Bifurcation for positive solutions of nonlinear diffusive logistic equations in $\mathbb{R}^N$ with indefinite weight

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Abstract
We consider a diffusive $p$–logistic equation in the whole of $\mathbb{R}^N$ with absorption and an indefinite weight. Using variational and truncation techniques we prove a bifurcation theorem and describe completely the bifurcation point. In the semilinear case $p = 2$, under an additional hypothesis on the absorption term, we show that the positive solution is unique.

Keywords: diffusive $p$–logistic equation, positive solution, Hardy’s inequality, bifurcation theorem, nonlinear regularity, indefinite weight.

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1 Introduction

In this paper we are interested in non existence, existence and uniqueness (in the semilinear case $p = 2$) of positive solutions for the following $p$–logistic type equation with absorption defined in the whole of $\mathbb{R}^N (N > p)$:

\[
(P_\lambda) \quad \begin{cases} 
-\Delta_p u(z) = \lambda (\beta(z) u^{p-1} - f(z, u(z))) & \text{in } \mathbb{R}^N, \\
u(z) \to 0 \text{ as } \|z\| \to \infty, \ u > 0,
\end{cases}
\]

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where $\lambda > 0$, $\| \cdot \|$ is the Euclidean norm in $\mathbb{R}^N$ and $\Delta_p$ denotes the $p$–Laplace differential operator defined by $\Delta_p u = \text{div}(\|Du\|^{p-2}Du)$ for all $u \in W^{1,p}(\mathbb{R}^N)$. Moreover, $\beta$ is a sign changing weight function and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory perturbation (i.e., for all $x \in \mathbb{R}$ the map $z \mapsto f(z,x)$ is measurable, and for a.e. $z \in \mathbb{R}^N$ the map $x \mapsto f(z,x)$ is continuous).

Problems of this type arise as the steady state equation of population dynamics models (for example, see [3], [10], [11], [12], [13], [14], [26]). In this case the unknown $u$ corresponds to the density of a population, the weight $\beta$ corresponds to the birth rate of the population if self–limitation is ignored and the perturbation $-f$ expresses the fact that the population is self–limiting. The function $\beta$ is in general sign changing and the region where $\beta$ is positive (resp. negative) the population, ignoring self–limitation, has positive (resp. negative) birth rate. Since $u$ describes the population density, we are interested only in positive solutions of problem $\mathcal{P}_\lambda$, such solutions corresponding to possible steady state distributions of the population.

The result we prove establishes a relation between the diffusion parameter $\lambda > 0$ (extent of diffusion) and the existence of a nontrivial steady state population density. Thus, the population will prosper in those regions where the birth rate $\beta$ is positive, and if the diffusion is small (i.e. if $\lambda > 0$ is large) nontrivial steady state solutions are possible, even if the population is subject to some overall disadvantage (i.e. $\beta$ is predominantly negative). On the other hand, if the diffusion is big (i.e. $\lambda > 0$ is small), the population is not safely protected in small regions of disadvantage and a steady state solution cannot exist.

Problem $\mathcal{P}_\lambda$ settled in a bounded region with Neumann boundary condition was studied by Cantrell–Cosner–Hutson [4] and Umezu [25], and extensions to the $p$–Laplacian Dirichlet case can be found in the works of Dong [7], García Melian–Sabina de Lis [15], Guedda–Veron [17], Kamin–Veron [19] and Papageorgiou–Papalini [21]. A related Neumann problem can be found also in Cardinali–Papageorgiou–Rubbioni [5]. Diffusive logistic equations in the whole of $\mathbb{R}^N$ were studied by Afrouzi–Brown [2], who dealt with the semilinear equation (i.e. $p = 2$). More precisely, they considered the equation

$$-\Delta u = \lambda [\beta u - u^2] \quad \text{in} \, \mathbb{R}^N$$

with $\beta : \mathbb{R}^N \to \mathbb{R}$ a smooth function bounded above. Our semilinear result here (see Theorem 9) extends the result of Afrouzi–Brown [2], since we do not assume that $\beta$ is smooth and bounded above, and our absorption term $-f(z,x)$ is more general than the one considered therein.

2 The bifurcation parameter

Following Szulkin–Willem [24], we impose the following conditions on the weight function $\beta$:

$H(\beta) : \beta \in L^\infty(\mathbb{R}^N)$, $\beta^+ = \beta_1 + \beta_2 \neq 0$, with $\beta_1 \in L^{N/p}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ ($1 < p < N$) and

$$\|z\|^p \beta_2(z) \to 0 \text{ as } \|z\| \to \infty, \quad \lim_{z \to \hat{z}} \|z - \hat{z}\|^p \beta_2(z) = 0 \text{ for all } \hat{z} \in \mathbb{R}^N,$$
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\[ \|z\|^{\frac{p(N-1)}{N}} \beta_2(z) \to 0 \text{ as } z \to 0. \]

Here, as usual, $\beta^+ = \max\{\beta, 0\}$ and $\beta^- = \max\{-\beta, 0\}$.

