ON THE DE GIORGI—LETTA INTEGRAL WITH RESPECT TO MEANS WITH VALUES IN RIESZ SPACES

Abstract

A monotone integral is given for scalar function, with respect to Riesz space values means, and also a necessary and sufficient condition to obtain a Radon-Nikodym density for two means.

1 Introduction

Integrals like Kurzweil-Stieltjes, Riemann sums and Bochner have been studied in vector lattices by Duchoň, Riečan and Vrábelová, ([11], [21], [22]), Wright ([26], [27]), McGill ([19]), Šipoš ([24]), Maličký ([18]), Cristescu ([8]), Haluška ([15]), Boccuto ([3], [4]), and others.

In this paper we extend to such spaces the monotone integral, given by Choquet in 1953 ([6]), and developed by De Giorgi-Letta ([9]), Greco ([13]), Brooks-Martellotti ([5]), and others ([10], [12], [16], etc.).

Given a mean \( \mu : \mathcal{A} \to R \) and a measurable function \( f : X \to \mathbb{R}^+ \), we say that \( f \) is integrable (in the monotone sense) if the following limit exists in \( R \).

\[
(\sigma) - \lim_{a \to +\infty} \int_0^a \mu(\{x \in X : f(x) > t\}) \, dt.
\]

For this integral we obtain some elementary properties and we give some Vitali-type theorems. We note that in general this integral is different from the one...
introduced in [5] for Banach spaces. Finally we prove a version of Radon-Nikodym-type theorems for the introduced integral (see also [14]).

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2 Preliminaries

We begin with some definitions.

**Definition 2.1.** A Riesz space $R$ is called Archimedean if the following property holds.

(2.1.1) For every choice of $a, b \in R$, if $na \leq b$ for all $n \in \mathbb{N}$, then $a \leq 0$.

**Definition 2.2.** A Riesz space $R$ is said to be Dedekind complete (resp. σ-Dedekind complete) if every nonempty (countable) subset of $R$, bounded from above, has supremum in $R$.

The following results are well-known (see [1], [2]).

**Proposition 2.3.** Every σ-Dedekind complete Riesz space is Archimedean.

**Theorem 2.4.** Given an Archimedean (Dedekind complete) Riesz space $R$, there exists a compact Stonian topological space $\Omega$, unique up to homeomorphisms such that $R$ can be embedded as a (solid) subspace of

$$C_\infty(\Omega) = \{f \in \mathbb{R}^\Omega : f \text{ is continuous, and } \{\omega : |f(\omega)| = +\infty\}\}$$

is nowhere dense in $\Omega$. Moreover if $(a_\lambda)_{\lambda \in \Lambda}$ is any family such that $a_\lambda \in R \forall \lambda$ and $a = \sup_\lambda a_\lambda \in R$ (where the supremum is taken with respect to $R$), then $a = \sup_\lambda a_\lambda$ with respect to $C_\infty(\Omega)$ and the set $\{\omega \in \Omega : (\sup_\lambda a_\lambda)(\omega) \neq \sup_\lambda a_\lambda(\omega)\}$ is meager in $\Omega$.

**Definition 2.5.** A sequence $(r_n)_n$ is said to be order-convergent (or (o)-convergent) to $r$, if there exists a sequence $(p_n)_n \in R$ such that $p_n \downarrow 0$ and $|r_n - r| \leq p_n$, $\forall n \in \mathbb{N}$, and we will write $(o) - \lim_n r_n = r$.

As $|r_n| \leq |r| + p_1 \forall n$, every (o)-convergent sequence is bounded. We note that, if $R$ is a σ-Dedekind complete Riesz space, (o)-convergence can be formulated in the following equivalent ways (see also [25]).

**Proposition 2.6.** A sequence $(r_n)_n$, bounded in $R$, (o)-converges to $r$ if and only if $r = (o) - \lim sup_n r_n = (o) - \lim inf_n r_n$, where $(o) - \lim sup_n r_n = \inf_n[\sup_{m \geq n} r_m], \quad (o) - \lim inf_n r_n = \sup_n[\inf_{m \geq n} r_m]$. 

Proposition 2.7. Let $R$ be as above, $\Omega$ as in Theorem 2.4. A bounded sequence $(r_n)_n$, $r_n \in R$, $(o)$-converges to $r$ if and only if the set \{\omega \in \Omega : r_n(\omega) \not\to r(\omega)\} is meager in $\Omega$.

We recall some fundamental properties of the order convergence (see [25]).

Proposition 2.8. If $(r_n)_n(o)$-converges to both $r$ and $s$, then $r \equiv s$. If $(r_n)_n(o)$-converges to $r$, $(s_n)_n(o)$-converges to $s$ and $\alpha \in \mathbb{R}$, then $(r_n + s_n)_n$, $(r_n \lor s_n)_n$, $(r_n \land s_n)_n$, $(\alpha r_n)_n$, $(|r_n|)_n(o)$-converge respectively to $r + s$, $r \lor s$, $r \land s$, $\alpha r$, $|r|$.

Definition 2.9. A sequence $(r_n)_n$ is said to be $(o)$-Cauchy if there exists a sequence $(p_n)_n \in \mathbb{R}$ such that $p_n \downarrow 0$ and $|r_n - r_m| \leq p_n$, $\forall n \in \mathbb{N}$, and $\forall m \geq n$.

Definition 2.10. A Riesz space $R$ is called $(o)$-complete if every $(o)$-Cauchy sequence is $(o)$-convergent.

The following result holds (see [17], [28]).

Proposition 2.11. Every $\sigma$-Dedekind complete Riesz is $(o)$-complete.

We note that there are some cases, in which $(o)$-convergence is not “generated” by a topology. For example, $L^0(X, \mathcal{B}, \mu)$, where $\mu$ is a $\sigma$-additive non-atomic positive $\mathbb{R}$-valued measure. We recall that, in such spaces, $(o)$-convergence coincides with almost everywhere convergence. (Also see [25].)

