COMPARISON OF SOME HENSTOCK TYPE INTEGRALS ON THE CLASS OF RIESZ-SPACE-VALUED FUNCTIONS

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We compare three Henstock-Kurzweil type integrals defined with respect to an abstract differential basis on the class of Riesz-space-valued functions. In contrast with the real-valued case, it is shown that the generalized $H$-integral of Riemann type and the equivalent variational integral have only partial intersection with the descriptively defined $SL$-integral.

In this paper we compare three Henstock-Kurzweil type integrals defined with respect to an abstract differential basis: the Henstock-Kurzweil integral whose construction is based on generalized Riemann sums, see [1, p. 2], the variational integral, and the so-called $SL$-integral defined in [2]. The relationship between these integrals depends essentially on the way of our consideration: on a real-valued functions, on Banach-space-valued functions, or on Riesz-space-valued functions (i.e., the functions taking values in vector lattices). In the case of real-valued functions on a straight segment all three integrals are equivalent, see [3, p. 169; 4]. In the case of Banach-space-valued functions, as can be verified easily, the $SL$-integral is equivalent to the variational integral, but at the same time the both are strictly embedded into the Henstock-Kurzweil integral [5]. Concerning the integration of Riesz-space-valued functions, we show here that in this case the Henstock-Kurzweil integral is equivalent to the variational integral, but is not equivalent to the $SL$-integral. Only under strong additional conditions on the Riesz space all three integrals turn out to be equivalent.

Introduce some definitions and notations. For simplicity sake, we restrict ourselves with integration of functions determined on a segment of the real axis $\mathbb{R}$ although the results obtained here are valid in a more general case also. A differential basis (or simply a basis) $B$ on the segment $[a, b] \subset \mathbb{R}$ is a filter base on the Cartesian product $\mathcal{I} \times [a, b]$, where $\mathcal{I}$ is some family of subsegments of the segment $[a, b]$ called $B$-intervals. In other words, $B$ is a nonempty family of subsets $\beta$ of the set $\mathcal{I} \times [a, b]$ such that each $\beta \in B$ is a set of pairs $(I, x)$, where $I \in \mathcal{I}$, $x \in [a, b]$, and $B$ satisfies the filter base conditions: $\emptyset \notin B$ and for any $\beta_1, \beta_2 \in B$ there exists $\beta \in B$ such that $\beta \subset \beta_1 \cap \beta_2$. Thus, each differential base is a directed set ordered by the "inverse" inclusion. The elements $\beta$ of the set $B$ are called basis sets. In this paper we always assume that if $(I, x) \in B$, then $x \in I$, though in the general integration theory not any basis has this property (see [1, p. 54; 4]). If $E \subset [a, b]$ and $\beta \in B$, we use the notations

\[ \beta(E) = \{(I, x) \in \beta : I \subset E\}, \quad \beta[E] = \{(I, x) \in \beta : x \in E\}. \]

We assume that the base $B$ has a local character [6, Definition 1.1.7] in the following sense: for each element $\beta = \bigwedge_{x \in [a, b]} \alpha_x \subset \mathcal{I} \times [a, b]$ of the Cartesian product \( \prod_{x \in [a, b]} (\beta = \beta_x) \) there exists a basis set $\alpha \in B$ such that

\[ \alpha = \{x \in [a, b] : \alpha_x \subset \beta_x\}. \]

We also assume that $B$ is a Vitali basis, i.e., the set $\{I \times \beta((x)) : I \subset (x - \delta, x + \delta)\}$ is nonempty for any $x$, $\delta > 0$, and $\beta \in B$. The simplest example of a differential Vitali basis having a local character on $\mathbb{R}$ is the complete basis of intervals. In this case the set $\mathcal{I}$ is generated by all the segments of $\mathbb{R}$, and each basis set is defined by a positive function $\delta$ on $\mathbb{R}$ (which is referred to as a scale function) as follows

\[ \beta_\delta = \{(I, x) : I \in \mathcal{I}, x \in I \subset (x - \delta(x), x + \delta(x))\}. \]

Thus, the complete basis of intervals is the family $\beta_\delta$, where $\delta$ is an arbitrary scale function.