Moreover, in view of the Bifurcation Theorems 7 and 9, we also assume the following condition:

$H(\beta)_2$:

\[
\begin{cases}
(r\hat{a}(r))^{1/(p-1)} \in L^1(1, \infty) & \text{if } 1 < p \leq 2, \\
\frac{1}{(p-2)(N+1)} \hat{a}(r) \in L^1(1, \infty) & \text{if } p \geq 2,
\end{cases}
\]

where we have set $\hat{a}(r) := \sup_{\|z\|=R} \beta^+(z)$.

Now, for any $r > 0$, let $B_r = \{ z \in \mathbb{R}^N : \|z\| < r \}$. We introduce the following quantity:

\[
\hat{\lambda}_1(r) = \inf \left\{ \int_{B_r} \|Du\|^p dz : z \in W^{1,p}_0(B_r), \int_{B_r} \beta |u|^p dz = 1 \right\}. \tag{1}
\]

From Szulkin–Willem [24] we know that $\hat{\lambda}_1(r)$ is the principal eigenvalue of the nonlinear eigenvalue problem

\[
(E) \quad \begin{cases}
-\Delta_p u = \lambda \beta |u|^{p-2}u & \text{in } B_r, \\
u = 0 & \text{on } \partial B_r.
\end{cases}
\]

From (1) it is clear that the map $r \mapsto \hat{\lambda}_1(r)$ is decreasing on $(0, \infty)$ and so we can define

\[
\hat{\lambda}^* = \lim_{r \to \infty} \hat{\lambda}_1(r). \tag{2}
\]

In the following we will need the so called Hardy’s inequality, an extension of the Poincaré inequality to the case of singular potentials:

\[ \int_{\mathbb{R}^N} \left( \frac{|u(z)|}{\|z\|} \right)^p dz \leq \frac{p^p}{(N-p)^p} \int_{\mathbb{R}^N} \|Du\|^p dz \tag{3} \]

for all $u \in D^{1,p}(\mathbb{R}^N) = C^\infty_c(\mathbb{R}^N)$, the closure being taken in the norm $\|u\| = \|Du\|_p$.

**Proposition 1.** If hypotheses $H(\beta)$ hold, then $\hat{\lambda}^* > 0$.

**Proof.** For $r > 0$, let $u \in W^{1,p}_0(B_r)$ be such that $\int_{B_r} \beta |u|^p dz = 1$. We have

\[
1 = \int_{B_r} \beta |u|^p dz = \int_{B_r} (\beta^+ - \beta^-) |u|^p dz \tag{4}
\]

\[ \leq \int_{B_r} \beta^+ |u|^p dz = \int_{B_r} \beta_1 |u|^p dz + \int_{B_r} \beta_2 |u|^p dz. \]
By hypotheses $H(\beta)$, we have that $\beta_1 \in L^{N/p}(\mathbb{R}^N)$, and by the Sobolev Embedding Theorem we have that $u \in L^{p^*}(B_r)$, where $p^* = Np/(N - p)$ is the critical Sobolev exponent. Note that

$$\frac{p}{N} + \frac{p}{p^*} = \frac{p}{N} + \frac{N - p}{N} = 1. \quad (5)$$

So, using Hölder’s and Sobolev’s inequalities, we have

$$\int_{B_r} \beta_1 |u|^p dz \leq \|\beta_1\|_{L^{N/p}(B_r)} \|u\|_{L^{p^*}(B_r)}^{p^*} \leq c_1 \|\beta_1\|_{L^{N/p}(B_r)} \|Du\|_{L^p(B_r, \mathbb{R}^N)}^p \quad (6)$$

for some $c_1 > 0$. From the assumptions on $\beta_2$ in $H(\beta)$, given $\varepsilon > 0$, we can find $r_0 = r_0(\varepsilon) > \delta = \delta(\varepsilon) > 0$ such that

$$\|z\|^{\beta_2(z)} \leq \varepsilon \quad \text{for all } z \in \mathbb{R}^N, \|z\| \geq r_0 \quad (7)$$

and

$$\|z\|^{\frac{(N-1)}{r} \beta_2(z)} \leq \varepsilon \quad \text{for all } z \in \partial B_\delta. \quad (8)$$

Let $r > r_0$ be such that $1/r < \delta$ and set $\Omega_1 = B_r \setminus \overline{B_{r_0}}$. We have

$$\int_{\Omega_1} \beta_2 |u|^p dz \leq \varepsilon \int_{\Omega_1} \frac{|u|^p}{\|z\|^p} dz \quad (by \ (7))$$

$$\leq \left(\frac{p}{N - p}\right) \varepsilon \int_{\Omega_1} \|Du\|^p dz \quad (by \ (3))$$

$$= c_2 \varepsilon \int_{\Omega_1} \|Du\|^p dz \quad (9)$$

with $c_2 = \left(\frac{p}{N - p}\right)^p > 0$.

