Generalized logistic equation with indefinite weight driven by the square root of the Laplacian

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Abstract

We consider an elliptic problem driven by the square root of the Laplacian in presence of a general logistic function having an indefinite weight. We prove a bifurcation result for the associated Dirichlet problem via regularity estimates of independent interest when the weight belongs only to certain Lebesgue spaces.

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1 Introduction

One of the most popular equations in Biology, and in particular in population dynamics, is the logistic equation, which we write as

\[ cu_t(x,t) = \Delta u(x,t) + \lambda(\beta(x)u(x,t) - u^2(x,t)) \quad x \in \Omega \subset \mathbb{R}^N, \quad t > 0. \tag{1} \]

Here \( \Omega \) is a domain in \( \mathbb{R}^N \) where the population having nonnegative density \( u \) lives and \( c, \lambda \) are positive constants. However, it is well known that the heat operator may be too rigid to describe the possible interaction of the specie in the whole of \( \Omega \) (for instance, see [5], [6], [19], [21] and the references therein), and for this fact a nonlocal operator may be more useful than a local one.

Moreover, a first natural step consists in studying steady states for these new equations (like in [5]). For this reason, in this paper, the diffusion is not described by classical Laplacian operator (or its generalizations), but by the square root of \(-\text{Laplacian}, \sqrt{-\Delta}\). This operator can be defined in several ways, and we
follow the approach developed in the pioneering works of Caffarelli, Silvestre [10], Caffarelli, Vasseur [11], Cabr´e, Tan [9], to which we refer in Section 2 for the precise mathematical description and properties. Here, we only say that it is becoming more and more popular for its applications in Physics, Probability, Finance and Biology. In particular, we recall the pure periodic logistic fractional equation in \( \mathbb{R}^N \) studied in [7] - where the behaviour of level set of solutions is investigated - , the evolution Fisher-KKP equation in \( \mathbb{R}^N \) driven by the generator of a Feller semigroup (for instance, \( \sqrt{-\Delta} \)) studied in [8] - where the authors prove that the stable state invades the unstable one with a front position which grows exponentially in time - , and the general fractional porous medium equation in \( \mathbb{R}^N \) studied in [13] - where existence and uniqueness are established.

The problem we are concerned with has homogeneous Dirichlet boundary conditions, so that the population is confined in a bounded domain \( \Omega \subset \mathbb{R}^N, N \geq 2 \). Hence, we shall study the following problem:

\[
\begin{aligned}
\sqrt{-\Delta} u(x) &= \lambda(\beta(x)u(x) - g(x,u(x))) \quad &\text{in} & \quad \Omega \\
u(x) &= 0 \quad &\text{on} & \quad \partial \Omega.
\end{aligned}
\]

(P\textsubscript{\beta\lambda})

Here the nonlinear term \( g(x,u) \) is a self–limiting factor for the population which generalizes the classical quadratic nonlinearity in (1), and whose properties will be introduced later on; we only emphasize the fact that, unlike similar variational problems, we don’t impose any growth condition at infinity for \( g \).

The weight \( \beta : \Omega \rightarrow \mathbb{R} \) corresponds to the birth rate of the population and, as main contribution of this paper, we suppose that it is sign changing and (generally) unbounded, so that both contributions to development and limitation of the population are possible. To our best knowledge, no results are known for unbounded sign changing weights, while this situation is clearly possible. Under this setting, we will establishes a relation between the parameter \( \lambda > 0 \) and the existence of a nontrivial stationary solution of the problem, showing that if \( \lambda \) is large, non trivial steady state are possible, though the birth rate \( \beta \) is mainly negative in the domain (as it happens in regions with natural or artificial difficulties which prevent proliferation); on the other hand if \( \lambda \) is small, the population is not safely protected in regions of disadvantage and steady state cannot exist (see Theorem 4.1). Let us also remark that \( \lambda \) large corresponds to a small diffusion and \( \lambda \) small corresponds to a large diffusion in (1).

We recall that logistic–type equations have been extensively studied and, in addition to the papers cited above, any bibliography would be incomplete. We only add [20], where steady states of equation (1) where found when \( \Omega = \mathbb{R}^N \) and \( \beta \) is sign–changing. Hence our result, is the counterpart of the results therein in the case of problem (P\textsubscript{\beta\lambda}).

The paper is organized as follows. Section 2 introduces the mathematical background we shall need in the following sections. In particular we recall some basic facts on the square root of the Laplacian and some fundamental inequalities. In Section 3 we will introduce the main character of this paper (i.e. the bifurcation parameter \( \lambda^* \)), and we will prove a general maximum principles (see Proposition 3.2) and a regularity result (see Proposition 3.3) of independent interest. In the last section we will prove a bifurcation theorem for problem (P\textsubscript{\beta\lambda}): via critical
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point theory and truncation techniques we will show that problem \((P_\lambda)\) admits a nontrivial solution if and only if \(\lambda > \lambda^*\), and, with additional but natural conditions, we will show that nontrivial solutions bifurcate from \(\lambda^*\) (see Theorems 4.1 and 4.2).

2 Mathematical Background

In this section we briefly recollect some definitions and inequalities we will use throughout the paper.

From now on, \(\Omega \subseteq \mathbb{R}^N\) will denote a bounded smooth domain, and we denote by \(\mathcal{C} = \Omega \times (0, +\infty)\) the half cylinder with base \(\Omega \times \{0\}\) - which we shall identify with \(\Omega^-\), and lateral boundary \(\partial_L \mathcal{C} = \partial \Omega \times [0, +\infty)\).