3 The Monotone Integral

Definition 3.1. Let $X$ be any set, $R$ a Dedekind complete Riesz space, $\mathcal{A} \subset \mathcal{P}(X)$ an algebra. A map $\mu : \mathcal{A} \to R$ is said to be mean if $\mu(A) \geq 0$, $\forall A \in \mathcal{A}$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A \cap B = \emptyset$. A mean $\mu$ is countably additive (or $\sigma$-additive) if $\mu(\cap_n A_n) = \inf_n \mu(A_n)$, whenever $(A_n)_n$ is a decreasing sequence in $\mathcal{A}$, such that $\cap_n A_n \in \mathcal{A}$.

Given a mapping $f : X \to \mathbb{R}_0^+$ and a mean $\mu$ as above for all $A \in \mathcal{A}$ and $t \in \mathbb{R}_0^+$, set $E_{t,A}^f$ (or simply $E_{t,A}$, when no confusion can arise) \equiv $\{x \in A : f(x) > t\}$; $E_{t}^f(E_t) \equiv \{x \in X : f(x) > t\}$; and, for every $t > 0$, let $u_{A,f}(t) = \mu(E_{t,A}^f)$; $u_f(t) = u(t) = \mu(E_t^f)$.

Definition 3.2. With the same notation as above, we say that a function $f : X \to \mathbb{R}_0^+$ is measurable if $E_t^f \in \mathcal{A}$, $\forall t \in \mathbb{R}_0^+$.

Now we define a Riemann (Lebesgue)-type integral, for maps, defined in an interval of the real line and taking values in a Dedekind complete Riesz space. (For similar integrals existing in the literature, also see [21] and [20].)
Definition 3.3. Let \( a, b \in \mathbb{R}, a < b, \) and \( R \) be as above. We say that a map \( g : [a, b] \to R \) is a step function if there exist \( n + 1 \) points \( x_0 \equiv a < x_1 < \ldots < x_n \equiv b \) such that \( g \) is constant in each interval of the type \( [x_{i-1}, x_i) \) \( (i = 1, \ldots, n) \). We say that \( g \) is simple if there exist \( n \) elements of \( R, a_1, \ldots, a_n, \) and \( n \) pairwise disjoint measurable sets \( E_i \) such that \( g = \sum_{i=1}^{n} a_i \chi_{E_i} \). If \( g \) is a step (simple) function, we put \( \int_{a}^{b} g(t) \, dt = \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot g(\xi_i) \cdot \sum_{i=1}^{n} |E_i| \cdot g(\xi_i) \), where \( \xi_i \) is an arbitrary point of \( [x_{i-1}, x_i] \) \( |E_i| \).

Definition 3.4. Let \( u : [a, b] \to R \) be a bounded function. We call the upper integral (resp. lower integral) of \( u \) the element of \( R \) given by

\[
\inf_{v \in V_u} \int_{a}^{b} v(t) \, dt \left[ \sup_{s \in S_u} \int_{a}^{b} s(t) \, dt \right] ,
\]

where

\[ V_u \equiv \{ v : v \text{ is a step (simple) function, } v(t) \geq u(t), \, \forall t \in [a, b] \} \]

\[ S_u \equiv \{ s : s \text{ is a step (simple) function, } s(t) \leq u(t), \, \forall t \in [a, b] \} . \]

We say that \( u \) is Riemann (Lebesgue) integrable (or \( R \), resp. \( L \))-integrable if its lower integral coincides with its upper integral and, in this case, we call integral of \( u \) (and write \( \int_{a}^{b} u(t) \, dt \)) their common value.

It is easy to check that this integral is well-defined, and is a linear monotone functional, with values in \( R \).

The following result holds.

Proposition 3.5. Every bounded monotone map \( u : [a, b] \to R \) is Riemann integrable.

Proof. The proof is almost identical to the classical one. \( \square \)

Now we define an integral for extended real-valued functions with respect to \( R \)-valued means.

Definition 3.6. Let \( X, R, \mu, f : X \to \mathbb{R}_0^+ \), \( u = u_f \) be as above. We say that \( f \) is integrable if the quantity

\[
(3.6.1) \quad \int_{0}^{+\infty} u(t) \, dt \equiv \sup_{a > 0} \int_{0}^{a} u(t) \, dt = (o) - \lim_{a \to +\infty} \int_{0}^{a} u(t) \, dt ,
\]

exists in \( R \) where the integral in \( (3.6.1) \) is intended as in Definition 3.4. If \( f \) is integrable, we denote the element in \( (3.6.1) \) by \( \int_{X} f \, d\mu \). A measurable function \( f : X \to \mathbb{R} \) is integrable if both \( f^+ \), \( f^- \) are integrable and, in this case we set \( \int_{X} f \, d\mu = \int_{X} f^+ \, d\mu - \int_{X} f^- \, d\mu \).
Remark 3.7. We can extend Definition 3.6 when $\mu : A \to R$ is any finitely additive bounded map. A measurable function $f$ is integrable if and only if $f$ is integrable with respect to $\mu^+, \mu^-$, where for every $A \in A$

$$\mu^+(A) \equiv \lor_{B \subset A, B \in A} \mu(B),$$

$$\mu^-(A) \equiv -\lor_{B \subset A, B \in A} \mu(B),$$

and $\mu = \mu^+ - \mu^-$. In this case, we set $\int_A f d\mu \equiv \int_X f d\mu^+ - \int_X f d\mu^-$. (Also see [7].)

An immediate consequence of Definition 3.6 and monotonicity of $\mu$ is the following assertion.

**Proposition 3.8.** If $f$ is integrable, then for each $A \in A$, $\sup_{a>0} \int_0^a u_{A,f}(t)\, dt$ exists in $R$ and is denoted by $\int_A f d\mu$.