Now let $\Omega_2 = B_\delta - \overline{B_{1/r}}$, so that

$$\int_{\Omega_2} \beta_2 |u|^p dz \leq \varepsilon \int_{\Omega_2} \frac{|u|^p}{\|z\|^p} (\frac{1}{N-1})^{N/p} dz \quad (by \ (8))$$

$$\leq \varepsilon \int_{\Omega_2} \left(\frac{1}{\|z\|^{N/(N-1)}}\right)^{N/p} dz \quad \|u\|^{p^*}_{L^{p^*}(\Omega_2)} \quad (by \ \text{Hölder’s inequality, see} \ (5))$$

$$\leq c_3 \varepsilon \left(\int_{1/r}^{\delta} \frac{1}{s^{N-1} \omega_N ds}\right)^{p/N} \|Du\|_{L^p(B_r, \mathbb{R}^N)}^p \quad (by \ \text{Sobolev’s inequality and for some} \ c_3 > 0)$$

$$= c_4 \varepsilon \left(\frac{\delta - 1}{r}\right)^{p/N} \int_{B_r} \|Du\|^p dz \quad \text{for some} \ c_4 > 0; \quad (10)$$

Here $\omega_N$ denotes the measure of the unit sphere in $\mathbb{R}^N$. 


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Next let $\Omega_3 = B_{r_0} \setminus \overline{B}_0$. The compactness of $\overline{\Omega_3}$ and hypotheses $H(\beta)$ imply that we can find closed balls $\left\{ B_{\rho_k}(z_k) \right\}_{k=1}^m$ such that in correspondence of $\varepsilon > 0$ chosen above, there holds

$$\|z - z_k\| \leq \rho_k \Rightarrow \|z - z_k\|^3 \beta_2(z) \leq \varepsilon. \quad (11)$$

From (11) it follows that we can find $\rho > 0$ such that for all $k \in \{1, \ldots, m\}$ we have $\rho < \rho_k$ and

$$\|z - z_k\| \leq \rho \Rightarrow \|z - z_k\|^\frac{2N-1}{N} \beta_2(z) \leq \frac{\varepsilon}{m}. \quad (12)$$

Setting $U = \cup_{k=1}^m B_{\rho}(z_k)$, we have

$$\int_{B_{\rho}(z_k)} \beta_2 |u|^p \, dz \leq \frac{\varepsilon}{m} \int_{B_{\rho}(z_k)} \frac{|u|^p}{\|z - z_k\|^\frac{2N-1}{N}} \, dz \quad \text{by (12)}$$

$$\leq \frac{\varepsilon}{m} \left[ \int_{B_{\rho}(z_k)} \left( \frac{1}{\|z - z_k\|^\frac{2N-1}{N}} \right)^{p/N} \|u\|_{L^p(B_r)}^p \, dz \right] \quad \text{by Hölder’s inequality and (5)}$$

$$\leq \frac{c_5 \varepsilon}{m} \left[ \int_{B_{\rho}} \|z\|^{-p(N-1)/N} \, dz \right]^{p/N} \|Du\|_{L^p(B_r, \mathbb{R}^N)}^p \quad \text{by Sobolev’s inequality}$$

$$\leq \frac{c_6 \varepsilon}{m} \|Du\|_{L^p(B_r, \mathbb{R}^N)}^p,$$

for some $c_5 > 0$ and with $c_0 = (\omega_N \rho)^{p/N}$. Adding these inequalities over $k \in \{1, \ldots, m\}$ we obtain

$$\int_U \beta_2 |u|^p \, dz \leq c_0 \varepsilon \|Du\|_{L^p(B_r, \mathbb{R}^N)}^p. \quad (14)$$

Finally, if $z \in \Omega_3 \setminus U$, then we can find $k \in \{1, \ldots, m\}$ such that

$$\rho < \|z - z_k\| \leq \rho_k.$$

Hence from (11) it follows that

$$\beta_2(z) \leq \frac{\varepsilon}{\rho^\frac{2N-1}{N}}. \quad (15)$$

Therefore, we have

$$\int_{\Omega_3 \setminus U} \beta_2 |u|^p \, dz \leq \frac{\varepsilon}{\rho^\frac{2N-1}{N}} \int_{\Omega_3 \setminus U} |u|^p \, dz \quad \text{(see (15))}$$

$$\leq c_7 \varepsilon \|Du\|_{L^p(B_r, \mathbb{R}^N)}^p,$$
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where $c_7 > 0$ is a constant depending on $r_0$ and $\rho$, but not on $r$.