Then, we introduce the Sobolev Space

\[
H^1_{0,L}(\mathcal{C}) = \left\{ v \in H^1(\mathcal{C}) \mid v = 0 \text{ a.e. on } \partial_L \mathcal{C} \right\},
\]

equipped with the norm

\[
\|v\|_{H^1_{0,L}(\mathcal{C})} = \left( \int_{\mathcal{C}} |Dv(x,y)|^2 \, dx dy \right)^{\frac{1}{2}},
\]
defined by the scalar product

\[
\int_{\mathcal{C}} Dv \cdot Dz \, dx dy, \quad v, z \in H^1_{0,L}(\mathcal{C}).
\]

As usual, a crucial rôle is played by trace operator

\[
\text{Tr}_\Omega : H^1_{0,L}(\mathcal{C}) \longrightarrow H^{\frac{1}{2}}(\Omega),
\]

which is a continuous map (see [9, Lemma 2.6]), and which gives several information, which we recall in the following. In particular, we will repeatedly use the following

**Lemma 2.1** (Sobolev trace inequality, Lemma 2.3, [9]). Let \(N \geq 2\) and \(2^{\sharp} = \frac{2N}{N-1}\). Then there exists a constant \(C = C(N)\), such that, for all \(v \in H^1_{0,L}(\mathcal{C})\)

\[
\left( \int_{\Omega} |v(x,0)|^{2^\sharp} \, dx \right)^{\frac{1}{2^\sharp}} \leq C \left( \int_{\mathcal{C}} |Dv(x,y)|^2 \, dx dy \right)^{\frac{1}{2}}.
\]

(3)

From Lemma 2.1 it is immediate to prove

**Lemma 2.2** (Lemma 2.4, [9]). Let \(N \geq 2\) and \(1 \leq q \leq 2^\sharp = \frac{2N}{N-1}\). Then there exists a positive \(C = C(N, q, |\Omega|)\), such that, for all \(v \in H^1_{0,L}(\mathcal{C})\)

\[
\left( \int_{\Omega} |v(x,0)|^q \, dx \right)^{\frac{1}{q}} \leq C \left( \int_{\mathcal{C}} |Dv(x,y)|^2 \, dx dy \right)^{\frac{1}{2}}.
\]

(4)

If \(N = 1\), inequality (4) holds for all \(q \in [1, \infty)\).
Moreover, a stronger useful relation, which will be frequently used, is given by the following compact embedding:

**Lemma 2.3** (Lemma 2.5, [9]). Let \( 1 \leq q < 2^* \) if \( N \geq 2 \) and \( q \in [1, \infty) \) if \( N = 1 \); then

\[
Tr_\Omega(H^1_{0,L}(\mathcal{C})) \hookrightarrow L^q(\Omega).
\]

The trace operator in (2) permits to introduce the trace space on \( \Omega \times \{0\} \) of functions in \( H^1_{0,L}(\mathcal{C}) \) as

\[
V_0(\Omega) := \{ u = Tr_\Omega v \mid v \in H^1_{0,L}(\mathcal{C}) \} \subset H^{1/2}(\Omega),
\]
equipped with the norm

\[
\|u\|_{V_0(\Omega)} = \left[ \|u\|^2_{H^{1/2}} + \int_{\Omega} \frac{u^2}{d(x)} \right]^{1/2},
\]
where \( d(x) = dist(x, \partial \Omega) \). However, there exists another characterization of \( V_0(\Omega) \):

**Lemma 2.4** (Lemma 2.7, [9]). The following identity holds:

\[
V_0(\Omega) = \left\{ u \in H^1(\Omega) \mid \int_{\Omega} \frac{u^2}{d(x)} \, dx < \infty \right\}.
\]

Here, and in the following, we have denoted by \( x \) a generic point of \( \Omega \) and by \( y \) a generic point in \((0, \infty)\).

Next we introduce the following minimizing problem: fix \( u \in V_0(\Omega) \) and consider

\[
\inf \left\{ \int_{\mathcal{C}} |Dv|^2 \, dx \, dy \mid v \in H^1_{0,L}(\mathcal{C}), \, v(.,0) = u \text{ in } \Omega \right\}.
\]

**Proposition 2.1** (Lemma 2.8, [9]). For every \( u \in V_0(\Omega) \) there exists a unique minimizer \( v \) of (5), called the harmonic extension of \( u \) to \( \mathcal{C} \) which vanishes on \( \partial_L \mathcal{C} \).

Such a function is denoted by \( v = \text{h-ext}(u) \).

**Remark 2.1.** If \( \{e_k\}_k \) is the orthonormal frame of \( L^2(\Omega) \) obtained by the eigenfunctions of \(-\Delta\) with homogeneous Dirichlet boundary conditions and with associated eigenvalues \( \{\lambda_k\}_k \), given \( u \in V_0(\Omega) \), we can write \( u = \sum_{k=1}^{\infty} b_k e_k(x) \). Hence, by Proposition 2.1, we get that \( v = \text{h-ext}(u) \in H^1_{0,L}(\mathcal{C}) \) is given by

\[
v(x, y) = \sum_{k=1}^{\infty} b_k e_k(x) e^{-\lambda_k^2 y}.
\]

Finally, let us define the square root of the Laplacian (or half-Laplacian) in a bounded domain \( \Omega \) of \( \mathbb{R}^N \). For any continuous function \( u \) in \( \Omega \), there exists a unique function \( v \) such that

\[
\begin{cases}
\Delta_C v = 0 & \text{in } \mathcal{C} \\
v(x, 0) = u(x) & \text{on } \Omega \times \{0\} \simeq \Omega \\
v = 0 & \text{on } \partial_L \mathcal{C}
\end{cases}
\]
where $\Delta_C$ denotes the Laplacian operator in $C$, so that $v$ is the generalized harmonic extension of $u$ in $C$ which is 0 on $\partial_L C$.

Now, consider the operator $T$ defined as

$$T u(x) := - \frac{\partial v}{\partial y}(x, 0). \quad (7)$$

It is readily seen that the system

$$\begin{cases}
\Delta_C w = 0 & \text{in } C \\
w(x, 0) = - \frac{\partial v}{\partial y}(x, 0) = Tu(x) & \text{on } \Omega \times \{0\} \simeq \Omega, \\
w = 0 & \text{on } \partial_L C,
\end{cases}$$

admits the unique solution $w(x, y) = - \frac{\partial v}{\partial y}(x, y)$, and thus by (6) we immediately have

$$T(T u)(x) = - \frac{\partial w}{\partial y}(x, 0) = \frac{\partial^2 v}{\partial y^2}(x, 0) = (-\Delta v)(x, 0).$$

In conclusion, $T^2 = \Delta$, i.e the operator $T$ mapping the Dirichlet datum $u$ to the Neumann datum $- \frac{\partial v}{\partial y}(\cdot, 0)$ is a square root of the Laplacian $-\Delta$ in $\Omega$.

