A HAKE-TYPE THEOREM FOR INTEGRALS WITH RESPECT TO ABSTRACT DERIVATION BASES IN THE RIESZ SPACE SETTING

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Abstract. A Kurzweil-Henstock type integral with respect to an abstract derivation basis in a topological measure space, for Riesz space-valued functions, is studied. A Hake-type theorem is proved for this integral, by using technical properties of Riesz spaces.

1. Introduction

The known Hake theorem in integration theory (see [27, Chapter VIII, Lemma 3.1]) states that, in contrast to the Lebesgue integral, the Perron integral on a compact interval is equivalent to the improper Perron integral, i.e., Perron integrability of a function $f$ on $[a,b]$ is equivalent to Perron integrability on $[a,c]$ with $a < c < b$ together with the existence of the limit $\lim_{c \to b^-} \int_a^c f$. As the Perron integral on the real line is known to be equivalent to the Kurzweil-Henstock integral (see [21, 24]), the same property is true for the last integral.

The general idea of computing the improper integral as a limit of integrals over increasing families $\{A_\alpha\}$ of sets can be realized in the multidimensional case and in abstract measure spaces in several ways, depending on the type of the integral and on the family $\{A_\alpha\}$ chosen to generalize the compact intervals of the one-dimensional case. This gives rise to various versions of the Hake theorem. Some kinds of it for Kurzweil-Henstock type integrals in $\mathbb{R}^n$ and in more general spaces were studied in [9, 17, 19, 23, 31]. A version of the Hake theorem for Riesz space-valued functions was proved in [10]. For recent studies about measures and integrals in the Riesz space context see also [5, 6, 11, 12].

In this paper we consider a Kurzweil-Henstock type integral with respect to an abstract derivation basis in a topological space for Riesz space-valued functions.
A similar kind of integral was studied in [16]. We investigate what assumptions on the basis and on the Riesz space guarantee a generalized Hake property for this integral. A special attention is given to the kind of order convergence used in the definition of the integral.

The main result of the paper is an extension, to the Riesz space setting, of a Hake type theorem proved in [31] for the real-valued case. It also gives a generalization of results of [10], where a special type of derivation basis, defined by gauges, was considered.

2. Riesz spaces and modes of convergence

We begin with reminding some preliminary notions and results about Riesz spaces.

A Riesz space \( R \) is said to be Dedekind complete if every nonempty subset \( A \subset R \), bounded from above, has a supremum in \( R \). A Dedekind complete Riesz space \( R \) is said to be super Dedekind complete if for any nonempty set \( A \subset R \), bounded from above, there exists a countable subset \( A^* \subset A \), such that \( \bigvee A = \bigvee A^* \).

A non-empty ordered set \( \Lambda = (\Lambda, \succeq) \) is said to be directed if for every \( \lambda_1, \lambda_2 \in \Lambda \) there exists \( \lambda_3 \in \Lambda \) such that \( \lambda_3 \succeq \lambda_1 \) and \( \lambda_3 \succeq \lambda_2 \).

A net is any function defined on a directed set \( \Lambda = (\Lambda, \succeq) \) with values in a Riesz space \( R \), and we denote it by the symbol \((r_\lambda)_{\lambda \in \Lambda}\).

An \((O)\)-net \((r_\lambda)_{\lambda \in \Lambda}\) is a monotone decreasing net of elements of \( R \), such that \( \bigwedge_{\lambda \in \Lambda} r_\lambda = 0 \). In particular, in the case \((\Lambda, \succeq) = (\mathbb{N}, \geq)\), this defines the notion of \((O)\)-sequence.

A bounded double sequence \((a_{i,j})_{i,j}\) in \( R \) is called a regulator or a \((D)\)-sequence if \((a_{i,j})_{j}\) is an \((O)\)-sequence for each \( i \in \mathbb{N} \).

A subset \( W \) of a Riesz space \( R \) is \((PR)\)-bounded if there exists a strictly increasing sequence \((w_n)_{n}\) of positive elements of \( R \), such that for every \( w \in W \) there is \( n \in \mathbb{N} \) with \( |w| \leq w_n \) (see [25, 32]).

A Riesz space \( R \) is weakly \( \sigma \)-distributive if

\[
\bigwedge_{\varphi \in \mathbb{N}^\omega} \left( \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) = 0
\]  

for every \((D)\)-sequence \((a_{i,j})_{i,j}\).

We now recall some modes of convergence of nets in Riesz spaces. The most general notion of order convergence of a net in a Riesz space \( R \) seems to be introduced in [28] (see also [3, Definition 1.2]).

**Definition 2.1.** A net \((x_\lambda)_{\lambda \in \Lambda}\) in \( R \) is order convergent to \( x \in R \) if there exist a directed set \( \Gamma \) and an \((O)\)-net \((y_\gamma)_{\gamma \in \Gamma}\) such that for each \( \gamma \in \Gamma \) there exists \( \lambda_0 \in \Lambda \) with \( |x_\lambda - x| \leq y_\gamma \) for every \( \lambda \succeq \lambda_0 \).
For a Dedekind complete Riesz space this definition is equivalent (see [3]) to the following one:

**Definition 2.2.** A net \((x_\lambda)_{\lambda \in \Lambda}\) in \(R\) is order convergent to \(x \in R\) if there exist an \((O)\)-net \((y_\lambda)_{\lambda \in \Lambda}\) and \(\lambda_0 \in \Lambda\) with \(|x_\lambda - x| \leq y_\lambda\) for every \(\lambda \geq \lambda_0\).

Specifying the directed set \(\Gamma\) in Definition 2.1 we get more restrictive notions which we call \((O)\)- and \((D)\)-convergence (see also [8, 22, 26]).