A finite set $\pi \subset \beta$ is called a $\beta$-partition if for any two elements $(I', x')$ and $(I'', x'')$ of $\pi$ the segments $I'$ and $I''$ do not have common interior points. If for some $I \in \mathcal{I}$ a partition $\pi = \{(I_1, x_1)\} \subset \beta(I)$ is such that $\bigcup_{I \in \mathcal{I}} I = I$, then we say that $\pi$ is a $\beta$-partition of the segment $I$. If $\pi \subset \beta[E]$, then $\pi$ is called a $\beta$-partition on the set $E$. We say that a basis $B$ possesses the partition property if for any $\beta \in B$ and for each
$B$-interval $I$ there exists its $\beta$-partition. In the particular case of the complete basis of intervals on $\mathbb{R}$ the partition property is known for a long time as the Cousin Lemma. But for some other bases this property was established not long ago [7] and several bases do not possess this property at all or possess it in a rather weakened form as in the case of a symmetric approximate basis [8]. We say that a basis $B$ ignores a set $E \subset [a, b]$ if there exists a basis set $\beta \in B$ such that $\beta(E)$ is empty. We say that $B$ is complete if it does not ignore any nonempty set. We say that $B$ is almost complete if it does not ignore any set of nonzero (exterior) Lebesgue measure $\mu$. Each complete basis $B$ on $[a, b]$ may be associated with an almost complete basis which ignores any set of measure zero. To do this, it is sufficient to include in $B$ together with each basis set $\beta$ all the sets of the form $\beta \setminus \beta[K]$, where $K$ is an arbitrary set of measure zero. By $B_0$ we denote such an extension of the basis $B$. Note that the partition property of the basis $B$ does not guarantee that the basis $B_0$ has the same property.

Let $R$ be some Dedekind-complete Riesz space, see [9, p. 124]. We add two additional elements $+\infty$ and $-\infty$ to $R$ and naturally extend the ordering relation and the linear operations to $R \cup \{+\infty, -\infty\}$. Using the basis $B$ as an index set, we consider directed sets (nets) of the form $\{p_\beta\}_{\beta \in B}$, where $p_\beta \in R$. If a net $\{p_\beta\}_{\beta \in B}$ is monotone decreasing and $\inf_{\beta \in B} p_\beta = 0$, then we call it an $(o)$-net.

We say that a Riesz space $R$ satisfies the condition $\sigma$ (see [9, p. 478]) if for each sequence $(u_n)_n$ in $R$ such that $u_n \geq 0$ for any $n \in \mathbb{N}$ there exists a sequence $(\lambda_n)_n$ of positive real numbers for which the sequence $(\lambda_n u_n)_n$ is bounded in $R$. A Riesz space $R$ satisfies the Schwarz condition if there exists a sequence $(h_n)_n$ in $R$ such that for any $x \in R$ there exist $k, n \in \mathbb{N}$ satisfying $|x| \leq k h_n$, see [10].

Now we introduce the Henstock type integral corresponding to a basis $B$ (see [1, 11] concerning the real-valued case). In the following definition and below $|I|$ stands for the length of an interval $I$.

**Definition 1.** Let $R$ be a Dedekind-complete Riesz space, $B$ be a fixed complete basis possessing the partition property, and $J \subset R$ be some $B$-interval. A function $f : J \to R$ is called integrable in the sense of Henstock–Kurzweil on $J$ with respect to the basis $B$ (in short, we say $H_B$-integrable) and the value of the integral is equal to $Y \in R$ if

$$\inf_{\beta \in B} \left( \sup \left\{ \left( \sum_{i=1}^{q} f(\xi_i) |I_i| - Y \right) : \{(I_i, \xi_i) : i = 1, \ldots, q\} \in \beta(J), \bigcup_{i=1}^{q} I_i = I \right\} \right) = 0. \quad (1)$$

We write down the value of the integral as $(H_B) \int_J f = Y$.

It is easy to verify that the value $Y$ of the integral in (1) is uniquely determined.

The following two assertions may be proved word for word as the corresponding assertions in [12].

**Assertion 1.** If $I \subset J$, where $I$ and $J$ are $B$-intervals, and $f$ is $(H_B)$-integrable on $J$, then $f$ is $(H_B)$-integrable on $I$ too.

**Assertion 2.** If $K = I \cup J$, where $K, I, J$ are $B$-intervals, and $I$ and $J$ do not overlap, and $f$ is $H_B$-integrable on $K$, then $(H_B) \int_K f = (H_B) \int_I f + (H_B) \int_J f$.

These assertions imply that for any $H_B$-integrable function $f : J \to R$ an additive $R$-valued function

$$F(I) = (H_B) \int_I f \quad (2)$$

is determined on the family of all $B$-intervals contained in $J$; we call this function the indefinite $H_B$-integral.

The following version of the so-called Saks–Henstock lemma may be obtained by the methods used in [13] for proving a rather less general version of this Lemma (see also [14, Lemma 12, pp. 353–354]).