In light of the previous “Dirichlet to Neumann” procedure, by Remark 2.1 we have that

$$\sqrt{-\Delta} u = \left. \frac{\partial v}{\partial y} \right|_{\Omega \times \{y=0\}} = - \left. \frac{\partial v}{\partial y} \right|_{\Omega \times \{y=0\}} = \sum_{k=1}^{\infty} b_k \lambda_k^{1/2} e_k(x),$$

where $\nu = -y$. Moreover, it is natural to give the following

**Definition 2.1.** We say that $u \in H^{1/2}_0(\Omega)$ is a weak solution of

$$\begin{cases}
\sqrt{-\Delta} u = g(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega, 
\end{cases} \quad (*)$$

whenever the harmonic extension $v \in H^1_{0,L}(C)$ of $u$ is a weak solution of

$$\begin{cases}
\Delta v = 0 & \text{in } C \\
v = 0 & \text{on } \partial_L C \\
\frac{\partial v}{\partial\nu} = g(x, u) & \text{on } \Omega \times \{0\},
\end{cases} \quad (**),$$

that is if

$$\int_C Dv \cdot D\xi \, dx \, dy = \int_{\Omega \times \{0\}} g(x, v(x, 0)) \xi(x, 0) \, dx \quad \forall \xi \in H^1_{0,L}(C).$$

An important regularity result for $\sqrt{-\Delta}$ is available:

**Theorem 2.1 (Theorem 1.9, [9])).** Let $\Omega$ be a bounded domain of $\mathbb{R}^N$ of class $C^{2,\alpha}$, $\alpha \in (0, 1)$ . If $g \in V_0(\Omega)^*$ and $u$ is a weak solution of

$$\begin{cases}
\sqrt{-\Delta} u = g(x) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases} \quad (P)$$

then:
1. If \( g \in L^2(\Omega) \), then \( u \in H^1_0(\Omega) \).

2. If \( g \in H^1_0(\Omega) \), then \( u \in H^2(\Omega) \cap H^1_0(\Omega) \).

3. If \( g \in L^\infty(\Omega) \), then \( u \in C^{\alpha}(\Omega) \).

4. If \( g \in C^1(\Omega) \) and \( g|_{\partial \Omega} \equiv 0 \), then \( u \in C^{2,\alpha}(\Omega) \).

5. If \( g \in C^{1,\alpha}(\Omega) \) and \( g|_{\partial \Omega} \equiv 0 \), then \( u \in C^2(\Omega) \).

Of course, applying Theorem 2.1 to weak solutions of

\[
\begin{cases}
\sqrt{-\Delta} u = f(x,u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

we can obtained related results for nonlinear problems.

3 The bifurcation parameter

We consider the problem

\[
\begin{cases}
\sqrt{-\Delta} u = \lambda(\beta u - g(x,u)) & \text{in } \Omega \\
v = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( N \geq 2 \), \( \lambda \in \mathbb{R} \), \( \Omega \subset \mathbb{R}^N \) is a bounded domain of class \( C^{2,\alpha} \) for some \( 0 < \alpha \leq 1 \), \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory perturbation (i.e., for all \( s \in \mathbb{R} \) the map \( x \mapsto g(x,s) \) is measurable and for a.e. \( x \in \Omega \) the map \( x \mapsto g(x,s) \) is continuous). Concerning the function \( \beta \), we assume that it is a sign changing measurable weight (for a first presentation of such classes of weights, we refer to [24]).

In view of the considerations of the previous section, studying problem \((P_\lambda^\beta)\) is equivalent to studying the extension problem in the cylinder

\[
\begin{cases}
-\Delta v = 0 & \text{in } C \\
v = 0 & \text{on } \partial L(C) \\
\frac{\partial v}{\partial \nu} = \lambda(\beta v - g(x,v)) & \text{in } \Omega \times \{0\}.
\end{cases}
\]

Of course, \((Q_\lambda^\beta)\) has a variational structure, so that \( v \) solves \((Q_\lambda^\beta)\) if and only if it is a critical point for the functional \( I : H^1_{0,L}(C) \to \mathbb{R} \) defined as

\[
I(u) = \frac{1}{2} \int_{C} |Du|^2 \, dx dy - \lambda \int_{\Omega \times \{0\}} F(x,u) \, dx,
\]

where

\[
F(x,u) = \int_{0}^{u} [\beta s - g(x,s)] \, ds.
\]

From now on, all integrals of the form

\[
\int_{\Omega \times \{0\}}
\]

will be simply denoted by \( \int_{\Omega} \).
Now, we set
\[ \lambda^* = \inf \left\{ \int_C |Du|^2 \, dx \, dy : u \in H^1_{0,L}(C) \text{ with } \int_{\Omega} \beta u^2 = 1 \right\}. \] (9)

In order to ensure that \( \lambda^* \) is well defined, we impose the following condition on the weight function \( \beta \):
\( H(\beta): \beta \in L^q(\Omega) \) with \( q > N \) and \( \beta^+ = \max\{\beta, 0\} \neq 0 \).

**Remark 3.1.** The assumption \( \beta^+ = \max\{\beta, 0\} \neq 0 \) implies that the set
\[ \left\{ u \in H^1_{0,L}(C) \text{ with } \int_{\Omega} \beta u^2 = 1 \right\} \]
is not empty, so that \( \lambda^* \) is well defined.

**Proposition 3.1.** If hypothesis \( H(\beta) \) holds, then \( \lambda^* > 0 \).

**Proof.** Let \( u \in H^1_{0,L}(C) \) be such that \( \int_{\Omega} \beta u^2 = 1 \). Since \( u \in H^1_{0,L}(C) \), then \( u \in L^2(\Omega) \), and we note that
\[ \frac{1}{N} + \frac{1}{N-1} = \frac{1}{N} + \frac{N-1}{N} = 1. \]

Then, using H"older’s inequality and Lemma 2.1, we get
\[ 1 = \int_{\Omega} \beta u^2 \, dx \leq \|eta^+\|_{L^\infty(\Omega)} \|u\|_{L^{2^*(\Omega)}}^2 \leq |\Omega|^\frac{N}{2^* N} \|eta^+\|_{L^{2^*(\Omega)}} \|u\|_{L^{2^*(\Omega)}}^2 \\ \leq C_1 \|eta^+\|_{L^1(\Omega)} \|Du\|_{L^2(\Omega)}^2 \]
for some universal constant \( C_1 > 0 \).

Hence,
\[ \inf_{u \in H^1_0(C)} \left( \int_C |Du|^2 \, dx \, dy \right) = \lambda^* \geq \left( C_1 \|eta^+\|_{L^{1}(\Omega)} \right)^{-1} > 0. \]

We introduce the following general maximum principles.