**Definition 2.3.** A net \((x_\lambda)_{\lambda \in \Lambda}\) is \((O)\)-convergent to \(x \in R\) (and we write \(\lim_{\lambda \in \Lambda} x_\lambda = x\)) if there exists an \((O)\)-sequence \((b_n)\) such that for each \(n \in \mathbb{N}\) there is \(\lambda_0 \in \Lambda\) with \(|x_\lambda - x| \leq b_n\) for every \(\lambda \geq \lambda_0\).

**Definition 2.4.** Let \(R\) be a weakly \(\sigma\)-distributive Riesz space. A net \((x_\lambda)_{\lambda \in \Lambda}\) is \((D)\)-convergent to \(x \in R\) (shortly, \((D)\)-convergence) if there exists an \((D)\)-sequence \((a_{i,j})_{i,j}\) in \(R\), such that for every \(\varphi \in \mathbb{N}^\mathbb{N}\) there is \(\lambda_0 \in \Lambda\) with

\[
|x_\lambda - x| \leq \sum_{i=1}^{\infty} a_{i,\varphi(i)} \quad \text{for every } \lambda \geq \lambda_0.
\]

It is clear that both \((O)\)- and \((D)\)-convergence of a net imply order convergence in the sense of Definitions 2.1 and 2.2 to the same limit (indeed, choose as \(\Gamma\) the directed sets \((\mathbb{N}, \geq)\) for \((O)\)-convergence and \(\mathbb{N}^\mathbb{N}\) with the increasing pointwise ordering for \((D)\)-convergence). Moreover, note that order convergence and \((O)\)-convergence coincide if and only if the Riesz space involved is super Dedekind complete. Indeed (see [1, p. 20]), a Dedekind complete Riesz space is super Dedekind complete if and only if for any \((O)\)-net \((x_\lambda)_{\lambda \in \Lambda}\) there is a strictly increasing sequence \((\lambda_n)\) such that \((x_{\lambda_n})_n\) is an \((O)\)-sequence.

A Riesz space \(R\) is said to satisfy property \((\sigma)\) if, given any sequence \((u_n)_n\) in \(R\) with \(u_n \geq 0\) for all \(n \in \mathbb{N}\), there exist an element \(u \in R\), \(u \geq 0\), and a sequence \((\lambda_n)_n\) of positive real numbers, such that \(\lambda_n u_n \leq u\) for all \(n \in \mathbb{N}\).

A Riesz space \(R\) is said to be regular if it satisfies property \((\sigma)\) and for every sequence \((x_n)_n\), \((O)\)-convergent to \(x \in R\), there exists \(w \in R\) such that for any real number \(\varepsilon > 0\) there is \(n_0 \in \mathbb{N}\) with

\[
|x_n - x| \leq \varepsilon w \quad \text{for each } n \geq n_0. \tag{2.2}
\]

**Remark 1.** Observe that, if \((X, \mathcal{M}, \mu)\) is a measure space, with \(\mu\) positive, \(\sigma\)-finite and \(\sigma\)-additive, then the space \(L^0(X, \mathcal{M}, \mu)\) of all \(\mu\)-measurable real-valued functions, with identification up to \(\mu\)-null sets, is super Dedekind complete, weakly \(\sigma\)-distributive and regular (see [4, 22]).

We now turn to a relation between \((O)\)- and \((D)\)-convergence. The following result is an extension of [4, Theorem 3.4] and will be useful in the sequel.

**Proposition 2.1.** In every Riesz space \(R\), any \((O)\)-convergent net is \((D)\)-convergent too.
Proof. Let \((x_\lambda)_{\lambda \in \Lambda}\) be a net in \(R\), \((O)\)-convergent to \(x \in R\), and \((b_n)_n\) be an \((O)\)-sequence which defines \((O)\)-convergence. For every \(i, j \in \mathbb{N}\) define \(a_{i,j} := b_j\). It is clear that this double sequence \((a_{i,j})_{i,j}\) is a regulator. Choose arbitrarily \(\varphi \in \mathbb{N}^\mathbb{N}\). By hypothesis, in correspondence with \(n = \varphi(1)\) there exists \(\lambda \in \Lambda\) such that, for all \(\lambda \geq \lambda\),
\[
|x_\lambda - x| \leq b_{\varphi(1)} = a_{1,\varphi(1)} \leq \sqrt[n]{a_{i,\varphi(i)}}.
\]
Thus the assertion follows. \(\square\)

The converse of Proposition 2.1 holds if \(R\) is a super Dedekind complete and weakly \(\varphi\)-distributive Riesz space. However, as we have already mentioned above, \((D)\)-convergence always implies the order convergence in the sense of Definition 2.1 and, in the case of a Dedekind complete space, also the convergence in the sense of Definition 2.2, and here we need not the assumption of super Dedekind completeness. So we have the following

**Proposition 2.2.** Let \(R\) be a Dedekind complete and weakly \(\sigma\)-distributive Riesz space, \((\Lambda, \geq)\) be a directed set, and suppose that \((D)\lim_{\lambda \in \Lambda} x_\lambda = x\). Then there exist an \((O)\)-net \((y_\lambda)_{\lambda \in \Lambda}\) and \(\lambda_0 \in \Lambda\) with \(|x_\lambda - x| \leq y_\lambda\) for every \(\lambda \geq \lambda_0\).

We now state a technical lemma, which gives a property of regular Riesz spaces.\(^\Box\)

**Lemma 2.1.** If \(R\) is a regular Riesz space and \((x^{(k)}_\lambda)_{\lambda \in \Lambda}, k \in \mathbb{N}\), is a sequence of nets in \(R\) \((O)\)-convergent to \(x^{(k)}\), then there is \(w \in R\), \(w \geq 0\), such that for every \(k \in \mathbb{N}\) there is an element \(\lambda = \lambda_k \in \Lambda\) with \(|x^{(k)}_\lambda - x^{(k)}| \leq w\) for each \(\lambda \geq \lambda_k\).

