ADDENDUM TO: “SOME NEW TYPES OF FILTER LIMIT THEOREMS FOR TOPOLOGICAL GROUP-VALUED MEASURES”

Abstract

The purpose of this note is to point out some corrections to the paper: A. Boccuto and X. Dimitriou, “Some new types of filter limit theorems for topological group-valued measures”, Real Anal. Exchange 39 (1) (2014), 139-174.

We use the notation and terminology developed in [1].

On page 145, the definitions of $m^L$ and $m^+$ should be formulated as follows:

$$m^L(A) := \{m(B) : B \in L, B \subset A\}, \quad A \in L,$$
$$m^+(A) := m^\Sigma(A) = \{m(B) : B \in \Sigma, B \subset A\}, \quad A \in \Sigma.$$

On page 148, formula (7), the definition of “positive” measure defined in a $\sigma$-algebra $\Sigma$ and with values in a topological group $R$ should be stated as follows:

A finitely additive measure $m : \Sigma \to R$ is said to be positive iff every neighborhood $W$ of 0 contains a neighborhood $U_0$ of 0 such that, for every $A \in \Sigma$ with $m(A) \in U_0$ and for each $B \in \Sigma$ with $B \subset A$, we also get $m(B) \in U_0$. 
On page 148, Proposition 2.10 should be formulated as follows.

**Proposition 2.10** Let \( m : \Sigma \to R \) be a \( \sigma \)-additive measure. Then

\[
\lim_k m^+(H_k) = 0
\]

for each decreasing sequence \((H_k)_k\) in \( \Sigma \), satisfying

\[
m\left( B \cap \left( \bigcap_{k=1}^\infty H_k \right) \right) = 0 \quad \text{for every } B \in \Sigma.
\]

On page 150, Theorem 2.13 should be stated as follows.

**Theorem 2.13** Let \( m : \Sigma \to R \) be an \((s)\)-bounded measure. Then for each disjoint sequence \((C_k)_k\) in \( \Sigma \) there exists an infinite subset \( P_0 \subset \mathbb{N} \), with

\[
\lim_r \left\{ m\left( \bigcup_{k \in Y, k \geq r} C_k \right) : Y \subset P_0 \right\} = 0,
\]

(1)

and \( m \) is \( \sigma \)-additive on the \( \sigma \)-algebra generated by the sets \( C_k \), \( k \in P_0 \).

On page 150, Theorem 2.14 should be formulated as follows.

**Theorem 2.14** Let \( m_j : \Sigma \to R, j \in \mathbb{N} \), be a sequence of finitely additive \((s)\)-bounded measures. Then for any disjoint sequence \((C_k)_k\) in \( \Sigma \) there exists an infinite subset \( P \subset \mathbb{N} \), with

\[
\lim_h \left\{ m_j\left( \bigcup_{k \in Y, k \geq h} C_k \right) : Y \subset P \right\} = 0
\]

for every \( j \in \mathbb{N} \), and each \( m_j \) is \( \sigma \)-additive on the \( \sigma \)-algebra generated by the sets \( C_k \), \( k \in P \).

The following result, which is more general than Theorem 2.17 on page 152, was proved in [2, Corollary 6.2].

Let \((R, +)\) be a group, \( G \) be a locally compact Hausdorff topological space and \( m \) be a finitely additive \( R \)-valued set function, defined on the \( \delta \)-ring of the relatively compact Baire subsets of \( G \). Then \( m \) is regular if and only if \( m \) is \( \sigma \)-additive.
On page 156, formula (17), instead of
\[ m_{j_{2h-1}}^+([l\{n_{2h}, +\infty]\} \subset U_{2h} \subset U_2 \]
there should be
\[ m_{j_{2h-1}}^+([l\{n_{2h}, +\infty]\} \subset U_{2h} \subset U_2. \]

On page 159, formula (26) should be written as follows:
\[
m_j^+(H_k) = \{ m_j(B) : B \in \Sigma, B \subset H_k \} = \\
= \{ m_j(B \setminus H_\infty) : B \in \Sigma, B \subset H_k \} = \\
= \{ m_j(C) : C \in \Sigma, C \subset H_k \setminus H_\infty \} = \\
= m_j^+(H_k \setminus H_\infty) = m_j^+\left(\bigcup_{l=k}^{\infty} C_l\right)
\]
for every \( j, k \in \mathbb{N}. \)

On page 159, two lines above formula (28), instead of
\[
\nu_j^+([k, +\infty]) := \bigcup \{ \nu_j(D) : D \subset [k, +\infty] \subset m_j^+\left(\bigcup_{l=k}^{\infty} C_l\right) \}
\]
there should be
\[
\nu_j^+([k, +\infty]) := \{ \nu_j(D) : D \subset [k, +\infty] \subset m_j^+\left(\bigcup_{l=k}^{\infty} C_l\right) \}.
\]

On pages 160-161, Theorem 3.5 should be formulated as follows.