Now, we return to (15) and use (6), (9), (10), (14) and (15), obtaining

$$1 \leq \left[ c_1 \| \beta \|_{L^{N/p}(B_r)} + c_2 \varepsilon + c_4 \varepsilon \left( \delta - \frac{1}{r} \right)^{p/N} + (c_6 + c_7) \varepsilon \right] \int_{B_r} \| Du \|^p dz.$$ 

Since $u \in W^{1,p}_0(B_r)$ with $\int_{B_r} |\beta|^p u^p dz = 1$ was arbitrary, from (1) it follows

$$0 < \left[ c_1 \| \beta \|_{L^{N/p}(\mathbb{R}^N)} + c_2 \varepsilon + c_4 \varepsilon \left( \delta - \frac{1}{r} \right)^{p/N} + (c_6 + c_7) \varepsilon \right]^{-1} \leq \hat{\lambda}_1(r).$$ 

Passing to the limit as $r \to \infty$ in the previous inequality, we obtain

$$0 < \left[ c_1 \| \beta \|_{L^{N/p}(\mathbb{R}^N)} + c_2 \varepsilon + c_4 \varepsilon \left( \delta - \frac{1}{r} \right)^{p/N} + (c_6 + c_7) \varepsilon \right]^{-1} \leq \hat{\lambda}^*,$$

as claimed. □

**Remark 1.** For Proposition 1, the requirement $\beta_2 \in L^\infty_{loc}(\mathbb{R}^N)$ is sufficient, but in the future, namely for Proposition 2 and its consequences, we shall use $\beta \in L^\infty_{loc}(\mathbb{R}^N)$.

We end up this section recalling that, if $Y$ is a Banach space, $Y^*$ is its topological dual and $E : Y \to Y^*$ is a map, we say that $E$ is of type $(S)_+$ if

$$y_n \to y \text{ in } Y \text{ and } \limsup_{n \to \infty} \langle E(y_n), y_n - y \rangle_{Y^*,Y} \leq 0$$

(by $\langle \cdot, \cdot \rangle_{Y^*,Y}$ we denote the duality brackets for the pair $(Y^*,Y)$), then $y_n \to y$ in $Y$ (for example, see Gasinski–Papageorgiou [16]).

3 The bifurcation theorem

The hypotheses on the absorption term $f$ are:

**H(f):** $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.e. $z \in \mathbb{R}^N$ there holds $f(z,0) = 0$, $f(z,x) > 0$ for all $x > 0$ and

(i): for every $\rho > 0$ there exists $a_\rho \in L^\infty_{loc}(\mathbb{R}^N) \cap L^{p^*/2'}(\mathbb{R}^N)$ ($1/p^* + 1/(p^*)' = 1$) such that

$$f(z,x) \leq a_\rho(z) \text{ for a.e. } z \in \mathbb{R}^N \text{ and all } 0 \leq x \leq \rho;$$

(ii): there exists $\eta \in L^\infty(\mathbb{R}^N)$ such that

$$\theta = \text{ess inf}_{\mathbb{R}^N} (\eta - \beta) > 0$$

and

$$\liminf_{x \to \infty} \frac{f(z,x)}{x^{p-1}} \geq \eta(z) \text{ uniformly for a.e. } x \in \mathbb{R}^N;$$

(iii): we have

$$\lim_{x \to 0^+} \frac{f(z,x)}{x^{p-1}} = 0 \text{ uniformly for a.e. } z \in \mathbb{R}^N.$$
Remark 2. Since we are interested in positive solutions and hypotheses $H(f)$ concern the positive semiaxis $\mathbb{R}^+ = [0, \infty)$ only, by truncating $f$ if necessary, we may (and will) suppose that $f(z, x) = 0$ for a.e. $z \in \mathbb{R}^N$ and all $x \leq 0$.

Examples of functions satisfying $H(f)$ are easy to construct; as an example, if $\beta^+ \in L^\infty(\mathbb{R}^N) \cap L^{p^*'}(\mathbb{R}^N)$, let us present

$$f(z, x) = \begin{cases} 
\beta^+(z)x^{p-1} & \text{if } x \geq 1, \\
\beta^+(z)x^{p-1} & \text{if } x < 1,
\end{cases}$$

where $\varphi$ is any number strictly greater than $p$.

In what follows a fundamental rôle will be played by the suitably weighted space

$$V = \left\{ u \in D^{1,p}(\mathbb{R}^N) : \|u\|^p_V = \|Du\|^p_{L^p(\mathbb{R}^N, \mathbb{R}^N)} + \int_{\mathbb{R}^N} \beta^-|u|^p dz < \infty \right\},$$

which turns out to be a reflexive Banach space, and in which we will look for solutions to problem $(P_\lambda)$.