**Proof.** For each \(k \in \mathbb{N}\) let \((b^{(k)}_n)_n\) be the \((O)\)-sequences, which correspond to the net \((x^{(k)}_\lambda)_{\lambda \in \Lambda}\) according to Definition 2.3. As \(R\) is regular, for each \(k \in \mathbb{N}\) and for the sequences \((b^{(k)}_n)_n\) there is a positive element \(w_k \in R\), such that for every \(\varepsilon > 0\) there exists \(n_0 \in \mathbb{N}\) with
\[
|x_n - x| \leq \varepsilon w_k \quad \text{for all} \quad n \geq n_0. \tag{2.3}
\]
Since, by hypothesis, \(R\) satisfies property \((\sigma)\), there exist a sequence \((\zeta_k)\) of positive real numbers and an element \(w \in R\), \(w \geq 0\), with \(\zeta_k w_k \leq w\) for every \(k\). Choose now arbitrarily \(\varepsilon > 0\) and \(k \in \mathbb{N}\). Using (2.3) for the sequence \((b^{(k)}_n)_n\) and with \(\varepsilon\) replaced by \(\zeta_k \varepsilon\), we find a number \(n_0 = n_0(\varepsilon, k) \in \mathbb{N}\) with \(b^{(k)}_{n_0} \leq \varepsilon \zeta_k w_k \leq \varepsilon w\) whenever \(n \geq n_0\). By Definition 2.3, in correspondence with \(n_0\), an element \(\lambda \in \Lambda\) can be found, with
\[
|x^{(k)}_\lambda - x^{(k)}| \leq b^{(k)}_{n_0} \leq \varepsilon w \quad \text{for all} \quad \lambda \geq \lambda_k.
\]
Taking \(\varepsilon = 1\), we get the assertion. \(\Box\)
The following result (Fremlin lemma, see [18, Lemma 1C], [26, Theorem 3.2.3, pp. 42-45]) allows us to replace a countable family of regulators with one regulator, and this technique will be useful in the proof of the main results of the paper.

**Lemma 2.2.** Let $R$ be any Dedekind complete Riesz space and $(a^{(k)}_{i,j})$, $k \in \mathbb{N}$, be a sequence of regulators in $R$. Then for every $u \in R$, $u \geq 0$, there exists a $(D)$-sequence $(a_{i,j})$, $i,j$ with

$$u \land \left( \sum_{k=1}^{q} \left( \bigvee_{i=1}^{\infty} a^{(k)}_{i,\varphi(i+k)} \right) \right) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

for every $q \in \mathbb{N}$ and $\varphi \in \mathbb{N}^{\mathbb{N}}$.

**Definition 2.5.** Let $R_1, R_2, R$ be three Dedekind complete Riesz spaces. We say that $(R_1, R_2, R)$ is a product triple if there exists a function $\cdot : R_1 \times R_2 \to R$, which we call product, such that for all $r_j, s_j \in R_j$, $j = 1, 2$, we have:

2.5.1): $(r_1 + s_1) \cdot r_2 = r_1 \cdot r_2 + s_1 \cdot r_2$,

2.5.2): $r_1 \cdot (r_2 + s_2) = r_1 \cdot r_2 + r_1 \cdot s_2$,

2.5.3): $[r_1 \geq s_1, r_2 \geq 0] \Rightarrow [r_1 \cdot r_2 \geq s_1 \cdot r_2]$,

2.5.4): $[r_1 \geq 0, r_2 \geq s_2] \Rightarrow [r_1 \cdot r_2 \geq r_1 \cdot s_2]$;

2.5.5): if $(a_{\lambda})_{\lambda \in \Lambda}$ is any net in $R_2$, $(O) \lim_{\lambda \in \Lambda} a_{\lambda} = 0$ and $0 \leq b \in R_1$, then $(O) \lim_{\lambda \in \Lambda} (b \cdot a_{\lambda}) = 0$;

2.5.6): if $(a_{\lambda})_{\lambda \in \Lambda}$ is any net in $R_1$, $(O) \lim_{\lambda \in \Lambda} a_{\lambda} = 0$ and $0 \leq b \in R_2$, then $(O) \lim_{\lambda \in \Lambda} (a_{\lambda} \cdot b) = 0$;

2.5.7): if $a$ is a real number, then $(a r_1) \cdot r_2 = r_1 \cdot (a r_2) = a (r_1 \cdot r_2)$ for all $r_j \in R_j$, $j = 1, 2$.

3. The abstract derivation bases

We now introduce some notions involving abstract bases and some related topics (for the case $R_1 = R_2 = R = \mathbb{R}$ see also [31]).

Let $X$ be a Hausdorff topological space, $\mathcal{M}$ be the class of all Borel subsets of $X$, $R_2$ be as in Definition 2.5 and $\mu : \mathcal{M} \to R_2$ be a finitely additive positive measure. We say that $\mu$ is an outer regular measure if for every $E \in \mathcal{M}$ there exists a strictly positive element $z \in R_2$ with the property: for any $\varepsilon > 0$ there is an open set $G \supset E$ with $\mu(G \setminus E) \leq \varepsilon$.

For any set $I \subset X$ we use the notation $int(I)$ and $\partial I$ for the interior and the boundary of $I$, respectively.

A derivation basis (or simply a basis) $\mathcal{B}$ in $(X, \mathcal{M}, \mu)$ is a filter base on the product space $\mathcal{I} \times X$, where $\mathcal{I}$ is a family of closed subsets of $X$ having strictly positive measure $\mu$ and called generalized intervals or B-intervals (see also [16, 24, 31, 33]). That is $\mathcal{B}$ is a nonempty collection of subsets of $\mathcal{I} \times X$ so that
each $\beta \in \mathcal{B}$ is a set of pairs $(I, x)$, where $I \in \mathcal{I}$, $x \in X$, and $\mathcal{B}$ has the filter base property: $\emptyset \notin \mathcal{B}$ and for every $\beta_1, \beta_2 \in \mathcal{B}$ there exists $\beta_3 \in \mathcal{B}$ such that $\beta_3 \subseteq \beta_1 \cap \beta_2$. So, $\mathcal{B} = (\mathcal{B}, \subseteq)$ is a directed set with the order given by the “reversed” inclusion $\subseteq$. We refer to the elements $\beta$ of $\mathcal{B}$ as basis sets. Some particular examples of derivation basis in different types of topological spaces can be found in [2, 16, 24, 29, 30, 34].