**Proposition 3.9.** With the same notation as above, if $f$ is integrable, then $\int_A f d\mu = \int_X f \cdot \chi_A d\mu$, $\forall A \in A$.

**Proof.** For each fixed $t > 0$ and $x \in X$, we have $[f \cdot \chi_A(x) > t]$ if and only if $[x \in A]$ and $[f(x) > t]$. So, $u_{X,f \chi_A} = u_{A,f}$. Thus, the assertion follows.

It is easy to check that this integral is a linear $R$-valued functional and that, for every positive integrable map $f$, $\int f d\mu$ is a mean.

We now list a number of technical results.

**Proposition 3.10.** If $f$ is integrable, then $\lim_{t \to +\infty} \mu(E_t) = 0$ and hence $\mu(E_\infty) = 0$, where $E_\infty = \{x \in X : f(x) = +\infty\}$.

**Proof.** For every $t > 0$, we have

$$0 \leq \mu(E_\infty) \leq \mu(E_t) = \frac{\int_{E_t} f d\mu}{t} \leq \frac{\int_X f d\mu}{t} \leq \frac{\int_X f d\mu}{t}.$$ 

Taking the infimum, we obtain $0 \leq \mu(E_t) \leq \inf_{t>0} \frac{\int_X f d\mu}{t} = 0$. 

**Proposition 3.11.** Let $f : X \to \mathbb{R}_0^+$ be measurable. Then, $f$ is integrable if and only if $\sup_n \int_X (f \wedge n) d\mu \in R$, and in this case $\sup_n \int_X (f \wedge n) d\mu = \int_X f d\mu$.

**Proof.** Fix $n \in \mathbb{N}$ and pick $t < n$. Then $f(x) \wedge n > t$ if and only if $f(x) > t$ and so $\int_0^n u_f(t)\, dt = \int_0^n u_{f \wedge n}(t)\, dt = \int_0^{+\infty} u_{f \wedge n}(t)\, dt = \int_X (f \wedge n) d\mu$. So the first part of the assertion follows immediately. Moreover taking the suprema, we get $\sup_n \int_X (f \wedge n) d\mu = (o) - \lim_{n \to +\infty} \int_0^n u_f(t)\, dt = \int_X f d\mu$. 

$\square$
Proposition 3.12. Let \( f : X \to \mathbb{R}_0^+ \) be measurable and bounded and set \( S_f \) (resp. \( V_f \)) \( \equiv \{ g : X \to \mathbb{R} : g \leq f, \ g \text{ is simple} \} \) (resp. \( \{ h : X \to \mathbb{R} : h \geq f, \ h \text{ is simple} \} \)). Then \( \int_X f \, d\mu = \sup_{g \in S_f} \int_X g \, d\mu = \inf_{h \in V_f} \int_X h \, d\mu \), and \( f \) is integrable.

**Proof.** It suffices to prove the part involving \( S_f \). Let \( L = \sup_{x \in X} f(x) \) and, for every fixed \( n \in \mathbb{N} \), let \( s_n(0) \equiv u(0) \), and \( s_n(t) \equiv u \left( \frac{t - 1}{2^n} \right) \) whenever \( t \in \left[ \frac{L(i - 1)}{2^n}, \frac{L i}{2^n} \right] \) \((i = 1, \ldots, 2^n)\). We have \( \int_0^L s_n(t) \, dt = \sum_{i=1}^{2^n} \frac{L i}{2^n} u \left( \frac{L i}{2^n} \right) \). Put

\[
U_i^{(n)} \equiv \left\{ x \in X : f(x) > \frac{L i}{2^n} \right\};
\]

\[
g_n \equiv \sum_{i=1}^{2^n} \frac{L}{2^n} \chi_{U_i^{(n)}}, \forall n \in \mathbb{N}, \ i = 1, 2, \ldots, 2^n.
\]

Then (Also see [9].) \( \int_X g_n \, d\mu = \sum_{i=1}^{2^n} \frac{L}{2^n} \mu(U_i^{(n)}) = \sum_{i=1}^{2^n} \frac{L}{2^n} u \left( \frac{L i}{2^n} \right) \). Taking the supremum, we get

\[
\int_X f \, d\mu = \int_0^L u(t) \, dt = \sup_n \int_X g_n \, d\mu = (o) - \lim_n \int_X g_n \, d\mu.
\]

If \( g \in S_f \), then \( \int_X g \, d\mu \leq \int_X f \, d\mu \), and so \( \int_X f \, d\mu = \sup_{n \in \mathbb{N}} \int_X g_n \, d\mu \leq \sup_{g \in S_f} \int_X g \, d\mu \leq \int_X f \, d\mu \), which completes the proof. \( \Box \)

Proposition 3.13. If \( f : X \to \mathbb{R}_0^+ \) is integrable, then \( \int_X f \, d\mu = \sup_{g \in S_f} \int_X g \, d\mu \).

Conversely, if \( g \geq 0 \) is such that the quantity \( \sup_{g \in S_f} \int_X g \, d\mu \) exists in \( R \), then \( f \) is integrable and \( \int_X f \, d\mu = \sup_{g \in S_f} \int_X g \, d\mu \).

**Proof.** The assertion follows by Propositions 3.11 and 3.12. \( \Box \)

The following result is easy also.

