)| : D, D_0 \text{ are rational,} \delta(D_0) \leq 1/k, D_{ref}D_0\})_{k}) = 0,
\]

and the theorem is proved, thanks to Proposition 3.3. As we already mentioned, if \( q \) is additive, then \( q \) is integrable, and so the conclusion of the previous theorem holds. But additivity of \( q \) implies that \( q \) is the jump function of some map \( g : [0, 1] \rightarrow R_2 \), hence the last theorem contains also the following (more classical) version of the Kondurar theorem.

Remark 3.9 Assume that \( W \) is any Stochastic Process, with Hölder continuous trajectories, on a probability space \((X, B, P)\), and \( g : [0, 1] \rightarrow L^0(X, B, P) \) is the function which associates to each \( t \in [0, 1] \) the random variable \( W_t \in L^0 \). We can endow the Riesz space \( L^0 \) with the convergence in probability (which satisfies all assumptions in Definition 2.1); if we assume that \( W \) is homogeneous and with independent increments, that \( W_t \in L^4 \) for all \( t \), and \( E(W_t) = 0 \) for all \( t \), then we deduce that \( E(W_t^2) \) is a linear function of \( t \). Moreover, if \( E(W_t^2) \leq Kt^2 \) for some positive constant \( k \), one can prove that the interval function \( q(I) = (\Delta(g)(I))^2 \) is integrable (see also [7]). So, integrability of any other process \( f \) with respect to \( \Delta(g)^2 \) can be deduced by Theorem 3.7.

References