In this paper we suppose that the pairs $(I, x)$ forming each $\beta \in \mathcal{B}$ have the property that $x \in I$, although it is not the case in the general theory (see for instance [20, 24]). We assume that $\mu(\partial I) = 0$ for any $\mathcal{B}$-interval $I$. We say that two $\mathcal{B}$-intervals $I'$, $I''$ are non-overlapping if $\mu(I' \cap I'') = 0$. We call $\mathcal{B}$-figure a finite union of non-overlapping $\mathcal{B}$-intervals. We suppose that the intersection of two overlapping $\mathcal{B}$-intervals is a $\mathcal{B}$-figure, so the intersection of two overlapping $\mathcal{B}$-figures is also a $\mathcal{B}$-figure. For a set $E \subseteq X$ and $\beta \in \mathcal{B}$ we put

$$\beta(E) := \{(I, x) \in \beta : I \subseteq E\} \text{ and } \beta[E] := \{(I, x) \in \beta : x \in E\}.$$ 

If the basis is considered exclusively on a $\mathcal{B}$-interval or a $\mathcal{B}$-figure $J$ (instead of the whole of $X$), we refer to it as basis in $J$ using the same notations $\mathcal{B}$ for it and $\beta$ for its elements.

From now on, we assume that the basis involved $\mathcal{B}$ has the following properties:

a) $\mathcal{B}$ ignores no point, i.e., $\beta([x]) \neq \emptyset$ for any point $x \in X$ and for any $\beta \in \mathcal{B}$.

b) $\mathcal{B}$ has a local character, that is for any family of basis sets $(\beta_r)_r$ in $\mathcal{B}$ and for any pairwise disjoint sets $E_r$ there exists $\beta \in \mathcal{B}$ such that $\beta[\bigcup_r E_r] \subseteq \bigcup_r \beta_r[E_r]$.

c) $\mathcal{B}$ is a Vitali basis, namely for every $x \in X$ and for any neighborhood $U(x)$ of $x$ there exists $\beta_x \in \mathcal{B}$ such that $I \subseteq U(x)$ for each pair $(I, x) \in \beta_x$.

For a fixed basis set $\beta$, a finite collection $\pi$ of $\beta$, where the distinct elements $(I', x')$ and $(I'', x'')$ in $\pi$ have $I'$ and $I''$ non-overlapping, is called a $\beta$-partition.

Let $L \subseteq X$. If $\pi \subseteq \beta(L)$, then $\pi$ is called a $\beta$-partition in $L$; if $\pi \subseteq \beta[L]$, then $\pi$ is called a $\beta$-partition on $L$; if $\bigcup_{(I, x) \in \pi} I = L$, then $\pi$ is called a $\beta$-partition of $L$. For a set $E$ and a $\beta$-partition $\pi$, let $\pi[E] := \{(I, x) \in \pi : (I, x) \in \beta[E]\}$.

We say that a basis $\mathcal{B}$ has the partitioning property if the following conditions hold:

(i) for each $\mathcal{B}$-interval $I_0$ and each finite collection $I_1, \ldots, I_n$ of pairwise non-overlapping $\mathcal{B}$-intervals with $I_1, \ldots, I_n \subseteq I_0$ there exists a finite number of $\mathcal{B}$-intervals $I_{n+1}, \ldots, I_m$ such that $I_0 = \bigcup_{r=1}^m I_r$ and $I_r$ are pairwise non-overlapping;

(ii) for each $\mathcal{B}$-interval $I$ and for any $\beta \in \mathcal{B}$ there exists a $\beta$-partition of $I$. 

Note that condition (i) of partitioning property implies that the union of any two $B$-figures is a $B$-figure, while condition (ii) implies the existence of a $\beta$-partition of any $B$-figure. The union of all $B$-intervals involved in a $\beta$-partition $\pi$ is called the $B$-figure generated by $\pi$.

One of the simplest derivation bases is the so-called full interval basis on the real line $\mathbb{R}$, defined by the basis sets $\beta_\delta := \{(I, x) : I \in \mathcal{I}, \ x \in I \subset (x - \delta(x), x + \delta(x))\}$, where $\delta$ is a so-called gauge, i.e., a positive function defined on $\mathbb{R}$.

The following lemma on extension of the $\beta$-partition holds.

**Lemma 3.1.** Let $\pi'$ be a $\beta$-partition in a $B$-figure $L$. Then there exists a $\beta$-partition $\pi''$ such that $\pi = \pi' \cup \pi''$ is a $\beta$-partition of $L$.

**Proof.** Let $\pi' := \{(I_j, x_j)\}_{j=1}^m$. By condition (i) of the partitioning property we get that $L = (\bigcup_{j=1}^m I_j) \bigcup (\bigcup_{r=1}^n K_r)$, where the $K_r$'s are pairwise non-overlapping $B$-intervals. By virtue of the partitioning property (ii) we obtain the existence of a $\beta$-partition $\pi_r$ of $K_r$ for every index $r$. Thus $\pi'' := \bigcup_{r=1}^n \pi_r$ is the required $\beta$-partition. \qed

The $\beta$-partition $\pi''$ of the above lemma is called a $\beta$-complementary partition to $\pi'$ in $L$. If $F$ is the $B$-figure generated by the partition $\pi'$, then the $B$-figure generated by the partition $\pi''$ is called the $B$-figure complementary to $F$ in $L$, and it is denoted by the symbol $C_\beta(F)$.

4. The $H_B$-integral and the main results

Let $(R_1, R_2, R)$ be a product triple. We now introduce the Kurzweil-Henstock type integral with respect to an abstract basis in the Riesz space setting.