Proposition 3.14. Let \( f : X \to \mathbb{R}_0^+ \) be an integrable map, \( g : X \to \mathbb{R}_0^+ \) measurable such that \( 0 \leq g(x) \leq f(x), \ \forall x \in X \). Then \( g \) is integrable, and

\[
\int_X g \, d\mu \leq \int_X f \, d\mu.
\]

Now we note that if \( \mu : X \to R \) is a mean and \( C_\infty(\Omega) \) is as in Theorem 2.4, then there exists a nowhere dense set \( \Omega' \subset \Omega \) such that \( \mu(A)(\omega) \) is real, \( \forall \omega \notin \Omega' \), \( \forall A \in A \).
Proposition 3.15. Let \( R \subset C_\infty(\Omega) \) be a Dedekind complete Riesz space where \( \Omega' \) is as above and set \( \mu_\omega(A) \equiv \mu(A)(\omega), \ \forall \omega \notin \Omega' \). Assume that \( f : X \to \mathbb{R} \) is an integrable map. Then there exists a meager set \( N \subset \Omega \) such that \( f \) is integrable with respect to \( \mu_\omega \) and \( \int_A f \, d\mu_\omega = \left( \int_A f \, d\mu \right)(\omega), \ \forall \omega \in N^c, \ \forall A \in \mathcal{A} \).

**Proof.** Without loss of generality, we can assume that \( f \) is nonnegative. First suppose that \( f \) is bounded. There exists a sequence of simple functions \( (s_n)_n \) such that \( s_n \uparrow f \) and \( \int_A s_n \, d\mu \uparrow \int_A f \, d\mu \). So we have, for every \( n \in \mathbb{N} \), up to the complement of a meager set, depending only on \( X \)

\[
0 \leq \left| \int_A f \, d\mu_\omega - \left( \int_A f \, d\mu \right)(\omega) \right| \\
\leq \left| \int_A f \, d\mu_\omega - \int_A s_n \, d\mu_\omega \right| + \left| \int_A s_n \, d\mu_\omega - \left( \int_A f \, d\mu \right)(\omega) \right| \\
= \left| \int_A f \, d\mu_\omega - \int_A s_n \, d\mu_\omega \right| + \left| \left( \int_A s_n \, d\mu \right)(\omega) - \left( \int_A f \, d\mu \right)(\omega) \right| \\
\leq \int_X f - s_n \, d\mu_\omega + \left( \int_X f - s_n \, d\mu \right)(\omega).
\]

Then

\[
0 \leq \left| \int_A f \, d\mu_\omega - \left( \int_A f \, d\mu \right)(\omega) \right| \\
\leq \limsup_n \int_X f - s_n \, d\mu_\omega + \limsup_n \left( \int_X f - s_n \, d\mu \right)(\omega) \\
= \inf_n \int_X f - s_n \, d\mu_\omega + \inf_n \left( \int_X f - s_n \, d\mu \right)(\omega) = 0.
\]

Assume now that \( f \) is integrable. By the previous step, there exists a meager set \( N^* \) such that, \( \forall n \in \mathbb{N}, \forall \omega \notin N^*, \forall A \in \mathcal{A} \)

\[
\int_A (f \wedge n) \, d\mu_\omega = \left( \int_A f \wedge n \, d\mu \right)(\omega).
\]

The proof is now analogous to the first part. It will be enough to replace \( s_n \) with \( f \wedge n \).

Now we prove the following theorem.
Theorem 3.16. Let \( f : X \to \bar{\mathbb{R}}_0^+ \) be an integrable map. Then there exists a meager set \( N \) such that for every \( A \in \mathcal{A} \) and for every \( \omega \notin N \),
\[
\left( \int_A f \, d\mu \right)(\omega) \in (\mu(A) \bar{\mu} \{ f(x) : x \in A \})(\omega).
\]

Proof. By Proposition 3.15 and classical results we have, up to the complement of a meager set
\[
\left( \int_A f \, d\mu \right)(\omega) = \int_A f \, d\mu_\omega \in \mu_\omega(A) \bar{\mu} \{ f(x), x \in A \}
= \bar{\mu} \{ f(x)\mu_\omega(A), x \in A \} = (\mu(A) \bar{\mu} \{ f(x), x \in A \})(\omega). \]

For the definition of absolute continuity and related remarks, see ([4]).

Proposition 3.17. If \( f : X \to \bar{\mathbb{R}}_0^+ \) is integrable, then the integral \( \int f \, d\mu \) is absolutely continuous; that is, \((o) - \lim_n \int_{A_n} f \, d\mu = 0\) whenever \((A_n)\) is a sequence in \( \mathcal{A} \) such that \((o) - \lim_n \mu(A_n) = 0\).

Proof. The assertion is trivial when \( f \) is bounded. So we prove absolute continuity in the general case. Fix \( n, k \in \mathbb{N} \), and pick \((A_n)\), with \((o) - \lim_n \mu(A_n) = 0\). We have
\[
0 \leq \int_{A_n} f \, d\mu = \int_{A_n} (f \land k) \, d\mu + \int_{A_n} (f - (f \land k)) \, d\mu \\
\leq \int_{A_n} (f \land k) \, d\mu + \int_X (f - (f \land k)) \, d\mu.
\]
As \((o) - \lim_k \int_X (f - (f \land k)) \, d\mu = 0\) and \((o) - \lim_n \int_{A_n} (f \land k) \, d\mu = 0\) for each \( k \in \mathbb{N} \), there exist a sequence \((r_k)\) in \( R \), \( r_k \downarrow 0 \), and a double sequence \((r'_{n,k})\) in \( R \), \( r'_{n,k} \downarrow 0 \) \((n \to +\infty, k = 1, 2, \ldots)\) such that
\[
0 \leq \int_{A_n} f \, d\mu \leq r'_{n,k} + r_k, \quad \forall n, k \in \mathbb{N}.
\]
It follows that
\[
0 \leq (o) - \lim_{n \to +\infty} \int_{A_n} f \, d\mu \leq ((o) - \lim_{n \to +\infty} r'_{n,k}) + r_k = r_k, \quad \forall k \in \mathbb{N}.
\]
By the arbitrariness of \( k \), we get \((o) - \lim_{n \to +\infty} \int_{A_n} f \, d\mu = 0\) and hence
\[
(o) - \lim_{n \to +\infty} \int_{A_n} f \, d\mu = 0. \]

Now we will prove a Vitali-type theorem for our integral.
Definition 3.18. Let \((f_n : X \to \mathbb{R})_n\) be a sequence of integrable functions. We say that \((f_n)_n\) is uniformly integrable if \(\sup_n \int_X |f_n|\,d\mu \in \text{Rand} (o)\) and
\[
\lim_n \sup_{k \geq n} \left( \int_{A_n} |f_k|\,d\mu \right) = 0, \text{whenever } (o) - \lim_k \mu(A_k) = 0.
\]

Definition 3.19. Under the same hypotheses and notation as above, we say that \((f_n)_n\) converges in \(L^1\) to \(f\) if \((o) - \lim_n \int_X |f_n - f|\,d\mu = 0\).