**Lemma.** If $f$ is $(H_B)$-integrable on $J$ and $F$ is the indefinite integral given by equality (2), then

$$\inf_{\beta \in B} \left( \sum_{(I, x) \in \pi} |f(x)| |I| - F(I) : \pi \in \beta(J) \right) = 0. \quad (3)$$

For a fixed basis $B$ and a function $F : I \times J \to R$, for each set $E \subset J$ let us define

$$\Var(\beta, F, E) = \sup_{\pi \subset \beta(E)} \sum_{(I, x) \in \pi} |F(I, x)|.$$
In this case assume that \( \text{Var} (\beta, F, E) = 0 \) providing the set \( \beta[E] \) is empty. For a complete basis \( B \) we also define

\[
V (B, F, E) = \inf_{\beta \in B} \text{Var} (\beta, F, E).
\]

The function \( \text{Var} (\beta, F, \cdot) \) considered as a set function determined on the family of all subsets of the \( B \)-interval \( J \) is called the \( \beta \)-variation and \( V (B, F, \cdot) \) is called the variational measure with respect to the basis \( B \) generated by the function \( F \) (the term “measure” is used here because in the real-valued case the variational measure is a metric exterior measure). If in this definition we replace a complete basis \( B \) by the almost complete basis \( B_0 \) obtained as the extension of the basis \( B \), then we get the definition of the essential variational measure:

\[
V_{ess} (B_0, F, E) = \inf_{\beta \in B_0} \sup_{\pi \subset \beta[E]} \sum_{(I, x) \in \pi} |F(I, x)|.
\]

It is clear that

\[
V_{ess} (B_0, F, E) \leq V (B, F, E).
\] (4)

**Definition 2.** Two functions \( F \) and \( G \) determined on \( I \times J \) are said to be variationally equivalent if \( V (B, F - G, J) = 0 \).

**Definition 3.** A function \( f : J \to R \) is said to be \( B \)-variationally integrable (VB-integrable) if there exists an additive \( B \)-interval function \( \tau \) which (when considered as a function on \( I \times J \) and being constant with respect to the second variable for each fixed value of the first variable) is variationally equivalent to the function \( f(x)|I| \). In this case we assume by definition \( \text{VB} \int_J f = \tau (J) \), and the function \( \tau \) is called the indefinite VB-integral of the function \( f \).

The following result is a direct corollary of the Saks–Henstock lemma.

**Theorem 1.** The VB-integral is equivalent to the \( H_B \)-integral.

**Definition 4.** Two functions \( F \) and \( G \) determined on \( I \times J \) are said to be variationally almost equivalent if \( V_{ess} (B_0, F - G, J) = 0 \).

Due to (4), the variational equivalence implies the variational almost equivalence. Now we introduce the concept of the \( SL \)-integral with respect to a basis \( B \). In the case of a complete basis this integral was considered in [2] and [3, p. 155] for real-valued functions and in [12] for Riesz-space-valued functions.

**Definition 5.** A variational measure \( V (B, F, \cdot) \) is said to be absolutely continuous (with respect to the Lebesgue measure \( \mu \)) if \( V (B, F, N) = 0 \) for any set \( N \subset E \) such that \( \mu (N) = 0 \).

**Definition 6.** We say that an \( R \)-valued \( B \)-interval function \( \tau \) belongs to the class \((SL) \) or has the property \((SL) \) on a \( B \)-interval \( J \) if the variational measure generated by this function is absolutely continuous on \( J \).

Note that the notation \((SL) \) was introduced in [2] as an abbreviation of the words “strong Luzin property.”

**Definition 7.** Let \( R \) be a Dedekind-complete Riesz space. A function \( f : J \to R \) is called \( SL \)-integrable on the \( B \)-interval \( J \) if there exists an additive \( B \)-interval function \( \tau \) from the class \((SL) \) which (when considered as a function on \( I \times J \) and being constant with respect to the second variable for each fixed value of the first variable) is variationally almost equivalent to the function \( f(x)|I| \). In this case we assume by definition \((SL) \int_J f = \tau (J) \), and the function \( \tau \) is called the indefinite \( SL \)-integral of the function \( f \).

**Assertion 3.** If a function \( f \) is \( H_B \)-integrable and its indefinite \( H_B \)-integral \( F \) belongs to the class \((SL) \), then \( f \) is also \( SL \)-integrable and \( F \) is its indefinite \( SL \)-integral.

**Proof.** The assertion follows from the fact that the variational equivalence evidently implies the variational almost equivalence.

The following result is obtained directly from the definitions.