**Definition 4.1.** (see also [24]) Let $B$ be a basis in $(X, M, \mu)$ having the partitioning property, $\mu : M \to R_2$ be an outer regular measure, and $L$ be a $B$-figure. Let $\Pi(\beta, L)$ denote, for each $\beta \in B$, the set of all $\beta$-partitions of $L$. A function $f : L \to R_1$ is said to be Kurzweil-Henstock integrable with respect to $\mathcal{B}$ (or $H_B$-integrable) on $L$, with $H_B$-integral $A \in R$, if \((O) \lim_{\beta \in B} p_\beta = 0\), where

$$p_\beta := \sqrt{\left\{ \sum_{(I, x) \in \pi} f(x) \mu(I) - A : \pi \in \Pi(\beta, L) \right\}}, \ \beta \in B. \quad (4.1)$$

We denote the integral value $A$ by $(H_B) \int_L f$.

We say that a function $f : L \to R_1$ is $H_B$-integrable on a set $E \subset L$ if the function $f \cdot \chi_E$, where $\chi_E$ is the characteristic function of the set $E$, is $H_B$-integrable on $L$, and we set $\int_E f := \int_L f \cdot \chi_E$. 
Remark 2. Observe that in [16] the concept of $H_B$-integral was given by requiring that
\[ \bigwedge_{\beta \in B} p_\beta = 0, \quad (4.2) \]
where $(p_\beta)_{\beta \in B}$ is the decreasing net in (4.1), i.e., on the basis of Definition 2.2 of order convergence. However, we give our definition of $H_B$-integral as above, to prove our main results, because we will often deal with $(O)$-sequences, and this is more natural for the techniques used in the proofs. Note that, when $R$ is super Dedekind complete, the definitions of $H_B$-integral in (4.1) and (4.2) are equivalent, since, as we have already mentioned, Definition 2.2 of order convergence is equivalent in this case to Definition 2.3 of $(O)$-convergence.

The next result is a condition under which a Riesz space-valued function, vanishing outside a $\mu$-null set, has integral value zero. Note that, as we shall see below, there are Riesz space-valued functions, vanishing outside a countable subset of $[0,1]$, which are not Kurzweil-Henstock integrable with respect to the basis formed by all closed subintervals of $[0,1]$. The following statement is a generalization of a result which is always true for the real case (see [21] and [31, Proposition 1]).

Proposition 4.1. Let $(R_1, R_2, R)$ be a product triple, assume that $R$ satisfies property $(\sigma)$, and suppose that $f: X \to R_1$ is equal to zero $\mu$-almost everywhere on a $B$-figure $L$ and has $(PR)$-bounded range. Then $f$ is $H_B$-integrable on $L$ with integral value zero.

Proof. Since the range of $f$ is $(PR)$-bounded, there is a strictly increasing sequence $(w_n)_n$ of positive elements of $R_1$, such that for each $x \in L$ there is an index $n$ with $|f(x)| \leq w_n$. Let $E := \{x \in L : f(x) \neq 0\}$ and $E_n = \{x \in E : |f(x)| \leq w_n\}$, $n \in \mathbb{N}$. We put $D_n = E_n \setminus E_{n-1}$ with $E_0 := \emptyset$. Then $E = \bigcup_n D_n$.

In correspondence with $E$ there is $z \in R_2$ according to outer regularity of the measure $\mu$. By virtue of property $(\sigma)$, it is possible to associate with the sequence $(w_n z)_n$ a sequence $(\lambda_n)_n$ of positive real numbers and an element $0 \leq v \in R$ with $\lambda_n w_n z \leq v$ for all $n \in \mathbb{N}$. Now, in correspondence with $n \in \mathbb{N}$ and $\varepsilon \lambda_n / 2^n$, there exists an open set $G_n \supset E$ with $\mu(G_n) \leq \varepsilon \lambda_n / 2^n$. As our basis is a Vitali basis, for any $x \in E$ there is a basis set $\beta^n_x$ in $L$ with $I \subset G_n$ for each $(I, x) \in \beta^n_x([x])$. By the local character of the basis, we can find $\beta$ in $L$ such that
\[ \beta[E] \subset \bigcup_{n=1}^{\infty} \left( \bigcup_{x \in D_n} \beta^n_x([x]) \right), \]
with $\beta[L \setminus E]$ being defined arbitrarily. Take any $\beta$-partition $\pi$ of $L$, we get:

$$\left| \sum_{(I,x) \in \pi} f(x) \cdot \mu(I) \right| \leq \sum_{n=1}^{\infty} \left| \sum_{(I,x) \in \pi[D_{n}]} f(x) \cdot \mu(I) \right| \leq \sum_{n=1}^{\infty} \left( \sum_{(I,x) \in \pi[D_{n}]} w_{n} \cdot \mu(I) \right) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2n} \lambda_{n} w_{n} z \leq \left( \sum_{n=1}^{\infty} \frac{\varepsilon}{2n} \right) v = \varepsilon v.$$  

From this the assertion follows. □

Observe that the hypothesis that $R$ fulfils property (\sigma) in Proposition 4.1 in general cannot be dropped. To see this we use the full interval basis $\mathcal{B}$ on $X = [0,1]$ mentioned above. Let $R_{2} = \mathbb{R}$, $\mu$ be the Lebesgue measure on all measurable subsets of $[0,1]$, and $R_{1} = R = c_{00}$ be the space of all eventually null real-valued sequences. Since $\mathbb{R}^{N}$ is super Dedekind complete and every solid subspace of a super Dedekind complete Riesz space is super Dedekind complete too (see also [22]), then $c_{00}$ is super Dedekind complete. Moreover, observe that every $c_{00}$-valued function $f$ has (PR)-bounded range, since $c_{00}$ is (PR)-bounded (see also [32]). Furthermore it is easy to check that in $c_{00}$ the topology generated by pointwise convergence is locally convex and Hausdorff. Finally, if an increasing sequence $(y_{n})_{n}$ in $c_{00}$ has supremum $y \in c_{00}$ according to the order convergence, then $(y_{n})_{n}$ is pointwise convergent to $y$. Therefore, by [36, Corollary J], the space $c_{00}$ is weakly $\sigma$-distributive.