**Proposition 3.2.** Let \( \Omega \) be a smooth bounded domain of \( \mathbb{R}^N \) and let \( u \in C^1(\overline{\Omega}) \) be a solution of
\[
\begin{cases}
\sqrt{-\Delta} u + B(x, u, Du) \geq 0 & \text{in } \Omega, \\
u \geq 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
with \( B = B(x, s, \xi) \) of class \( C^1 \) in \( \xi \), \( B \in Lip_{loc} \) in \( s \) and such that \( B(x, 0, 0) = 0 \) for a.e. \( x \in \Omega \). Then \( u > 0 \) in \( \Omega \) or \( u \equiv 0 \) in \( \Omega \).
Proof. Let \( v = \text{h-ext}(u) \) be the harmonic extension of \( u \), so that \( v \) satisfies
\[
\begin{cases}
-\Delta v = 0 & \text{in } C, \\
v \geq 0 & \text{in } C, \\
v = 0 & \text{on } \partial L(C), \\
\frac{\partial v}{\partial \nu} \geq -B(x, v(x,0), Dv(x,0)) & \text{in } \Omega.
\end{cases}
\]
Suppose by contradiction that \( u \) is not identically zero and that there exists \( x_0 \in \Omega \) such that \( u(x_0) = 0 \), so that \( v(x_0,0) = 0 \). By [22, Theorem 2.7.1] we find
\[\partial_{\nu} v(x_0,0) < 0.\] (10)
By (10), the assumptions on \( B \) and the equivalence between problem \((P)\) and \((P')\), we find
\[0 > \sqrt{-\Delta} u(x_0) \geq -B(x_0, u(x_0), Du(x_0)) = 0,\]
and a contradiction arises. \( \square \)

Remark 3.2. In the case of weak solution, the previous result still holds true by [22, Theorem 5.4.1].

Next, let us consider the problem
\[
\begin{cases}
\sqrt{-\Delta} u = \lambda^* \beta u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

\((P_{\lambda^*})\)

We start with the following regularity result (see [16] for a related result in \(\mathbb{R}^N\)).

Proposition 3.3. Let \( \Omega \) be a smooth bounded domain of \(\mathbb{R}^N\) and let \( u \) be a weak solution of \((P_{\lambda})\). If \( H(\beta) \) holds, then \( u \in C^\alpha(\overline{\Omega}) \).

Proof. Set \( v = \text{h-ext}(u) \), so that \( v \in H^1_{0,L}(C) \) solves
\[
\begin{cases}
-\Delta v = 0 & \text{in } C, \\
v = 0 & \text{on } \partial L(C), \\
\frac{\partial v}{\partial \nu} = \lambda \beta v & \text{in } \Omega \times \{0\}.
\end{cases}
\]
Now, we will use the Moser iteration technique. If \( M > 0 \) we define \( v_M = \min \{v^+, M\} \) and thus \( v_M \in H^1_{0,L}(C) \cap L^\infty(C) \). For any nonnegative number \( k \) we set \( \phi = v_M^{2k+1} \), so that \( D\phi = (2k+1)v_M^{2k} Dv_M \).

By definition of weak solution with \( \phi = v_M^{2k+1} \) as test function, we get
\[ (2k+1) \int_C v_M^{2k} Dv \cdot Dv_M dx dy = \lambda \int_\Omega \beta v_M^{2k+1} dx, \]
that is
\[ (2k+1) \int_C v_M^{2k} |Dv_M|^2 dx dy = \lambda \int_\Omega \beta v_M^{2k+2} dx. \] (11)
We note that
\[
(2k + 1) \int_{\mathcal{C}} v_M^{2k} |Dv_M|^2 \, dx \, dy = \frac{(2k + 1)}{(k + 1)^2} < Dv_M^k, Dv_M^k > \nonumber \\
= \frac{(2k + 1)}{(k + 1)^2} \int_{\mathcal{C}} |Dv_M^{k+1}|^2 \, dx \, dy. \tag{12}
\]

In addition, by applying the Hölder inequality to the right-hand side of (11), we obtain
\[
\lambda \int_{\Omega} \beta v_M^{2k+2} \, dx \leq \lambda \|\beta^+\|_{L^q(\Omega)} \|v^+(k+1)\|^2_{L^{2q'}(\Omega)}. \tag{13}
\]

By applying (12) and (13), from (11) one gets
\[
\frac{(2k + 1)}{(k + 1)^2} \int_{\mathcal{C}} |Dv_M^{k+1}|^2 \, dx \, dy \leq \lambda \|\beta^+\|_{L^q} \|v^+(k+1)\|^2_{2q'}. \tag{14}
\]

Since \( \lim_{M \to \infty} v_M(x, y) = v^+(x, y) \) for a.e. \((x, y) \in \mathcal{C}\), applying Fatou’s Lemma, we get
\[
\|v^+(k+1)\|_{H_{0, L}^l(\mathcal{C})} \leq \frac{k + 1}{\sqrt{2k + 1}} \sqrt{\lambda} \|\beta^+\|_{L^q}^{1/2} \|v^+(k+1)\|_{2q'}. \tag{15}
\]

By Lemma 2.1, we get the existence of \( S > 0 \) such that
\[
\|v^+(k+1)\|_{2q'} \leq \frac{k + 1}{\sqrt{2k + 1}} S \sqrt{\lambda} \|\beta^+\|_{L^q}^{1/2} \|v^+(k+1)\|_{2q'}. \tag{16}
\]

where, with some abuse of notation, we have set \( \|v\|_{L^q(\Omega)} = \|v(\cdot, 0)\|_{L^q(\Omega \times \{0\})} \) for any \( v \in H_{0, L}^l(\mathcal{C}) \) and all \( q \geq 1 \).

Since \( q > N \), then \( 2q' < 2^q \). Now, define \( k_0 = 0 \) and by recurrence
\[
2(k_n + 1)q' = (k_{n-1} + 1)2^q.
\]

It is easy see that \( k_i > k_{i-1} \) and that \( k_n \to \infty \) as \( n \to \infty \). Moreover, (16) implies that
\[
\text{if } (v^+) \in L^{2(k_{i-1}+1)q'}(\Omega) \text{ then } (v^+) \in L^{(k_{i-1}+1)2^i}(\Omega) = L^{2(k_i+1)q'}(\Omega),
\]

and hence \( v^+ \in L^r(\Omega) \) for every \( r \geq 1 \). Moreover, by (15) we obtain that
\[
\int_{\mathcal{C}} |D(v^+)^{k+1}|^2 \leq C, \text{ for every } i \geq 0,
\]

or equivalently \( (v^+)^{k+1} \in H_{0, L}^1(\mathcal{C}) \) for every \( i \geq 0 \).