**Assertion 4.** Let \( R \) be a Dedekind-complete Riesz space, \( N \subset J \) be a set of zero measure, and \( f : J \to R \) be a function such that \( f(x) = 0 \) for all \( x \notin N \). Then \( f \) is \( SL \)-integrable on \( J \) and its indefinite \( SL \)-integral is the identical zero function.

**Corollary.** Let \( R \) be a Dedekind-complete Riesz space and two functions \( f, g : J \to R \) coincide almost everywhere on \( J \). Then the \( SL \)-integrability of one of the functions implies the \( SL \)-integrability of another one and \((SL) \int_J f = (SL) \int_J g \).

Generally speaking, the \( SL \)-integral is not equivalent to the \( H_B \)-integral. An example of an \( SL \)-integrable function being not \( H_B \)-integrable may be obtained from Example 4.21 presented in [12]. In this example the
space $R$ does not satisfy the condition $\sigma$. Below we construct an example of an $H\delta$-integrable function being not $SL$-integrable. This example also uses a space $R$ not satisfying the condition $\sigma$. Thus, this condition is necessary for the equivalence of the $SL$-integral and the $H\delta$-integral.

**Example.** Consider binary intervals $\Delta_j^k = \left[\frac{j}{2^k}, \frac{j+1}{2^k}\right)$, $0 \leq j \leq 2^k - 1$, $k \in \mathbb{N} \cup \{0\}$. Note that $(0, 1) = \bigcup_{k=1}^{\infty} \Delta_2^k$ and $\Delta_1^k = \Delta_2^{k+1} \cup \Delta_3^{k+1}$. Define a basis $B$ on $[0, 1)$ as follows. Let $I$ consist of right open intervals $I = [u, v)$ satisfying the following conditions: if $u = 0$, then $I = \Delta_0^k$, $k \in \mathbb{N} \cup \{0\}$; if $u \in \Delta_2^{k+1}$, $j = 2$ or $3$, $k \in \mathbb{N}$, then $I \cap \Delta_j^{k+1}$, where $j = 2$ or $3$, respectively. Consider the family of scale functions $\delta$ each of which is constant on each interval $\Delta_j^{k+1}$, $j = 2, 3$, $k \in \mathbb{N}$. Define the basis $B$ as a system of the basis sets

$$
\beta_\delta = \{(I, x) : I = [u, v) \in I, x = u, \mu(I) \leq \delta(x)\}
$$

corresponding to the scale functions from the above family. It is easy to verify that this basis possesses the partition property and so the $H\delta$-integral is defined for it. Further, assuming that $R$ is a Riesz space not satisfying the condition $\sigma$, we choose a sequence $(p_k)_k$ of elements from $R$ so that $p_k \geq 0$ for all $k \in \mathbb{N}$ and for any sequence $(\lambda_k)_k$ of positive numbers the sequence $(\lambda_k p_k)_k$ is not bounded in $R$. Define a function $f : [0, 1) \to R$ as follows: $f(x) = 0$ for $x = 0$ and $f(x) = (-1)^j p_k$ for $x \in \Delta_j^{k+1}$, $j = 2, 3$, $k \in \mathbb{N}$. It is easy to verify that the function $f$ is $H\delta$-integrable with respect to the basis considered. Besides,

$$
(H\delta) \int_{\Delta_0^k} f = 0 \text{ for any } k \in \mathbb{N} \cup \{0\}
$$

and

$$
(H\delta) \int_I f = f(u) \frac{\mu(I)}{2k} \text{ for any } I = [u, v) \in I, u \neq 0.
$$

In particular, denoting the indefinite integral of the function $f$ by $F$, we get the equality

$$
F(I) = (H\delta) \int_I f = p_k \frac{\mu(I)}{2k}
$$

for $I \subset \Delta_2^{k+1}$. Now choose a countable set $E = \{x_k : k \in \mathbb{N}\}$ so that $x_k \in \Delta_2^{k+1}$. Note that for any $\beta_\delta$-partition $\pi \subset \{(I_k, x_k)\}_k$ on $E$ the inclusion $I_k \subset \Delta_2^{k+1}$ holds and, therefore,

$$
\sum_{(I, x) \in \pi} |F(I, x)| = \sum_{(I, x) \in \pi} p_k \frac{\mu(I)}{2k}.
$$

Due to the choice of the sequence $(p_k)_k$, the set of the values of these sums is not bounded. Hence $\text{Var}(\beta_\delta, F, E) = \text{V}(B, F, E) = +\infty$ and $F$ does not thus belong to the class $(SL)$.