Remark 3.20. It is easy to check that \((f_n)_n\) converges in \(L^1\) to \(f\) if and only if \(\int_A f\,d\mu = (o) - \lim_{n \to +\infty} \int_A f_n\,d\mu \) uniformly with respect to \(A \in \mathcal{A}\).

Theorem 3.21. (Vitali’s theorem) Under the same notation as above, let \((f_n)_n\) be a uniformly integrable sequence of functions, convergent in measure to \(f\). Then \(f\) is integrable and \((f_n)_n\) converges in \(L^1\) to \(f\).

Conversely, every sequence \((f_n)\) of integrable functions, convergent in \(L^1\) to an integrable map \(f\), is convergent in measure to \(f\) and uniformly integrable.

Proof. To obtain the integrability of \(|f|\), it is enough to prove that
\[
\sup S_{|f|} \equiv \sup \left\{ \int_X \varphi\,d\mu : 0 \leq \varphi \leq |f| \text{ and } \varphi \text{ is simple} \right\} \in R, \tag{1}
\]
by virtue of Proposition 3.13. Let \(\varphi \in S_{|f|}\), \(\varphi = \sum_{j=1}^k c_j \chi_{B_j}\). Fix \(j = 1, 2, \ldots, k\) and for every \(n \in \mathbb{N}\), set \(A_n \equiv E_{1}^{||f| - f_n||}\). If \(x \in A_n \cap B_j\), we have \(\varphi(x) = c_j \leq |f_n(x)| + 1\) and hence \(\int_{B_j \cap A_n} \varphi(x)\,d\mu \leq \int_{B_j} |f_n(x)|\,d\mu + \mu(B_j)\).

As to \(A_n \cap B_j\), we have \(\int_{B_j \cap A_n} \varphi(x)\,d\mu \leq c_j \mu(A_n)\). Thus
\[
\int_{B_j} \varphi(x)\,d\mu \leq \int_{B_j} |f_n(x)|\,d\mu + \mu(B_j) + c_j \mu(A_n),
\]
\[
\int_X \varphi(x)\,d\mu \leq \int_X |f_n(x)|\,d\mu + \mu(X) + \mu(A_n) \sum_{j=1}^k c_j.
\]

By convergence in measure, \((o) - \lim_{n \to +\infty} \mu(A_n) \sum_{j=1}^k c_j = 0\) and since \(n\) is arbitrary, \(\int_X \varphi\,d\mu \leq \sup_n \int_X |f_n|\,d\mu + \mu(X) \in R\). Since the right hand side does not depend on \(\varphi\), (1) follows. So \(|f|\) is integrable. By Proposition 3.14, \(f^+\) and \(f^-\) are integrable and so is \(f\).

Now fix \(\varepsilon > 0\) and \(n \in \mathbb{N}\). As \(f_n\) is integrable by hypothesis, so is \(f - f_n\).
We have
\[
\int_X |f_n - f| \, d\mu \leq \int_{\{x \in X : |f_n - f| \leq \varepsilon\}} |f_n - f| \, d\mu + \int_{\{x \in X : |f_n - f| > \varepsilon\}} |f_n - f| \, d\mu
\leq \int_X \varepsilon \, d\mu + \int_{\{x \in X : |f_n - f| > \varepsilon\}} |f_n| \, d\mu + \int_{\{x \in X : |f_n - f| > \varepsilon\}} |f| \, d\mu
\leq \varepsilon \cdot \mu(X) + \sup_{k \geq n} \int_{\{x \in X : |f_n - f| > \varepsilon\}} |f_k| \, d\mu + \int_{\{x \in X : |f_n - f| > \varepsilon\}} |f| \, d\mu.
\]

As \(o - \lim_{n \to +\infty} \mu(\{x \in X : |f - f_n| > \varepsilon\}) = 0\), by virtue of uniform integrability of \((f_k)_k\), integrability of \(f\) and absolute continuity of the integral we get
\[
(o - \lim_{n \to +\infty} \left[ \sup_{k \geq n} \int_{\{x \in X : |f_n - f| > \varepsilon\}} |f_k| \, d\mu + \int_{\{x \in X : |f_n - f| > \varepsilon\}} |f| \, d\mu \right]) = 0.
\]

Thus we obtain
\[
0 \leq (o - \lim_{n \to +\infty} \int_X |f_n - f| \, d\mu) \leq \varepsilon \cdot \mu(X) + \lim_{n \to +\infty} \sup_{n \in \mathbb{N}} r_n
= \varepsilon \cdot \mu(X) + \inf_{n \in \mathbb{N}} r_n = \varepsilon \cdot \mu(X).
\]

Since \(\varepsilon > 0\) was arbitrary, we get \(o - \lim_{n \to +\infty} \int_X |f_n - f| \, d\mu = 0\).

Conversely, suppose that \((f_n)_n\) converges in \(L^1\) to \(f\). Fix \(\varepsilon > 0\) and set
\[
E_\varepsilon^{|f-f_n|} \equiv \{x \in X : |f_n(x) - f(x)| > \varepsilon\}, \quad \forall n \in \mathbb{N}.
\]

Then
\[
\frac{\int_X |f_n - f| \, d\mu}{\varepsilon} \geq \frac{\int_{E_\varepsilon^{|f-f_n|}} |f_n - f| \, d\mu}{\varepsilon} \geq \mu(E_\varepsilon^{|f-f_n|}) \geq 0,
\]

and hence \((o - \lim_{n \to +\infty} \mu(E_\varepsilon^{|f-f_n|}) = 0)\).