Note that $c_{00}$ does not fulfil property (\sigma). Indeed, for all $n \in \mathbb{N}$, set $u_{n} := (0,1,0,\ldots)$, where the value 1 is assumed at the $n$-th coordinate. Note that, for any sequence $(\lambda_{n})_{n}$ of positive real numbers, the sequence $(\lambda_{n}u_{n})_{n}$ is not order bounded by any element of $c_{00}$.

Example 1. Using the above sequence $(u_{n})_{n}$ we consider a function $f: [0,1] \to c_{00}$, defined by

$$f(x) = \begin{cases} u_{n}, & \text{if } x = 1/n, \quad n = 1,2,\ldots, \\ 0, & \text{otherwise}. \end{cases} \quad (4.3)$$

This function vanishes almost everywhere (with respect to the Lebesgue measure), but is not $(KH)$-integrable on $[0,1]$ (see also [8, Example 7.33], [7, Example 2.8]). Indeed, if $\pi$ is any $\delta$-fine partition with an interval $I$ tagged at $1/n$, then, as $f$ is non-negative, we get $\sum_{\pi} f \geq |I|u_{n}$, so the set of such $\sum_{\pi} f$ is not bounded in $R$.

We now examine some other properties of the $H_{F}$-integral.

It is easy to check that the family of all $H_{F}$-integrable functions on a $\mathcal{B}$-figure is a linear space, and the correspondent linearity equalities for integrals hold.
Observe that, by means of a technique similar to that used in [15] and [16], it is possible to check that our integral satisfies a Cauchy-type criterion.

It is not difficult to see that, if $f \in H_B$, then it is $H_B$-integrable also on any $B$-figure $J \subset L$. The proof is analogous to the one of [15, Proposition 3.5] (see also [16, Proposition 3.6]). The $B$-figure function $\Phi: J \to (H_B) \int_J f$ is called the indefinite $H_B$-integral of $f$. If we suppose that $(H_B) \int_V f = 0$ for each set $V \subset L$ with $\mu(V) = 0$ (this is true, for example, if $R$ and $R_1$ satisfy the hypotheses of Proposition 4.1), the function $\Phi$ is additive in the following sense: $\Phi(A) + \Phi(B) = \Phi(A \cup B)$ if $A, B$ are any two non-overlapping $B$-figures.

Using the mentioned above additivity of the integral and the local character of the basis, we can give a version of the Saks-Henstock lemma for our integral, whose proof is analogous to that of [15, Theorem 3.3] (see also [20, Theorem 3.2.1], [21, Lemma 3.8] and [24, Theorem 1.6.1] for the real-valued case).

**Lemma 4.1.** Under the hypotheses of Proposition 4.1, if an $R_1$-valued function $f$ is $H_B$-integrable on a $B$-figure $L$, and $\Phi$ is its indefinite $H_B$-integral, then

$$\left( O \lim_{\beta \in B} \right) \bigvee \left\{ \left| \sum_{(I,x) \in \pi} f(x) \mu(I) - \Phi(F) \right| : \pi \subset \beta(L) \right\} = 0,$$

where $F$ is the $B$-figure generated by the $\beta$-partition $\pi$ in $L$.

We now recall the following lemma.

**Lemma 4.2.** For any $\beta \in B$ in a $B$-figure $L$ and a closed set $E \subset L$ there exists a basis set $\beta' \subset \beta$ in $L$ such that $I \cap E = \emptyset$ for each $(I,x) \in \beta'[L \setminus E]$.

**Definition 4.2.** Given a $B$-figure $L$, a closed set $E \subset L$ and a basis set $\beta$ in $L$, we say that a $B$-figure $O_E = \bigcup_{j=1}^k I_j$ is a $\beta$-halo of $E$ if $O_E$ is generated by a $\beta[E]$-partition $\{(I_j, x_j)\}_{j=1}^k$ and $C_{\beta}(O_E) \cap E = \emptyset$, where $C_{\beta}(O_E)$ is the $B$-figure complementary to $O_E$ in $L$.

The following result is a straightforward consequence of the previous lemma and definition.

**Lemma 4.3.** For any $\beta \in B$ in a $B$-figure $L$ and a closed set $E \subset L$ there exists $\beta' \subset \beta$ in $L$ such that for any $\beta'$-partition $\pi$ of $L$, the $B$-figure generated by the partition $\pi[E]$ is a $\beta$-halo $O_E$.

We now turn to our main theorem, which is a Hake-type result in the context of abstract derivation basis and extends to the Riesz space setting a result in [31, Theorem 1], which was proved in the particular case $R_1 = R_2 = R = \mathbb{R}$. 
Similar versions of Hake-type theorems were proved in [9, 10, 13, 14], where the basis defined by gauges was considered. In the formulation of the next theorem and in its proof, \( \text{int}_L(F) \) denotes the interior of the set \( F \) with respect to the topology in \( L \) induced by the original topology in the space \( X \).

**Theorem 4.2.** Let \( R \) be a super Dedekind complete weakly \( \sigma \)-distributive Riesz space satisfying property \( \sigma \), and \( L \subset X \) be a \( \mathcal{B} \)-figure. Suppose that there exist a closed set \( E \subset L \) and an increasing sequence of \( \mathcal{B} \)-figures \( (F_k)_k \) such that \( F_k \cap E = \emptyset, \ k \in \mathbb{N}, \) and \( \bigcup_{k=1}^{\infty} \text{int}_L(F_k) = L \setminus E \). Assume that a function \( f: L \to R_1 \) has \( \text{PR} \)-bounded range and is \( H_\mathcal{B} \)-integrable on the set \( E \) and on any \( \mathcal{B} \)-figure \( F \subset L \) with \( F \cap E = \emptyset \).