Proposition 2. If hypotheses $H(\beta)$, $H(\beta)_2$, $H(f)$ hold and $\lambda > \hat{\lambda}^* > 0$, then problem $(P_\lambda)$ has a positive solution $u_0 \in C^1(\mathbb{R}^N) \cap V$ such that

$$u_0(z) > 0 \text{ for all } z \in \mathbb{R}^N \text{ and } u_0(z) \to 0 \text{ as } \|z\| \to \infty.$$

Proof. The proof is rather long, and will need several intermediate steps.

By virtue of hypothesis $H(f)(ii)$, given $\varepsilon > 0$, there exists $M = M(\varepsilon) > 0$ such that

$$f(z, x) \geq (\eta(z) - \varepsilon)x^{p-1} \text{ for a.e. } z \in \mathbb{R}^N \text{ and all } x \geq M. \quad (17)$$

Let $\xi > M$. For any $r > 0$ and any $h \in W^{1,p}_0(B_r)$, $h \geq 0$, we have

$$\int_{B_r} (\beta(z)\xi^{p-1} - f(z, \xi))h(z) dz \leq \int_{B_r} (\beta(z)\xi^{p-1} - \eta(z)\xi^{p-1} + \varepsilon\xi^{p-1})h(z) dz \quad (\text{see (17)})$$

$$\leq \xi^{p-1}(\varepsilon - \vartheta) \int_{B_r} h dz \quad (\text{see hypothesis } H(f)(ii)).$$

Choosing $\varepsilon \in (0, \vartheta)$, from (18) we infer that

$$\int_{B_r} (\beta(z)\xi^{p-1} - f(z, \xi))h(z) dz \leq 0 \text{ for all } h \in W^{1,p}_0(B_r), h \geq 0. \quad (19)$$

Now, we consider the following truncation for the reaction term:

$$g(z, x) = \begin{cases} 
0 & \text{if } x \leq 0, \\
\beta(z)x^{p-1} - f(z, x) & \text{if } 0 \leq x \leq \xi, \\
\beta(z)\xi^{p-1} - f(x, \xi) & \text{if } x > \xi.
\end{cases} \quad (20)$$
Of course, \( q \) is a Carathéodory function. We set \( G(z, x) = \int_0^r q(z, s) \, ds \) for all \((z, x) \in B_r \times \mathbb{R}, \) and we consider the \( C^1 \) functional \( \psi_\lambda : W_0^{1, p}(B_r) \to \mathbb{R} \) defined by
\[
\psi_\lambda (u) = \frac{1}{p} \| Du \|_p^p - \lambda \int_{B_r} G(z, u(z)) \, dz
\]
for all \( u \in W_0^{1, p}(B_r). \)

It is clear from (20) that \( \psi_\lambda \) is coercive. Moreover, exploiting the compact embedding of \( W_0^{1, p}(B_r) \) into \( L^p(B_r), \) we can easily check that \( \psi_\lambda \) is sequentially weakly lower semicontinuous. Therefore, by the Weierstrass Theorem, we can find \( u_0^r \in W_0^{1, p}(B_r) \) such that
\[
\psi_\lambda (u_0^r) = \inf \{ \psi_\lambda (u) : u \in W_0^{1, p}(B_r) \} = m_r.
\]

**Lemma 3.** \( u_0^r \neq 0. \)

**Proof.** By virtue of hypothesis \( H(f)(iii), \) given \( \varepsilon > 0, \) we can find \( \delta = \delta(\varepsilon) > 0 \) such that
\[
f(z, x) \leq \varepsilon x^{p-1} \text{ for a.e. } z \in B_r \text{ and all } x \in [0, \delta],
\]
so that
\[
F(z, x) \leq \frac{\varepsilon}{p} x^p \text{ for a.e. } z \in B_r \text{ and all } x \in [0, \delta].
\]

The ordered Banach space \( C_0^1(\overline{B_r}) = \{ u \in C^1(B_r) : u = 0 \text{ on } \partial \Omega \} \) has a positive cone given by \( C_+^r = \{ u \in C_0^1(\overline{B_r}) : u(z) \geq 0 \text{ for all } z \in \overline{B_r} \}, \) and this cone has a nonempty interior given by
\[
\text{int } C_+^r = \{ u \in C_+^r : u(z) > 0 \text{ for all } z \in B_r \text{ and } \frac{\partial u}{\partial \nu} < 0 \text{ for all } z \in \partial B_r \},
\]
where \( \nu(z) \) denotes the unit outward normal to \( \partial B_r \) at \( z. \)

Let \( \hat{u}_r \in \text{int } C_+^r \) be the principal eigenfunction of the eigenvalue problem \((E)\) (see Szulkin–Willem [24] for the existence and Vazquez [27] for the regularity). We can find \( t \in (0, 1) \) so small that \( t\hat{u}_r(z) \in [0, \delta] \) for all \( z \in \overline{B_r}. \) Then
\[
\psi_\lambda (t\hat{u}_r) = \frac{t^p}{p} \| D\hat{u}_r \|_{L^p(B_r, \mathbb{R}^N)}^p - \frac{\lambda t^p}{p} \int_{B_r} \beta \hat{u}_r^p \, dz + \lambda \int_{B_r} F(z, t\hat{u}_r) \, dz \text{ (see (20))}
\]
\[
\leq \frac{(\lambda_{\lambda}(r) - \lambda) t^p}{p} \int_{B_r} \beta \hat{u}_r \, dz + \frac{\lambda t^p}{p} \varepsilon \| \hat{u}_r \|_p^p \text{ (see (22)).}
\]