Formulate some indirect characteristics of spaces $R$ which the $H\delta$-integrability implies the $SL$-integrability for.

**Assertion 5.** The indefinite $H\delta$-integral of an $R$-valued function $f$ being $H\delta$-integrable on a $B$-interval $J$ belongs to the class $(SL)$ if and only if for any set $N \subset J$ of zero Lebesgue measure the function $f \chi_N$ is $H\delta$-integrable on $J$ and its integral is equal to zero.

We omit a simple proof of this assertion based on Lemma.

**Theorem 2.** An $R$-valued $H\delta$-integrable on a $B$-interval $J$ function $f$ is $SL$-integrable on $J$ and the values of the integrals coincide with each other if and only if for any set $N \subset J$ of zero Lebesgue measure the function $f \chi_N$ is $H\delta$-integrable on $J$ and its integral is equal to zero.

**Proof.** It is sufficient to apply Assertions 3 and 5. \qed

This theorem implies the following result.

**Theorem 3.** If a space $R$ is such that any $R$-valued function being equal to zero almost everywhere on a $B$-interval $J$ is $H\delta$-integrable on $J$ and its integral is equal to zero, then any $R$-valued $H\delta$-integrable on $J$ function $f$ is $SL$-integrable on $J$ and the values of the integrals coincide with each other.

For a very strong restriction to the class of the spaces $R$ considered here being a sufficient condition for an arbitrary $R$-valued function equal to zero almost everywhere on a $B$-interval $J$ to be $H\delta$-integrable on $J$ and the integral to be equal to zero, one may take the assumption that $R$ possesses the property $\sigma$ and the Schwarz property simultaneously.
Now consider conditions under which the \( SL \)-integrability implies the \( H_B \)-integrability.

**Theorem 4.** If an \( R \)-valued function \( f \) is \( SL \)-integrable on a \( B \)-interval \( J \) with the \( SL \)-integral \( F \) and if for any set \( N \subset J \) of zero Lebesgue measure the function \( f_X \) is \( H_B \)-integrable on \( J \) with zero integral, then \( f \) is \( H_B \)-integrable on \( J \) and \( F \) is its indefinite \( H_B \)-integral.

**Proof.** In accordance with the \( SL \)-integrability of the function \( f \), there exists an \((o)\)-net \( \{r_\theta\}_{\theta \in \mathcal{B}_0} \) such that for any partition \( \pi \subset \theta \) of the interval \( J \) we have:

\[
\sum_{(I,x) \in \pi} |f(x)| I - F(I)| \leq r_\theta.
\]

Each basis set \( \theta \in \mathcal{B}_0 \) defines a set \( N \) of zero measure such that the set \( \theta[N] \) is empty. By the assumption, the function \( f_X \) is \( H_B \)-integrable with zero integral, and this together with Lemma imply the existence of an \((o)\)-net \( \{p_{\theta,\beta}\}_{\beta \in \mathcal{B}} \) such that for any partition \( \pi \subset \beta[N] \) we have

\[
\sum_{(I,x) \in \pi} |f(x)| \chi_{\theta}(x)|I| \leq p_{\theta,\beta}.
\]

Another \((o)\)-net \( \{s_{\theta,\beta}\}_{\beta \in \mathcal{B}} \) is defined by the fact that \( F \) belongs to the class \((SL)\). For this net and for any partition \( \pi \subset \beta[N] \) the inequality \( \sum_{(I,x) \in \pi} |F(I)| \leq s_{\theta,\beta} \) holds.

Now define a basis set \( \gamma \in \mathcal{B} \) satisfying the condition

\[
\gamma([x]) \subset \begin{cases} \beta([x]), & x \in N; \\ \theta([x]) \cap \beta([x]), & x \notin N. \end{cases}
\]

(Such a basis set exists due to a local character of the basis \( \mathcal{B} \).)

Summing the estimates obtained, we get the following inequality for an arbitrary partition \( \pi \subset \gamma \):

\[
\sum_{(I,x) \in \pi} |f(x)| I - F(I)| \leq \sum_{(I,x) \in \pi, x \in J \setminus N} |f(x)| I - F(I)| + \sum_{(I,x) \in \pi, x \in N} |F(I)| \leq r_\theta + p_{\theta,\beta} + s_{\theta,\beta}.
\]

It is clear that \( \inf \{r_\theta + p_{\theta,\beta} + s_{\theta,\beta} : \theta \in \mathcal{B}_0, \beta \in \mathcal{B} \} = 0 \), which easily implies that \( f \) is \( H_B \)-integrable and \( F \) is its indefinite \( H_B \)-integral.

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