Furthermore, suppose that there is a positive element \( u \in R \) such that for every \( k \in \mathbb{N} \) there exist \( \hat{\beta}_k \in \mathcal{B} \) with

\[
\left| \sum_{(I,x) \in \pi_k} f(x)\mu(I) - \int_{S_k} f \right| \leq u \quad \text{(H1)}
\]

for every \( \hat{\beta}_k \)-partition \( \pi_k \) in \( F_k \), where \( S_k \) is the \( \mathcal{B} \)-figure generated by \( \pi_k \).

Then \( f \) is \( H_\mathcal{B} \)-integrable on \( L \) with integral value \( A + \int_E f \) if and only if

\[
(\text{O}) \lim_{\beta \in \mathcal{B}} \left[ \left\{ \left| \int_{C_\beta(O_E)} f - A \right| : O_E \text{ is a } \beta\text{-halo of } E \right\} \right] = 0. \quad (4.4)
\]

**Proof.** We begin with the "only if" part. Let \( f \) be \( H_\mathcal{B} \)-integrable on \( L \), and \( \int_L f = A + \int_E f \). Then \( f \) is also \( H_\mathcal{B} \)-integrable on \( L \setminus E \) with integral value \( A \), and thus there exists an \((\text{O})\)-sequence \((p_n)_n\), with the property that for every \( n \in \mathbb{N} \) there is \( \beta_n \in \mathcal{B} \) such that, for every \( \beta \)-partition \( \pi \) of \( L \), with \( \beta \subset \beta_n \), we have

\[
\left| \sum_{(I,x) \in \pi} f(x)\chi_{L\setminus E}(x)\mu(I) - A \right| \leq p_n. \quad (4.5)
\]

Fix any \( \beta \)-halo \( O_E \) corresponding to the above chosen \( \beta \), according to Lemma 4.3. It is a \( \mathcal{B} \)-figure generated by \( \pi[E] \). If \( \pi_0 \) is any \( \beta' \)-partition of the complementary \( \mathcal{B} \)-figure \( C_\beta(O_E) \) with \( \beta' \subset \beta \), then \( \pi_0 \cup \pi[E] \) is a new \( \beta \)-partition of \( L \) for which (4.5) also holds. The function \( f \) is \( H_\mathcal{B} \)-integrable on the \( \mathcal{B} \)-figure \( C_\beta(O_E) \), and so there is an \((\text{O})\)-sequence \((q_m)_m\), such that for any \( m \in \mathbb{N} \) there exists \( \beta_m \in \mathcal{B} \) such that for any \( \beta'_m \)-partition \( \pi_m \) of \( C_\beta(O_E) \) we get

\[
\left| \sum_{(I,x) \in \pi_m} f(x)\mu(I) - \int_{C_\beta(O_E)} f \right| \leq q_m. \quad (4.6)
\]
We can suppose that $\beta'_m \subset \beta$, and so $\pi_m$ is also a $\beta$-partition of $C_\beta(O_E)$. Then, as above, (4.5) holds for the $\beta$-partition $\pi = \pi_m \cup \pi[E]$ of $L$. Note that, for this $\pi$,
\[
\sum_{(I,x) \in \pi} f(x)\chi_{L \setminus E}(x)\mu(I) = \sum_{(I,x) \in \pi_m} f(x)\mu(I). \tag{4.7}
\]
Now combining (4.5), (4.6) and (4.7) we get for any $m$
\[
\left| \int_{C_\beta(O_E)} f - A \right| \leq p_n + q_m. \tag{4.8}
\]
Since $n$ is fixed and the left-hand side of (4.8) does not depend on $m$, we can take in (4.8) the infimum with respect to $m$, getting
\[
\left| \int_{C_\beta(O_E)} f - A \right| \leq p_n.
\]
Since $\beta \subset \beta_n$ and the $\beta$-halo for this $\beta$ was chosen arbitrarily, we obtain (4.4).

So the "only if" part is proved. Note that, in this implication, super Dedekind completeness of $R$ is not used.

We now turn to the "if" part. Suppose that (4.4) holds for some $A \in R$.

Consider the increasing sequence of $\mathcal{B}$-figures $(F_k)_k$ given by assumption and put $T_k = \text{int}_L(F_k) \setminus \text{int}_L(F_{k-1})$ (where $F_0 = \emptyset$). The assumption of the theorem implies that $\bigcup_{k=1}^{\infty} T_k = L \setminus E$ and $T_r \cap T_s = \emptyset$ for every $r \neq s$. It is enough to prove that $f\chi_{L \setminus E}$ is $H\mathcal{B}$-integrable with the integral value $A$.

Since $f$ is $H\mathcal{B}$-integrable on $F_k$ for each $k \in \mathbb{N}$, then, taking into account Lemma 4.1, there is a countable family of $(O)$-sequences $(b^{(k)}_n)_n$, $k \in \mathbb{N}$, such that to every $k, n \in \mathbb{N}$ there corresponds a basis set $\beta^{(k)}_n$ with
\[
\left| \sum_{(I,x) \in \pi_k} f(x)\mu(I) - \int_{S_k} f \right| \leq b^{(k)}_n \tag{4.9}
\]
for all $\beta^{(k)}_n$-partitions $\pi_k$ in $F_k$, where $S_k$ is the $\mathcal{B}$-figure generated by $\pi_k$. Thus, if we consider the directed set $(\mathcal{B}, \subset)$, the net
\[
\left( \bigvee \left\{ \left| \sum_{(I,x) \in \pi_k} f(x)\mu(I) - \int_{S_k} f \right| : \pi_k \text{ is a $\beta$-partition in } F_k \right\} \right)_{\beta \in \mathcal{B}}
\]
is $(O)$-convergent, and hence also $(D)$-convergent. So, for every $k \in \mathbb{N}$ there exists a $(D)$-sequence $(a^{(k)}_{i,j})_{i,j}$ with the property that for every $\varphi \in \mathbb{N}^\mathbb{N}$ there is an element $\beta^{(k)}_k = \beta^{(k)}_{k,\varphi} \in \mathcal{B}$, such that
\[
\left| \sum_{(I,x) \in \pi_k} f(x)\mu(I) - \int_{S_k} f \right| \leq \bigvee_{s=1}^{\infty} a^{(k)}_{i,\varphi(i+k)} \tag{4.10}
\]
for any $\beta^*_k$-partition $\pi_k$ in $F_k$.