But \( \int_{B_r} \beta \hat{u}_r \, dz = 1, \) and for \( r > 0 \) large enough we have \( \lambda_{\lambda}(r) < \lambda \) (see Proposition 1 and recall that \( \lambda > \lambda^* \)). So, if in (23) we choose \( \varepsilon > 0 \) sufficiently small, then we have \( \psi_\lambda (t\hat{u}_r) < 0, \) and by (21)
\[
m_r = \psi_\lambda (u_0^r) < \psi_\lambda (0) = 0,
\]
and therefore \( u_0^r \neq 0, \) as claimed. \( \square \)
Lemma 4. \( u_0^r \in \text{int} \ C_r^c \), and in addition
\[
\left\{ u_0^r \in [0, \xi] \mid u \in W_0^{1,p}(B_r) : 0 \leq u(z) \leq \xi \text{ a.e. in } B_r \right\}.
\]

Proof. From (21) we have \((\psi_\lambda)'(u_0) = 0\), that is
\[
A(u_0^r) = \lambda N_g(u_0^r),
\]
where \(A : W_0^{1,p}(B_r) \to W^{-1,p'}(B_r)\) is the nonlinear map defined by
\[
\langle A(u), y \rangle = \int_{B_r} \|Du\|^{p-2}(Du, Dy)_{\mathbb{R}^N} dz
\]
for all \(u, y \in W_0^{1,p}(B_r)\), and \(N_g(u)(\cdot) = g(\cdot, u(\cdot))\) for all \(u \in W_0^{1,p}(B_r)\).

On (24) we first act with \(-u_0^r \in W_0^{1,p}(B_r)\), and recalling (20), we obtain
\[
\int_{B_r} \|D(u_0^r)^-\|^pdz = 0,
\]
that is \(u_0^r \geq 0\) and \(u_0^r \neq 0\) (from above).

Next we act on (24) with \((u_0^r - \xi)^+ \in W_0^{1,p}(B_r)\), obtaining
\[
\langle A(u_0^r), (u_0^r - \xi)^+ \rangle = \lambda \int_{B_r} g(z, u_0^r)(u_0^r - \xi)^+ dz
\]
\[
= \lambda \int_{B_r} (\beta \xi^{p-1} - f(z, \xi))(u_0^r - \xi)^+ dz \quad \text{(by (20))}
\]
\[
\leq 0 \quad \text{(by (19)),}
\]
that is
\[
\int_{\{u_0^r > \xi\}} \|Du_0^r\|^pdz \leq 0.
\]
Thus, either \(|\{u_0^r > \xi\}|_N = 0\) where \(| \cdot |_N\) denotes the Lebesgue measure in \(\mathbb{R}^N\), or \(u_0^r\) is constant in the set \(\{u_0^r > \xi\}\). Nonlinear regularity theory (see, for example, Gasinski-Papageorgiou [16, pp. 737–738]) implies that \(u_0^r \in C_0^1(\overline{B_r})\), so that the second possibility is neglected, and thus \(u_0^r \leq \xi\).

Therefore, we have proved that
\( u_0^r \in [0, \xi] \).

Then (20) implies that (24) becomes
\[
A(u_0^r) = \lambda \beta(u_0^r)^{p-1} - \lambda N_f(u_0^r),
\]
with \(N_f(u)(\cdot) = f(\cdot, u(\cdot))\) for all \(u \in W_0^{1,p}(B_r)\), that is
\[
\begin{cases}
-\Delta_p u_0^r(z) = \lambda \beta(u_0^r(z))^{p-1} - f(z, u_0^r(z)) & \text{in } B_r, \\
u_0^r = 0 & \text{on } \partial B_r.
\end{cases}
\]
\[\text{(25)}\]
Then
\[ \Delta_p u_0^p(z) \leq \lambda \| \beta^+ \|_{L^\infty(B_r)} u_0^p(z) \leq f(z, x) \text{ a.e. in } B_r. \] (26)

The assumptions on \( f \) say that there exists \( C, \delta > 0 \) such that
\[ 0 \leq f(z, x) \leq \left( 1 + \frac{C}{\delta} \right) |x|^{p-1} \text{ for a.e. } z \in B_r \text{ and all } x \geq 0. \] (27)

Then, from (26) and (27) we have
\[ \Delta_p u_0^p(z) \leq \lambda \left[ \| \beta^+ \|_{L^\infty(B_r)} + \left( 1 + \frac{C}{\delta} \right) \right] u_0^p(z) \text{ a.e. in } B_r, \]
and so \( u_0^p \in \text{int } C_+^r \), see Vazquez [27] and Pucci–Serrin [22]. □

Lemma 5. Problem (25) has a smallest solution \( u \) belonging to \([0, \xi] \cap \text{int } C_+^r\).