Now we prove uniform integrability. By convergence in \(L^1\), it follows immediately that \(\sup_{n} \int_X |f_n| \, d\mu \in R\). Let \((A_n)_n\) be a sequence in \(A\) such that \((o - \lim_{n \to +\infty} \mu(A_n) = 0)\). Fix \(n \in \mathbb{N}\). For every \(k \geq n\) we have
\[
\int_{A_n} |f_k| \, d\mu \leq \int_{A_n} |f_k - f| \, d\mu + \int_{A_n} |f| \, d\mu \leq \int_X |f_k - f| \, d\mu + \int_{A_n} |f| \, d\mu.
\]
By convergence in $L^1$, there exists a sequence $(r_k)_{k}$ in $R$, $r_k \downarrow 0$ such that $\int_{X} |f_k - f| \, d\mu \leq r_k \leq r_n$. Thus $\sup_{k \geq n} \int_{A_n} |f_k| \, d\mu \leq r_n + \int_{A_n} |f| \, d\mu$. So

$$0 \leq (o) - \lim_{n \to \infty} \sup_{k \geq n} \int_{A_n} |f_k| \, d\mu \leq \inf_{n} r_n + (o) - \lim_{n \to \infty} \int_{A_n} |f| \, d\mu = 0$$

and hence $(o) - \lim_{n \to \infty} \sup_{k \geq n} \int_{A_n} |f_k| \, d\mu = 0.$

A consequence of Vitali’s theorem is the following theorem.

**Theorem 3.22.** (Lebesgue dominated convergence theorem) Let $(f_n)_{n}$ be a sequence of measurable functions and suppose that there exists an integrable map $h$ such that $|f_n(x)| \leq |h(x)|$ for all $n \in \mathbb{N}$ and almost everywhere with respect to $x$. Furthermore assume that $(f_n)_{n}$ converges in measure to $f$. Then for every $n \in \mathbb{N}$, $f_n$ is integrable and $(f_n)_{n}$ converges in $L^1$ to $f$.

**Proof.** Without loss of generality, we suppose that $|f_n(x)| \leq |h(x)|$, $\forall n \in \mathbb{N}$, $\forall x \in X$.

By integrability of $|h|$ and Proposition 3.14, $f_n$ is integrable for every $n \in \mathbb{N}$. Moreover by virtue of absolute continuity of the integral of $h$, the hypotheses of Theorem 3.21 hold. So the assertion follows.

As a consequence of Theorem 3.22, we prove the following theorem, that is a sufficient condition for the convergence in $L^1$, inspired by a well-known result of Scheffé’s ([23]):

**Theorem 3.23.** With the same notation as above, let $(f_n)_{n} : X \to \mathbb{R}_0^+$ be a sequence of integrable functions, convergent in measure to a nonnegative integrable mapping $f$. Assume that $\int_{X} f_n \, d\mu(o)$-converges to $\int_{X} f \, d\mu$. Then $(f_n)_{n}$ converges in $L^1$ to $f$.

**Proof.** Let $h_n(x) = f_n(x) - f(x)$, $\forall x \in X$. Thus $0 \leq |h_n(x)| \leq f(x)$, $\forall x$. Let $H_n(x) = |h_n(x)|$, $\forall x$. Then $f$, $H_n$ are integrable for every $n$ and $(H_n)_{n}$ converges in measure to $0$. By Theorem 3.22, we have $0 = (o) - \lim_{n} \int_{X} |h_n(x)| \, d\mu$ and so $(o) - \lim_{n} \int_{X} |h_n(x)| \, d\mu = (o) - \lim_{n} \int_{X} h_n \, d\mu = 0$, by hypothesis. Finally we get

$$\lim_{n} \int_{X} |h_n| \, d\mu = (o) - \lim_{n} \int_{X} |h_n(x)| \, d\mu = (2)$$

$$+ (o) - \lim_{n} \int_{X} |h_n(x)| \, d\mu = 0.$$
Theorem 3.24. With the same notation as above, let \((f_n)_n\) be an increasing sequence of nonnegative integrable maps, convergent in measure to an integrable function \(f\). Then \(\int_X f \, d\mu = (\circ) - \lim_n \int_X f_n \, d\mu\) and therefore \(f_n \to f\) in \(L^1\).

Proof. It is an immediate consequence of Vitali’s Theorem.

4 Countably Additive Case

If \(\mu\) is countably additive, convergence almost everywhere implies convergence in measure; this can be proved along classical lines. Hence we simply state the results. So both Levi’s theorem and Fatou’s lemma hold.

Proposition 4.1. Let \(R\) be a Dedekind complete Riesz space, \(A \subset \mathcal{P}(X)\) a \(\sigma\)-algebra, and assume that \(\mu : A \to R\) is a \(\sigma\)-additive mean. Set

\[ A_n^\varepsilon = \{ x \in X : |f_n(x) - f(x)| > \varepsilon \}, \quad \forall \varepsilon > 0. \]

Then, \(f_n\) converges almost everywhere to \(f\) if and only if \(\mu(\limsup_n A_n^\varepsilon) = 0\), \(\forall \varepsilon > 0\).

It is easy to prove the following.

Proposition 4.2. Let \(R\), \(A\) and \(\mu\) be as above, and assume that \(\mu\) is \(\sigma\)-additive. Then for each sequence \((A_n)_n\) in \(A\) one has

\[ \mu(\liminf_n A_n) \leq \liminf_n \mu(A_n) \leq \limsup_n \mu(A_n) \leq \mu(\limsup_n A_n). \]

A straightforward consequence of Proposition 4.2 is the following.