Furthermore, by condition (H1), there exists $u \in R$, $u \geq 0$, such that to every $k \in \mathbb{N}$ there corresponds a basis set $\hat{\beta}_k$ with

$$\left| \sum_{(I,x) \in \pi_k} f(x) \mu(I) - \int_{S_k} f \right| \leq u$$

(4.11)

for all $\hat{\beta}_k$-partitions $\pi_k$ in $F_k$. By virtue of Lemma 2.2, there is a regulator $(a_{i,j})_{i,j}$ such that

$$u \land \left( \sum_{k=1}^{q} \left( \bigvee_{i=1}^{\infty} a_{i,\varphi(i+k)}^{(k)} \right) \right) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

(4.12)

for all $q \in \mathbb{N}$ and $\varphi \in \mathbb{N}^\mathbb{N}$.

By Lemma 4.2, for each $k \in \mathbb{N}$ there is $\beta'_k \subset \beta^*_k \cap \hat{\beta}_k$ such that $I \subset \text{int}_{L}(F_k)$ for all $(I,x) \in \beta'_k[\text{int}_{L}(F_k)]$. Now, by the local character of $B$, there exists $\hat{\beta}$ in $L$ with $\hat{\beta}[T_k] \subset \beta'_k[T_k]$ for any $k \in \mathbb{N}$. Now we apply Proposition 2.1 to the $(O)$-net

$$\bigvee \left\{ \left| \int_{C_{O}(E)} f - A \right| : O_E \text{ is a } \beta\text{-halo of } E \right\}$$

and we obtain the existence of a regulator $(c_{i,j})_{i,j}$ with the property that, in correspondence with the function $\varphi$ in (4.10), there exists $\beta'' \in B$ such that

$$\left| \int_{C_{O''}(E)} f - A \right| \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}$$

(4.13)

for every $\beta''$-halo $O_E$.

Take now $\tilde{\beta} \subset \hat{\beta} \cap \beta''$, and pick any $\tilde{\beta}$-partition $\pi$ of $L$. From the definition of $\hat{\beta}$ and since $\tilde{\beta} \subset \hat{\beta}$, the $B$-figure generated by the partition $\pi[E]$ is a $\tilde{\beta}$-halo of $E$, say $O_E$. By construction of $\hat{\beta}$, for all $(I,x) \in \pi[T_k]$ we get $I \subset F_k$. So (4.10) holds with $\pi_k = \pi[T_k]$. Observe that $\int_{\bigcup S_k} f = \sum_k \int_{S_k} f$, because the $H_B$-integral is additive and the partition $\pi$ is finite, and thus there are only finitely many non-empty $S_k$’s, say $S_1, \ldots, S_q$. Moreover, we get

$$\int_{\bigcup S_k} f = \int_{C_{\tilde{\beta}}(O_E)} f.$$
Note that, since $\tilde{\beta} \subset \beta''$, then any $\tilde{\beta}$-halo is also a $\beta''$-halo. Therefore, from (4.10), (4.11), (4.12) and (4.13) we get

$$\left| \sum_{(I,x) \in \pi} f(x)\chi_{L \setminus E}(x)\mu(I) - A \right| \leq \left| \sum_{k=1}^{q} \sum_{(I,x) \in \pi[T_k]} f(x)\mu(I) - \sum_{k=1}^{q} \int_{S_k} f \right| +$$

$$+ \left| \int_{\bigcup S_k} f - A \right| \leq \left| \sum_{k=1}^{q} \sum_{(I,x) \in \pi[T_k]} f(x)\mu(I) - \int_{S_k} f \right| +$$

$$+ \left| \int_{C_{\beta}(OE)} f - A \right| \leq u \wedge \left( \sum_{k=1}^{q} \bigvee_{i=1}^{\infty} a_{i,\varphi(i+k)}^{(k)} \right) + \bigvee_{i=1}^{\infty} c_{i,\varphi(i)} \leq$$

$$\leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}.$$

From this, by Proposition 2.2, we obtain that the decreasing net $q_{\beta} := \bigvee \left\{ \left| \sum_{(I,x) \in \pi} f(x)\chi_{L \setminus E}(x)\mu(I) - A \right| : \pi \in \Pi(\beta, L) \right\}$, $\beta \in B$ is an $(O)$-net and so, by super Dedekind completeness of $R$, it admits a sub-$(O)$-sequence $(q_{\beta_n})_n$, implying the $H_B$-integrability of $f$ on $L \setminus E$ with integral value $A$. Hence $f$ is $H_B$-integrable also on $L$ and $\int_{L} f = A + \int_{E} f$. This completes the proof. \qed

**Remark 3.** Observe that in Lemma 2.1 we have proved that, if $R$ is regular, then condition (H1) is fulfilled. Thus, if in Theorem 4.2 we assume the regularity of $R$, (H1) becomes superfluous.

Moreover, note that in Example 1 the condition (H1) is not fulfilled, since the Riemann sums involved are not bounded in $c_{00}$, and the thesis of Theorem 4.2 is not satisfied: indeed the function $f$ in (4.3) is not Kurzweil-Henstock integrable on $[0, 1]$, though it is so on every subinterval $[a, 1], 0 < a < 1$, with integral equal to zero.

**Acknowledgement.** The authors would thank the referee for his/her valuable suggestions.

**References**


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