Proof. Set
\[ S_\xi^r = \left\{ u \in W_0^{1,p}(B_r) : u \text{ is a nontrivial solution of (25) with } u \in [0, \xi] \right\}. \]

We have just shown above that \( S_\xi^r \neq \emptyset \) and that \( S_\xi^r \subset \text{int } C_+^r \). Using Lemma 4.3 of Filippakis–Kristaly–Papageorgiou [9], we infer that \( S_\xi^r \) is downward directed, i.e., if \( u, v \in S_\xi^r \), then there exists \( y \in S_\xi^r \) such that \( y \leq \min\{u, v\} \). We now show that \( S_\xi^r \) has a minimal element. To this purpose, let \( C \subset S_\xi^r \) be a chain (i.e., a totally ordered subset of \( S_\xi^r \)). From Dunford–Schwartz [8, p. 336] we know that we can find a sequence \( \{u_n\}_{n \geq 1} \subset C \) such that
\[ \inf_{n \geq 1} \{u_n\}_{n \geq 1} = \inf C. \]

Evidently, \( \{u_n\}_{n \geq 1} \subset W_0^{1,p}(B_r) \) is bounded, and so we may assume without loss of generality that
\[ u_n \rightharpoonup u \text{ in } W_0^{1,p}(B_r), \quad u_n \rightarrow u \text{ in } L^p(B_r) \text{ and a.e. in } B_r \text{ as } n \rightarrow \infty. \] (28)

Since \( u_n \in S_\xi^r \) for any \( n \in \mathbb{N} \), we have
\[ A(u_n) = \lambda \beta u_n^{p-1} - \lambda N_f(u_n) \text{ for all } n \geq 1. \] (29)

On (29) we act with \( u_n - u \in W_0^{1,p}(B_r) \) and then pass to the limit as \( n \rightarrow \infty. \) By (28) we immediately obtain
\[ \lim_{n \rightarrow \infty} (A(u_n), u_n - u) = 0, \]
and so, \( A \) being of type \((S)_+\) (see, for example, [16]), we get that
\[ u_n \rightarrow u \text{ in } W_0^{1,p}(B_r). \] (30)
Thus, if in (29) we pass to the limit as \( n \to \infty \) and use (30), we obtain
\[
A(u) = \lambda \beta u^{p-1} - \lambda N_f(u). \tag{31}
\]

By (28) we have that \( u \in [0, \xi] \); we now show that \( u \neq 0 \). If \( u = 0 \), then from (30) we would have that \( \|u_n\| = \|u_n\|_{W_1^p(B_r)} \to 0 \) as \( n \to \infty \), and so, up to a subsequence, \( u_n \to 0 \) a.e. in \( B_r \).

Let \( y_n = u_n/\|u_n\|, \ n \geq 1 \). Then \( \|y_n\| = 1 \) for all \( n \geq 1 \) and so we may assume that
\[
y_n \to y \text{ in } W_0^{1,p}(B_r), \ y_n \to y \text{ in } L^p(B_r) \text{ and a.e. in } B_r \text{ as } n \to \infty. \tag{32}
\]

From (29) we have
\[
A(y_n) = \lambda \beta u_n^{p-1} - \lambda \frac{N_f(u_n)}{\|u_n\|^{p-1}}, \ n \geq 1. \tag{33}
\]

Acting on (33) with \( y_n - y \in W_0^{1,p}(B_r) \) and passing to the limit as \( n \to \infty \), by \( H(f)(iii) \) and (32) we obtain
\[
\lim_{n \to \infty} (A(y_n), y_n - y) = 0,
\]
and thus
\[
y_n \to y \text{ in } W_0^{1,p}(B_r), \text{ so that } \|y\| = 1 \text{ and } y \geq 0. \tag{34}
\]

From \( H(f)(i) \) it is clear that \( \left\{ \frac{N_f(u_n)}{\|u_n\|^{p-1}} \right\}_{n \geq 1} \subset L^p(B_r) \) is bounded, and so we may assume that \( \frac{N_f(u_n)}{\|u_n\|^{p-1}} \to h \) in \( L^p(B_r) \) as \( n \to \infty \). Moreover, hypothesis \( H(f)(iii) \) implies \( h = 0 \). Therefore, if in (33) we pass to the limit as \( n \to \infty \), we obtain
\[
A(y) = \lambda \beta y^{p-1}.
\]

Equivalently,
\[
\begin{cases}
-\Delta y = \lambda \beta y^{p-1} & \text{in } B_r, \\
y = 0 & \text{on } \partial B_r,
\end{cases}
\]

that is \((\lambda, y)\) solve problem \((E)\), and so \( y \in C_0^1(\overline{B_r}) \) must be nodal, since \( \lambda > \lambda_1(r) \) (for example, see [6, Theorem 3.2]).