Analogously, choosing \( v_M := \min\{M, v^-\} \), we can prove that \( v^- \in L^r(\Omega) \) for every \( r \geq 1 \).

Now, since \( v \in L^r(\Omega) \) for all \( r < \infty \), then \( \beta v \in L^q(\Omega) \) with \( q > q > N \).

Now, following the proof of Proposition 3.1 (iii) in [9], for \((x, y) \in \mathcal{C}\) we introduce the function \( w(x, y) = \int_0^y v(x, t) \, dt \), which turns out to solve the homogeneous Dirichlet problem
\[
\begin{cases}
-\Delta w(x, y) = \lambda \beta(x) v(x, 0) & \text{in } \mathcal{C}, \\
w = 0 & \text{on } \partial \mathcal{C}.
\end{cases}
\]

Then, performing the odd reflection \( w_{\text{odd}} \) of \( w \) in \( \Omega \times \mathbb{R} \), by the Calderón-Zygmund inequality, we obtain that \( w_{\text{odd}} \in W^{2,p}(\Omega \times (-R,R)) \) for every \( R > 0 \). In particular, \( w \in C^{1,\alpha}(\Omega) \), and so, by the very definition of \( v = w_y \), we get that \( v \in C^\alpha(\overline{\Omega}) \), and so \( u \in C^\alpha(\overline{\Omega}) \).

Now, let us consider

\[
C_+ = \{ u \in C^\alpha(\overline{\Omega}) : u(x) \geq 0 \ \forall \ x \in \overline{\Omega} \},
\]

whose interior set is

\[
\text{int} \ C_+ = \{ u \in C^\alpha(\overline{\Omega}) : u(x) > 0 \ \forall \ x \in \overline{\Omega} \}.
\]

Let \( e_1 \) be the first eigenfunction of \( \sqrt{-\Delta} \) with associated first eigenvalue \( \lambda^* \) and \( L^2 \) weight \( \beta \), i.e. a solution of problem \((P_{\lambda^*})\). We first prove that \( e_1 \) does exist. In fact, we have

**Proposition 3.4.** If \( H(\beta) \) holds, then there exists \( e_1 \in H^{1/2}_0(\Omega) \cap \text{int} \ C_+ \) which is the first eigenfunction with associated eigenvalue \( \lambda^* \) of problem \((P_{\lambda^*})\).

**Proof.** Let us consider the functional

\[
J(u) = \frac{1}{2} \int_C |Du|^2 \, dx \, dy
\]

constrained on the set

\[
M = \left\{ u \in H^1_{0,L}(C) \left| \int_{\Omega} \beta u^2 \, dx = 1 \right. \right\},
\]

and note that it is sequentially weakly lower semicontinuous and coercive; so by the Weierstrass Theorem we can find \( e_1 \in H^1_{0,L}(C) \) such that

\[
J(e_1) = \inf \left\{ J(u) | u \in H^1_{0,L}(C) \right\}.
\]

Thus, there exists \( \lambda^* \in \mathbb{R} \) such that \( J'(e_1)v = \lambda^* \int_{\Omega} \beta e_1 v \, dx \) for every \( v \in H^1_{0,L}(C) \), that is \( e_1 \) is the first eigenfunction for the problem

\[
\begin{aligned}
\Delta e_1 &= 0 & \text{in } C, \\
\frac{\partial e_1}{\partial \nu} &= \lambda^* \beta e_1 & \text{in } \Omega.
\end{aligned}
\]

Moreover, it is clear that we can assume \( e_1 \geq 0 \) in \( \Omega \), and by Proposition 3.3 we get that \( e_1 \in C_+ \).

Finally, by Remark 3.2, \( e_1 \in \text{int} \ C^+ \).

In general no other properties can be deduced for \( e_1 \) if no other hypotheses on \( \beta \) are given. However, we can recover the classical features of the first eigenvalue with additional assumptions. In particular, from Theorem 2.1.4-5 we have the following regularity result:

**Proposition 3.5.** If \( \beta \in C^{1,\alpha}(\overline{\Omega}) \), then \( e_1 \in C^{2,\alpha}(\overline{\Omega}) \).
4 The bifurcation theorem

On the absorption term $g$ we assume that $H(g) : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $g(x, s) = 0$ for a.e. $x \in \Omega$ and for all $s \leq 0$ and $g(x, s) \geq 0$ for a.e. $x \in \Omega$ and for all $s > 0$. Moreover:

1. there exists $g_0 \in L^\infty(\Omega)$ such that:
   \[
   \lim_{s \to \infty} \frac{g(x, s)}{s} \geq g_0(x) \quad \text{uniformly for a.e. } x \in \Omega \tag{17}
   \]
   and
   \[
   \text{ess inf}_{\Omega} (g_0 - \beta) = \mu > 0. \tag{18}
   \]

2. \[
   \lim_{s \to 0^+} \frac{g(x, s)}{s} = 0 \quad \text{uniformly for a.e. } x \in \Omega \tag{19}
   \]

3. for every $u \in L^\infty(\Omega)$ there exists $\rho = \rho(u) \geq 0$ such that
   \[
   |g(x, u(x))| \leq \rho \quad \text{for a.e. } x \in \Omega. \tag{20}
   \]

4. For a.e. $x \in \Omega$ the function $u \mapsto \frac{g(x, u)}{u}$ is strictly increasing in $(0, \infty)$.

\textbf{Remark 4.1.} Hypothesis (17) does not exclude the classical case, that is

\[
\lim_{s \to \infty} \frac{g(x, s)}{s} = \infty,
\]

as it happens, for example, if $g(x, s) = |s|^{p-2}s$, $p > 2$.

\textbf{Remark 4.2.} Hypothesis (20) is very general and does exclude any a priori bound on the growth of the function at infinity, as it is usual in variational problems of this type.

We shall prove the following result:

\textbf{Proposition 4.1.} If hypotheses $H(\beta)$, $H(g)$, (17), (18), (19) and (20) hold, then for all $\lambda > \lambda^*$ there exists a solution $u \in C^\alpha(\bar{\Omega})$ of problem $(P^\lambda_\beta)$ such that $u(x) > 0$ for all $x \in \Omega$.