Theorem 4.3. Let \(f_n\), \(f\) and \(\mu\) be as above. If \((f_n)\) converges to \(f\) almost everywhere, \((f_n)\) converges to \(f\) in measure.

From Theorems 3.24 and 4.3, and by the proceeding as in the classical case, the next theorem follows.

Theorem 4.4. With the same notation and hypotheses as above, let \((f_n)_n\) be an increasing sequence of nonnegative measurable maps. Then \(f(x) \equiv \lim_n f_n(x)\) is integrable if and only if \(\lim_n \int_X f_n \, d\mu \in R\) and in this case

\[ \int_X f \, d\mu = (\circ) - \lim_n \int_X f_n \, d\mu. \]

A consequence of Beppo Levi’s Theorem is the following version of Fatou’s Lemma.
Theorem 4.5. Let $X$, $R$, $\mu$ be as above, $(f_n)_n$ a sequence of nonnegative integrable maps, $f(x) \equiv \lim_{n} f_n(x)$, $\forall x \in X$. If $\lim_{n} \int_X f_n \, d\mu \in R$, then $f$ is integrable and $\lim_{n} \int_X f_n \, d\mu \geq \int_X f \, d\mu$.

5 Radon-Nikodym Theorem

In this section we give a Greco-type condition for the existence of a Radon-Nikodym derivative for the monotone integral introduced in the previous section (see [14]). We show that the Radon-Nikodym problem, in general, has no solutions. Indeed, there exist two $\mathbb{R}^2$-valued $\sigma$-additive means $\mu$ and $\nu$, with $\nu \ll \mu$, such that there is no function $f : X \equiv \{0,1\} \to \mathbb{R}$ such that $\nu = \int_X f \, d\mu$.

Let $X \equiv \{0,1\}$, $A \equiv \mathcal{P}(X)$, $R \equiv \mathbb{R}^2$ (endowed with componentwise ordering), $\mu, \nu : \mathcal{P}(X) \to \mathbb{R}^2$ defined by setting

$$
\mu(\{0\}) = (1,0), \quad \mu(\{1\}) = (0,1), \quad \nu(\{0\}) = (0,1), \quad \nu(\{1\}) = (1,0).
$$

It is easy to check that $\mu$ and $\nu$ are $\sigma$-additive, $\nu$ is absolutely continuous with respect to $\mu$ and $\mu$ is absolutely continuous with respect to $\nu$. However there is no function $f : X \to \mathbb{R}$ such that $\nu(A) = \int_A f \, d\mu$, $\forall A \in \mathcal{P}(X)$ for otherwise, we would have $(1,0) = \nu(\{1\}) = \int_{\{1\}} f \, d\mu = f(1)\mu(\{1\}) = (0, f(1))$, which is a contradiction.

Furthermore it is easy to see that for every $r > 0$ there exists no Hahn decomposition for the map $\nu - r\mu$.

Now we introduce two preliminary lemmas.

Proposition 5.1. Let $\mu, \nu : \mathcal{A} \to \mathbb{R}$ be two means with $\nu \ll \mu$. If there exists an $\mathcal{A}$-measurable function $f : X \to \mathbb{R}^+_0$ such that, for every $E \in \mathcal{A}$

$$
\nu(E) = \int_E f \, d\mu,
$$

then, for every $r > 0$, the set $A_r = \{x \in X : f(x) > r\}$ satisfies

(5.1.1) $\nu(E) \geq r\mu(E)$ for every $E \in A_r \cap \mathcal{A}$,

(5.1.2) $\nu(E) \leq r\mu(E)$ for every $E \in A_r^c \cap \mathcal{A}$,

(5.1.3) $\lim_{r \to +\infty} \nu(A_r) = 0$. 

Proof. \( A_r \in \mathcal{A} \) for every \( r > 0 \) since \( f \) is measurable. Moreover for every \( r > 0 \) and for every \( E \in A_r \cap \mathcal{A}, F \in A_r^c \cap \mathcal{A} \) we have

\[
\nu(E) = \int_E f \, d\mu \geq \int_E r \, d\mu = r\mu(E) \\
\nu(F) = \int_F f \, d\mu \leq \int_F r \, d\mu = r\mu(F).
\]

This proves (5.1.1) and (5.1.2).

(5.1.3) is a consequence of (5.1.1). In fact (5.1.1) yields

\[
\mu(A_r) \leq \frac{\nu(A_r)}{r} \leq \frac{\nu(X)}{r}, \quad \forall r > 0.
\]

So (o) \(- \lim_{r \to +\infty} \mu(A_r) = 0\), and hence (o) \(- \lim_{r \to +\infty} \nu(A_r) = 0\).

\[ \square \]

**Proposition 5.2.** Let \( \mu, \nu : \mathcal{A} \to \mathbb{R} \) be two means with \( \nu \ll \mu \). Let \( D = \left\{ \frac{i}{2^n}, i, n \in \mathbb{N} \right\} \). If there exists a decreasing family \( (A_r)_{r \in D} \) such that \( A_0 = X \) and satisfying (5.1.1) and (5.1.2), then the function \( f : X \to [0, +\infty] \), defined by \( f(x) \equiv \sup \{ r \in D : x \in A_r \} \), is integrable and \( \nu(E) = \int_E f \, d\mu, \quad \forall E \in \mathcal{A} \).

Proof. \( f \) is \( \mathcal{A} \)-measurable, since, \( \forall t > 0, \{ x \in X : f(x) > t \} = \cup_{r \in D, r > t} A_r \).

Let \( f_n = \frac{1}{2^n} \sum_{k=1}^{2^n} \chi_{A_{\frac{k}{2^n}}} \), for every \( n \in \mathbb{N} \). Then \( f_n - f \leq f_n \leq f_n + 1, \forall n \).