**Theorem 3.5** Let \( G \) be any infinite set, \( \Sigma \subset \mathcal{P}(G) \) be a \( \sigma \)-algebra, \( m_j : \Sigma \to R, j \in \mathbb{N}, \) be a sequence of positive \( (s) \)-bounded measures, \( \mathcal{F} \) be a diagonal filter of \( \mathbb{N}. \) Assume that \( m_0(E) = (\mathcal{F}) \lim m_j(E) \)
exists in \( R \) for every \( E \in \Sigma, \) and that \( m_0 \) is \( \sigma \)-additive and positive on \( \Sigma. \)

Then for every disjoint sequence \( (C_k)_k \) in \( \Sigma \) and \( I \in \mathcal{F}^* \) there exists \( J \in \mathcal{F}^*, J \subset I, \) with
\[
\lim_k \left( \bigcup_{j \in J} m_j^+(C_k) \right) = \lim_k \{ m_j(C_k) : j \in J \} = 0.
\]
On page 161-162, after formula (30), the proof of the equality

$$\lim_{j \in J} m_j(B) = m_0(B)$$

for all $B \in K$

in Theorem 3.5 should be as follows:

Choose arbitrarily $U \in J(0)$, and let $W \in I(0)$ be such that $\overline{5W} \subset U$. In correspondence with $W$, let $U_0 \in J(0)$, $U_0 \subset W$, satisfy the condition of positivity, that is $m(B) \in U_0$ whenever $A \in \Sigma$, $m(A) \in U_0$ and $B \in \Sigma$, $B \subset A$. In correspondence with $U_0$ there exists $k_0 \in \mathbb{N}$ with $m_0\left( \bigcup_{k > k_0} C_k \right) \in U_0$ and therefore, by positivity of $m_0$,

$$m_0\left( \bigcup_{k > k_0, k \in P} C_k \right) \in U_0.$$ 

Moreover there is $j_0 \in J$, $j_0 = j_0(U, k_0)$ such that for every $j \in J$ with $j \geq j_0$ we have:

$$m_j\left( \bigcup_{k \leq k_0, k \in P} C_k \right) - m_0\left( \bigcup_{k \leq k_0, k \in P} C_k \right) \in U_0,$$

$$m_j\left( \bigcup_{k \leq k_0} C_k \right) - m_0\left( \bigcup_{k \leq k_0} C_k \right) \in U_0,$$

$$m_j\left( \bigcup_{k=1}^{\infty} C_k \right) - m_0\left( \bigcup_{k=1}^{\infty} C_k \right) \in U_0,$$

$$m_j\left( \bigcup_{k > k_0} C_k \right) - m_0\left( \bigcup_{k > k_0} C_k \right) \in 2U_0,$$

and hence

$$m_j\left( \bigcup_{k > k_0} C_k \right) = m_j\left( \bigcup_{k > k_0} C_k \right) - m_0\left( \bigcup_{k > k_0} C_k \right) + m_0\left( \bigcup_{k > k_0} C_k \right) + m_0\left( \bigcup_{k > k_0} C_k \right) \in 3U_0.$$ 

By positivity of $m_j$, we have also $m_j\left( \bigcup_{k > k_0, k \in P} C_k \right) \in U_0$. Thus
for every $B \in \mathcal{K}$, $B = \bigcup_{k \in P} C_k$, we get

$$m_j(B) - m_0(B) = m_j\left(\bigcup_{k \in P} C_k\right) - m_0\left(\bigcup_{k \in P} C_k\right) =$$

$$= m_j\left(\bigcup_{k \leq k_0, k \in P} C_k\right) - m_0\left(\bigcup_{k \leq k_0, k \in P} C_k\right) +$$

$$+ m_j\left(\bigcup_{k > k_0, k \in P} C_k\right) - m_0\left(\bigcup_{k > k_0, k \in P} C_k\right) \in$$

$$\in U_0 + 3U_0 + U_0 = 5U_0 \subset 5W \subset U.$$

Thus, $\lim_{j \in J} m_j(B) = m_0(B)$ for all $B \in \mathcal{K}$.

On pages 162-163, Theorem 3.6 should be formulated as follows.

**Theorem 3.6** Let $\Sigma, \mathcal{F}$ be as in Theorem 3.5, $\tau$ be a Fréchet-Nikodým topology on $\Sigma$, $m_j : \Sigma \to R, j \in \mathbb{N}$, be a sequence of positive finitely additive $(s)$-bounded and $\tau$-continuous measures. Assume that $m_0(E) := (\mathcal{F}) \lim_j m_j(E)$ exists in $R$ for each $E \in \Sigma$, and that $m_0$ is $\sigma$-additive and positive on $\Sigma$.

Then for every set $I \in \mathcal{F}^*$ and for each decreasing sequence $(H_k)_k$ in $\Sigma$ with $\tau \lim_k H_k = \emptyset$ there exists a set $J \in \mathcal{F}^*$, $J \subset I$, with

$$\lim_k \left(\bigcup_{j \in J} m_j^+(H_k)\right) = \lim_k \{m_j(H_k) : j \in J\} = 0.$$ 

On pages 163, Theorem 3.7 should be formulated as follows.

**Theorem 3.7** Let $\Sigma, \mathcal{F}$ be as in Theorem 3.6, $m_j : \Sigma \to R, j \in \mathbb{N}$, be a sequence of positive $\sigma$-additive measures. If

$$m_0(A) := (\mathcal{F}) \lim_j m_j(A)$$

exists in $R$ for each $A \in \Sigma$, and $m_0$ is $\sigma$-additive and positive on $\Sigma$, then for each $I \in \mathcal{F}^*$ and for every decreasing sequence $(H_k)_k$ in $\Sigma$ with $\bigcap_{k=1}^{\infty} H_k = \emptyset$ there exists $J \in \mathcal{F}^*$, $J \subset I$, with

$$\lim_k \left(\bigcup_{j \in J} m_j^+(H_k)\right) = \lim_k \{m_j(H_k) : j \in J\} = 0.$$
On page 168, formula (34), instead of

\[
\lim_s \left( \bigcup_{j \in M^*} m_j(C_{k_{rs}}) \right) = 0
\]

there should be

\[
\lim_s \left( \bigcup_{j \in M^*} m_j^+(C_{k_{rs}}) \right) = 0.
\]

On page 169, formula (35), instead of

\[
\lim_k \left( \bigcup_{j \in M^*} m_j(C_k) \right) = 0,
\]

there should be

\[
\lim_k \left( \bigcup_{j \in M^*} m_j^+(C_k) \right) = 0.
\]

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**References**