\textbf{Proof.} By virtue of hypothesis (17), it follows that for all $\epsilon \in (0, \mu)$ there exists $M = M(\epsilon) > 0$ such that for all $s > M$

\[
   g(x, s) \geq (g_0(x) - \epsilon)s \quad \text{for a.e. } x \in \Omega. \tag{22}
   \]

Now we fix $\psi > M$ and we consider the following truncation for the reaction term:

\[
h(x, s) = \begin{cases} 
0 & \text{if } s \leq 0 \\
\beta(x)s - g(x, s) & \text{if } 0 \leq s \leq \psi \\
\beta(x)\psi - g(x, \psi) & \text{if } s \geq \psi.
\end{cases} \tag{23}
\]
Of course, $h$ is still a Carathéodory function. Setting $H(x, s) = \int_0^s h(x, s) ds$, by $H(g)$ the functional $I_\psi : H^1_{0, L}(C) \to \mathbb{R}$ defined by

$$I_\psi(v) = \frac{1}{2} \|Dv\|^2_{L^2(C)} - \lambda \int_\Omega H(x, v(x)) \, dx$$

is of class $C^1$ and sequentially weakly lower semicontinuous. Moreover, by the very definition of the truncation $h(x, s)$, we have that $I_\psi$ is coercive. Therefore, by the Weierstrass Theorem we can find $\varpi \in H^1_{0, L}(C)$ such that

$$I_\psi(\varpi) = \inf \{ I_\psi(v) \mid v \in H^1_{0, L}(C) \} = m, \quad (24)$$

that is $m$ is a critical value for $I_\psi$.

Now we must check that $\varpi \neq 0$. By virtue of hypothesis (19), in correspondence of the previous $\varepsilon$ there exists $\delta = \delta(\varepsilon) > 0$ with $\delta < \min\{M, \psi\}$ such that

$$g(x, s) \leq \varepsilon s \quad \forall s \in [0, \delta] \text{ and a.e. } x \in \Omega,$$

so that

$$G(x, s) = \int_0^s g(x, s) ds \leq \frac{\varepsilon^2}{2} s^2 \quad \forall s \in [0, \delta] \text{ and a.e. } x \in \Omega. \quad (25)$$

By Proposition 3.4, we can find $t \in (0, 1)$ such that $te_1 \in [0, \delta]$ for all $x \in \Omega$. Then, by (25)

$$I_\psi(te_1) = \frac{t^2}{2} \|De_1\|^2_2 - \lambda \int_\Omega H(x, te_1) dx$$

$$= \frac{t^2}{2} \|De_1\|^2_2 - \frac{\lambda t^2}{2} \int_\Omega \beta e_1^2 dx + \lambda \int_\Omega G(x, te_1) dx$$

$$\leq \frac{t^2}{2} \|De_1\|^2_2 - \frac{\lambda t^2}{2} \int_\Omega \beta e_1^2 dx + \frac{\lambda}{2} t^2 \|e_1\|^2_2$$

$$= \frac{t^2}{2} (\Lambda - \lambda) \int_\Omega \beta e_1^2 dx + \frac{\lambda}{2} t^2 \|e_1\|^2_2.$$ 

Recalling that $\int_\Omega \beta e_1^2 dx = 1$ and $\Lambda < \lambda$, choosing $\varepsilon > 0$ sufficiently small, then we have $I_\psi(te_1) < 0$ and by (24) we find that

$$m = I_\psi(\varpi) < I_\psi(0) = 0, \quad (26)$$

that is $\varpi \neq 0$, as claimed.

Since $\varpi$ is a critical point for $I_\psi$ then $I'_\psi(\varpi) = 0$, that is

$$A(\varpi) = \lambda h(x, \varpi), \quad (27)$$

where $A : H^1_{0, L}(C) \to (H^1_{0, L}(C))^\prime$ is the linear map defined by

$$(A(u), v) = \int_C Du \cdot Dv \, dx dy$$

for all $u, v \in H^1_{0, L}(C)$.
Now, we act on (27) with $-(\overline{v})^- \in H^1_{0,L}(\mathcal{C})$, obtaining

$$(A(\overline{v}), -(\overline{v})^-) = \int_\mathcal{C} |D(\overline{v})^-|^2 \, dx dy = 0,$$

so that $\overline{v}^+ = 0$, that is $\overline{v} \geq 0$.

Next we act on (27) with $(\overline{v} - \psi)^+ \in H^1_{0,L}(\mathcal{C})$ and by definition of $h$ we find

$$(A(\overline{v}), (\overline{v} - \psi)^+) = \lambda \int_\Omega h(x, \overline{v})(\overline{v} - \psi)^+ \, dx = \int_\Omega (\beta \overline{v} - g(x, \overline{v}))(\overline{v} - \psi)^+ \, dx \leq \int_\Omega (\beta \psi - g(x, \psi))(\overline{v} - \psi)^+ \, dx. \tag{28}$$

First, let us prove that

$$\int_\Omega (\beta \psi - g(x, \psi))(\overline{v} - \psi)^+ \, dx \leq 0. \tag{29}$$

Indeed, recalling that $\psi > M$ and $\epsilon < \mu$, using (22), by (18) we have

$$\beta \psi - g(x, \psi) \leq \beta \psi - g_0(x) \psi + \epsilon \psi \leq (\epsilon - \mu) \psi < 0. \tag{30}$$

Then, by (28),

$$\int_{\{x \in \mathcal{C} : \overline{v} > \psi\}} |D\overline{v}|^2 \, dxdy \leq 0,$$

so that, from (28), (29) and (30), we find that

$$(\beta \psi - g(x, \psi))(\overline{v} - \psi)^+ = 0 \text{ in } \Omega.$$

Using again (30), this implies that $(\overline{v} - \psi)^+ = 0$ in $\Omega$, that is $\overline{v} \leq \psi$ in $\Omega$. Therefore, $\overline{v}$ is not only a solution of the truncation problem, but also of problem $(Q_\lambda)$; then $\overline{v} = \text{tr}_\Omega \overline{v}$ is a solution of problem $(P)$. Finally, (20) and elliptic regularity for Neumann problems in Lipschitz domains (for instance, see [18]) imply that $\psi \in C^\alpha(\overline{\mathcal{C}})$, and so $u \in C^\alpha(\Omega)$.

**Proposition 4.2.** If hypotheses $H(\beta)$ and (21) hold, then the nontrivial nonnegative solution of problem $(P_\lambda)$ is unique.