By construction for every \( E \in \mathcal{A} \),

\[
\int_E f_n \, d\mu = \frac{1}{2^n} \sum_{k=1}^{2^n} \mu(A_{\frac{k}{2^n}}) \\
= \frac{1}{2^n} \sum_{k=1}^{2^n} k \frac{1}{2^n} \left[ \mu(A_{\frac{k}{2^n}} \cap E) - \mu(A_{\frac{k+1}{2^n}} \cap E) \right] + n\mu(A_n \cap E) \\
\leq \frac{1}{2^n} \sum_{k=1}^{2^n} \left[ \nu(A_{\frac{k}{2^n}} \cap E) - \nu(A_{\frac{k+1}{2^n}} \cap E) \right] + n\nu(A_n \cap E) \leq \nu(E).
\]

So \( \sup_n \int_X f_n \, d\mu \leq \nu(X) \in R \) and thus

\[
\sup_n \int_X (f \wedge n) \, d\mu \leq \sup_n \int_X (f_n + 1) \, d\mu \leq \nu(X) + \mu(X).
\]

So by Proposition 3.11, \( f \) is integrable and hence, by Proposition 3.8, \( f \cdot \chi_E \)
The De Giorgi–Letta Integral

807

is integrable, $\forall E \in \mathcal{A}$. Thus

$$(o) - \lim_n \left[ \int_E (f \wedge n) \, d\mu - \int_E \left( f \wedge \frac{1}{2^n} \right) \, d\mu \right] = (o) - \lim_n \int_E (f \wedge n) \, d\mu = \int_E f \, d\mu,$$

and therefore $(o) - \lim_n \int_E f_n \, d\mu = \int_E f \, d\mu$ and $\int_E f \, d\mu \leq \nu(E), \ \forall E \in \mathcal{A}$.

On the other hand,

$$\int_E f_n \, d\mu = \sum_{k=1}^{n^2-1} \frac{k + 1}{2^n} \left[ \mu(A_{k+1}^n \cap E) - \mu(A_{k+1}^n \cap E) \right] + n\mu(A_n \cap E) +$$

$$- \frac{1}{2^n} \sum_{k=1}^{n^2-1} \left[ \mu(A_{k+1}^n \cap E) - \mu(A_{k+1}^n \cap E) \right] +$$

$$\geq \nu(A_{k+1}^n \cap E) - \nu(A_n \cap E) - \frac{1}{2^n} \left( \mu(A_{k+1}^n) - \mu(A_n \cap E) \right).$$

Taking the (o)-limits as $n \to \infty$, we obtain $\int_E f \, d\mu = \nu(E)$. \qed

A consequence of Proposition 5.1 and 5.2 is the following Radon-Nikodym Theorem.

**Theorem 5.3.** Let $\mu, \nu : \mathcal{A} \to R$ be two means with $\nu \ll \mu$. Then the following are equivalent:

(5.3.a) there exists an $\mathcal{A}$-measurable function $f : X \to \bar{\mathbb{R}}_0^+$ such that, for every $E \in \mathcal{A}$ we have $\nu(E) = \int_E f \, d\mu$,

(5.3.b) there exists a family $(A_r)_{r>0}$ of measurable sets such that for every $r > 0$

(5.3.b.1) $\nu(E) \geq r\mu(E)$ for every $E \in A_r \cap \mathcal{A}$,

(5.3.b.2) $\nu(E) \leq r\mu(E)$ for every $E \in A_r \cap \mathcal{A}$.

The following is a different formulation of Theorem 5.3.

**Theorem 5.4.** Let $\mu, \nu : \mathcal{A} \to R$ be two means with $\nu \ll \mu$. Then the following are equivalent:

(5.4.a) there exists a $\mathcal{A}$-measurable function $f : X \to \bar{\mathbb{R}}_0^+$ such that, for every $E \in \mathcal{A}$ we have $\nu(E) = \int_E f \, d\mu$,
(5.4.b) for every $r > 0$ the measure $\nu - r\mu$ admits a Hahn decomposition, namely there exist two disjoint measurable sets $(B_r, C_r)$ such that, $\forall E \in \mathcal{A}$

\[
(\nu - r\mu)^+(E) = (\nu - r\mu)(E \cap B_r) \\
(\nu - r\mu)^-(E) = (\nu - r\mu)(E \cap C_r).
\]

Proof. (5.4.a) $\implies$ (5.4.b)

By Theorem 5.3, there exists a family $(A_r)_{r>0}$ of measurable sets such that, for every $r > 0$

(5.3.b.1) $\nu(E) \geq r\mu(E)$ for every $E \in A_r \cap \mathcal{A}$,
(5.3.b.2) $\nu(E) \leq r\mu(E)$ for every $E \in A_r^c \cap \mathcal{A}$.

Set $B_r \equiv A_r$, $C_r \equiv A_r^c$. For every $E \in A_r \cap \mathcal{A}$ we have

\[
(\nu - r\mu)^+(E) = (\nu - r\mu)^+(E \cap A_r) + (\nu - r\mu)^+(E \cap A_r^c)
\]

\[
(\nu - r\mu)^-(E \cap A_r) = (\nu - r\mu)(E \cap A_r)
\]

from (5.3.b.1), since $(\nu - r\mu)(F) \leq 0$, $\forall F \in E \cap A_r^c \cap \mathcal{A}$. So we obtain, for every $E \in \mathcal{A}$, $(\nu - r\mu)^+(E) = (\nu - r\mu)(E \cap B_r)$. Analogously for each $E \in \mathcal{A}$, $(\nu - r\mu)^-(E) = (\nu - r\mu)(E \cap C_r)$. (5.4.b) $\implies$ (5.4.a)

It is easy to check that, if (5.4.b) holds, then (5.3.b.1) and (5.3.b.2) are satisfied. The assertion follows by Proposition 5.2.

References