**Proof.** We consider two nontrivial nonnegative solutions $u, v \in C^\alpha(\overline{\Omega})$ of problem $P_\lambda^\beta$ and we test with $\frac{u^2 - v^2}{u + \epsilon}$ and $\frac{v^2 - u^2}{v + \epsilon}$, which clearly belong to $H^1_{0,L}(\mathcal{C})$. Then

$$\int_\mathcal{C} Du \cdot D\frac{u^2 - v^2}{u + \epsilon} \, dxdy = \lambda \int_\Omega \beta u \frac{u^2 - v^2}{u + \epsilon} \, dx - \lambda \int_\Omega g(x, u) \frac{u^2 - v^2}{u + \epsilon} \, dx \tag{31}$$

and

$$\int_\mathcal{C} Dv \cdot D\frac{v^2 - u^2}{v + \epsilon} \, dxdy = \lambda \int_\Omega \beta v \frac{v^2 - u^2}{v + \epsilon} \, dx - \lambda \int_\Omega g(x, v) \frac{v^2 - u^2}{v + \epsilon} \, dx. \tag{32}$$

Now

$$D\frac{u^2 - v^2}{u + \epsilon} = \frac{u^2 + 2\epsilon u}{(u + \epsilon)^2} Du - \frac{2\epsilon Dv}{u + \epsilon} + \frac{u^2}{(u + \epsilon)^2} Du, \tag{33}$$
and an analogous form for $D\frac{v^2 - u^2}{u + \epsilon}$.

Adding (31) and (32), we find
\[
\int_{C} Du \cdot D\frac{u^2 - v^2}{u + \epsilon} dxdy + \int_{C} Dv \cdot D\frac{v^2 - u^2}{u + \epsilon} dxdy = \lambda \int_{\Omega} \beta \left[ \frac{v^3}{v + \epsilon} - \frac{vu^2}{v + \epsilon} + \frac{u^3}{u + \epsilon} - \frac{v^2 u}{u + \epsilon} \right] dx
\]
\[
- \lambda \int_{\Omega} g(x,u) \frac{u^2 - v^2}{u + \epsilon} dx - \lambda \int_{\Omega} g(x,v) \frac{v^2 - u^2}{v + \epsilon} dx.
\]

By (33) this reads as
\[
0 = \int_{C} Du \cdot \left[ \frac{u^2 + 2v}{(u + \epsilon)^2} Du - \frac{2v Du}{u + \epsilon} + \frac{v^2}{(u + \epsilon)^2} Du \right] dxdy
+ \int_{C} Dv \cdot \left[ \frac{v^2 + 2u}{(v + \epsilon)^2} Dv - \frac{2u Dv}{v + \epsilon} + \frac{u^2}{(v + \epsilon)^2} Dv \right] dxdy
+ \lambda \int_{\Omega} g(x,u) \frac{u^2 - v^2}{u + \epsilon} dx + \lambda \int_{\Omega} g(x,v) \frac{v^2 - u^2}{v + \epsilon} dx
- \lambda \int_{\Omega} \beta \left[ \frac{v^3}{v + \epsilon} - \frac{vu^2}{v + \epsilon} + \frac{u^3}{u + \epsilon} - \frac{v^2 u}{u + \epsilon} \right] dx
- \lambda \int_{\Omega} \beta \left[ \frac{v^3}{v + \epsilon} - \frac{vu^2}{v + \epsilon} + \frac{u^3}{u + \epsilon} - \frac{v^2 u}{u + \epsilon} \right] dx + 2\epsilon \int_{C} \left[ |D\log(v + \epsilon)|^2 u + |D\log(v + \epsilon)|^2 v - (u + v) \langle D\log(v + \epsilon), D\log(u + \epsilon) \rangle \right] dxdy,
\]
i.e.
\[
\int_{C} |D\log(u + \epsilon) - D\log(v + \epsilon)|^2 (u^2 + v^2) dxdy
+ \lambda \int_{\Omega} \left[ g(x,u) \frac{u^2 - v^2}{u + \epsilon} - g(x,v) \frac{v^2 - u^2}{v + \epsilon} \right] dx
+ 2\epsilon \int_{C} \left[ |D\log(v + \epsilon)|^2 u + |D\log(v + \epsilon)|^2 v - (u + v) \langle D\log(v + \epsilon), D\log(u + \epsilon) \rangle \right] dx
- \lambda \int_{\Omega} \beta \left[ \frac{v^3}{v + \epsilon} - \frac{vu^2}{v + \epsilon} + \frac{u^3}{u + \epsilon} - \frac{v^2 u}{u + \epsilon} \right] dx = 0.
\]

Now, by (20) we have
\[
\left| g(x,u) \frac{u^2 - v^2}{u + \epsilon} - g(x,v) \frac{v^2 - u^2}{v + \epsilon} \right| \leq \left( \frac{\rho(\|u\|_\infty)}{\|u\|_\infty} + \frac{\rho(\|v\|_\infty)}{\|v\|_\infty} \right) |u^2 - v^2| \in L^1(\Omega),
\]
and by the Lebesgue Theorem we immediately find that
\[
\int_{\Omega} \left[ g(x,u) \frac{u^2 - v^2}{u + \epsilon} - g(x,v) \frac{v^2 - u^2}{v + \epsilon} \right] dx \rightarrow \int_{\Omega} \left[ \frac{g(x,u)}{u} - \frac{g(x,v)}{v} \right] (u^2 - v^2) dx.
\]
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as $\epsilon \to 0$.

Second,

$$2\epsilon \left| \int_C \left[ |D\log(u + \epsilon)|^2u + |D\log(v + \epsilon)|^2v - (u + v)(D\log(v + \epsilon), D\log(u + \epsilon)) \right] dxdy \right|$$

$$\leq 2\epsilon \left( \frac{|Du|^2}{(u + \epsilon)^2} + \frac{|Dv|^2}{(v + \epsilon)^2} + \frac{|Du||Dv|}{(u + \epsilon)(v + \epsilon)} \right)$$

$$\leq \frac{|Du|^2}{2} + \frac{|Dv|^2}{2} + |Du||Dv| \in L^1(C),$$

and again the Lebesgue Theorem implies that the integral under consideration tends to 0 as $\epsilon \to 0$.

Moreover, it is easy to see that, similarly,

$$\int_\Omega \beta \left[ \frac{v^3}{v + \epsilon} - \frac{vu^2}{v + \epsilon} + \frac{u^3}{u + \epsilon} - \frac{v^2u}{u + \epsilon} \right] dx \to 0$$

as $\epsilon \to 0$.

In conclusion, passing to the lim inf in (34), Fatou’s Lemma implies that

$$0 \geq \int_C |D\log(u) - D\log(v)|^2(u^2 + v^2)dxdy + \lambda \int_\Omega \left[ \frac{g(x, u)}{u} - \frac{g(x, v)}{v} \right] (v^2 - u^2)dx.$$ 

Since both integrals are nonnegative by (21), we immediately get that $u = v$, as desired.

Next we prove that:

**Proposition 4.3.** If hypotheses $H(\beta)$ and $H(g)$ hold, problem $(P^\lambda_\beta)$ has no positive solutions when $\lambda \in (0, \lambda^*].$

**Proof.** Suppose that there exists a positive solution $v_0 \neq 0$ for $(Q^\beta_\lambda)$, that is

$$\int_C Dv_0 \cdot Dv dx dy = \lambda \int_\Omega [\beta v_0 - g(x, v_0)]v dx \quad \forall v \in H^1_{0,L}(C).$$

In particular

$$\int_C |Dv_0|^2 dx dy = \lambda \int_\Omega [\beta v_0^2 - g(x, v_0)v_0]dx < \lambda \int_\Omega \beta v_0^2 dx.$$

Comparing with the definition of $\lambda^*$, we have

$$\lambda^* \int_\Omega \beta v_0^2 \leq \int_C |Dv_0|^2 dx dy < \lambda \int_\Omega \beta v_0^2 dx,$$

and we reach a contradiction, since $\lambda \leq \lambda^*$.

In conclusion, we can state the following theorem:

**Theorem 4.1.** If hypotheses $H(\beta)$, $H(g)$, (17), (18), (19), (20), (21) hold, then there exists $\lambda^* > 0$ such that
1. for all \( \lambda > \lambda^* \) problem \((P^\beta_\lambda)\) has a unique positive solution \( u \in C^\alpha(\Omega) \).

2. for all \( \lambda \leq \lambda^* \) problem \((P^\beta_\lambda)\) has no positive solutions.

Finally, we concentrate on the behaviour of solutions near \( \lambda^* \), i.e. we concentrate on

\[
\lim_{\lambda \to \lambda^*^+} u(\lambda).
\]

For this purpose, we suppose that there exist \( p \in (2, 2^\dagger) \) and \( C_0 > 0 \) such that

\[
g(x,s) \geq C_0 |s|^{p-1} \quad \text{for a.e } x \in \Omega \text{ and for all } s \in \mathbb{R}.
\] (35)

Then we have

**Proposition 4.4.** Assume that \( H(g) \) holds and that \( \beta^+ \in L^{\frac{2}{p-2}}(\Omega) \) with \( \beta^+ \neq 0 \). If hypothesis (35) holds, then every solution \( u_\lambda \) of problem \((P^\beta_\lambda)\) converges to 0 in \( H^2(\Omega) \) when \( \lambda \downarrow \lambda^* \).

**Proof.** Let \( (\lambda_n)_n \) be a sequence strictly decreasing to \( \lambda^* \), and let \( v_n \) be a corresponding solution of problem \((Q^\beta_{\lambda_n})\) (notice that under these assumptions no uniqueness is granted).

Then

\[
\int_{\mathcal{C}} |Dv_n|^2 \, dx \, dy = \lambda_n \int_{\Omega} [\beta v_n - g(x,v_n)] v_n \, dx \quad \forall n \in \mathbb{N}. \tag{36}
\]

By Lemma 2.2, the Hölder inequality and (35), we have that

\[
C ||v_n||_p^2 \leq \int_{\mathcal{C}} |Dv_n|^2 \, dx \, dy = \lambda_n \int_{\Omega} |v_n|^2 \, dx - \lambda_n \int_{\Omega} g(x,v_n) v_n \, dx \leq
\]

\[
\leq \lambda_n ||\beta^+||_{L^\infty(\mathcal{C})} ||v_n||_p^2 - C_0 ||v_n||_p^p.
\]

Being \( p > 2 \), the sequence \((v_n)_n\) is bounded in \( L^p(\Omega) \), and therefore in \( L^2(\Omega) \).

As a consequence, by (36), \((v_n)_n\) is bounded in \( H^1_{0,L}(\mathcal{C}) \).

Thus, up to a subsequence, we have that

\[
v_n \rightharpoonup v^* \text{ in } H^1_{0,L}(\mathcal{C}) \text{ and } v_n \to v^* \text{ in } L^p(\Omega) \text{ as } n \to \infty.
\]

Using the definition of solution for \( v_n \) with \( v_n - v^* \in H^1_{0,L}(\mathcal{C}) \) as test function, we get

\[
\int_{\mathcal{C}} Dv_n \cdot D(v_n - v^*) \, dx \, dy = \lambda_n \int_{\Omega} \beta v_n (v_n - v^*) \, dx - \lambda_n \int_{\Omega} g(x,v_n)(v_n - v^*) \, dx \to 0
\]

as \( n \to \infty \), so that

\[
\lim_{n \to \infty} \int_{\mathcal{C}} |Dv_n|^2 \, dx \, dy = \int_{\mathcal{C}} |Dv^*|^2 \, dx \, dy,
\]

that is

\[
v_n \to v^* \text{ in } H^1_{0,L}(\mathcal{C}).
\]
Setting \( u_n = Tr_{\Omega} v_n \) and \( u^* = Tr_{\Omega} v^* \), exploiting the continuous injection of \( Tr_{\Omega}(H^1_0(L(\mathcal{C}))) \subset H^{1/2}(\Omega) \), we obtain that

\[
  u_n \to u^* \text{ in } H^{1/2}(\Omega).
\]

Now, observe that

\[
  \frac{p}{p-2} = \left( \frac{p}{2} \right)' > N,
\]

since \( 2 < p < 2^* \). Then, by Proposition (4.3), we find \( u^* = 0 \).

\[ \square \]

**Remark 4.3.** Let us remark that in the previous case, in agreement with \( H(\beta) \), we have \( q = \frac{p}{p-2} > N \), since \( p < 2^* \).

In the light of Proposition 4.4, we emphasize the fact that if \( g \) and \( \beta \) satisfy all the assumptions above, we obtain a complete bifurcation result for our problem that we summarize in the following final

**Theorem 4.2.** Assume that \( \beta \in L^\infty(\Omega) \) with \( \beta^* \neq 0 \). Moreover, assume \( H(g) \) and (17)–(21) and (35). Then the conclusions of Theorem 4.1 and Proposition 4.4 hold true.

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**References**


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