This contradicts (34) and this proves that \( u \neq 0 \). As before, via the nonlinear maximum principle (see [22], [27]), we have that \( u \in \text{int } C_+^r \), and so \( u \in S_0^r \).

Since \( C \) was an arbitrary chain in \( S_0^r \), from the Kuratowski–Zorn Lemma we infer that \( S_0^r \) has a minimal element \( u_\ast \in S_0^r \subset \text{int } C_+^r \). If \( u \in S_0^r \), since \( S_0^r \) is downward directed and due to the minimality of \( u_\ast \), we have \( u_\ast \leq u \), and so \( u_\ast \in \text{int } C_+^r \) is the smallest element of \( S_0^r \). This proves the lemma. \( \square \)

Now, let \( r_n \uparrow \infty \) be such that \( \lambda_1(r_1) < \lambda \), and hence \( \lambda_1(r_n) < \lambda \) for all \( n \geq 1 \). Let \( u_0^n = u_0^{r_n} \), \( n \geq 1 \). Extending by zero outside \( \partial B_n \), we obtain a function \( u_0^n \in W_1^{1,p}(\mathbb{R}^N) \). From the claim above we know that
\[
0 \leq u_0^n \leq \xi \text{ in } \mathbb{R}^N \text{ for all } n \geq 1. \tag{35}
\]
We also know that $A(\bar{u}_0^n) = \lambda \beta(\bar{u}_0^n)^p - \lambda N_f(\bar{u}_0^n)$ in $W_0^{1,p}(B_{r_0})^*$. In view of the first part of the proof, we make the truncation

$$g(z, x) = \begin{cases} 
0 & \text{if } x \leq 0, \\
\beta(z)x^{p-1} - f(z, x) & \text{if } 0 \leq x \leq \bar{u}_0^n(z), \\
\beta(z)\bar{u}_0^n(z)^{p-1} - f(x, \bar{u}_0^n(z)) & \text{if } x > \bar{u}_0^n(z).
\end{cases}$$

In this way we obtain a solution of (26) with $r = r_1$ which is located in the order interval $[0, \bar{u}_2^0 |B_{r_1}|]$. But $\bar{u}_2^0 |B_{r_1}| = u_1^0$ is the smallest solution of (26) (with $r = r_1$) in $[0, \xi] \supset [0, \bar{u}_2^0 |B_{r_1}|]$ by (35). Thus $\bar{u}_1^0 \leq \bar{u}_2^0$. Continuing this way, by induction it follows that

$$0 < \bar{u}_n^0 \leq \bar{u}_{n+1}^0 \text{ in } \mathbb{R}^N \text{ for all } n \geq 1, \quad (36)$$

and thus there exists

$$u_0 = \lim_{n \to \infty} \bar{u}_n^0 \text{ in } \mathbb{R}^N.$$

For every open and bounded domain $\Omega \subset \mathbb{R}^N$ with $C^2$ boundary, there exists $n_0 \in \mathbb{N}$ such that $\Omega \subset B_{r_n}$, for all $n \geq n_0$; from the nonlinear regularity theory (see, for example \cite[p. 738]{16}) we know that there exists $\alpha \in (0, 1)$ and $M_0 > 0$ such that

$$\bar{u}_n^0 |\Omega| \in C^{1,\alpha}(\Omega) \text{ and } ||\bar{u}_n^0 |\Omega||_{C^{1,\alpha}(\Omega)} \leq M_0 \text{ for all } n \geq n_0. \quad (37)$$

From the compact embedding of $C^{1,\alpha}(\Omega)$ into $C^1(\Omega)$ and (36), (37), we have

$$\bar{u}_n^0 |\Omega| \to u_0 |\Omega| \text{ in } C^1(\Omega),$$

so that

$$u_0 \in C^1(\mathbb{R}^N) \text{ and } u_0 > 0 \text{ in } \mathbb{R}^N.$$

We also know that

$$A(u_0^n) = \lambda \beta(u_0^n)^p - \lambda N_f(u_0^n) \text{ in } W_0^{1,p}(B_{r_n})^* \text{ for all } n \geq 1. \quad (38)$$

In (38) we act with $u_0^n \in W_0^{1,p}(B_{r_n})$ and obtain

$$\int_{B_{r_n}} ||Du_0^n||^p dz = \lambda \int_{B_{r_n}} \beta(u_0^n)^p dz - \lambda \int_{B_{r_n}} f(z, u_0^n)u_0^0 dz$$

$$\leq \lambda \int_{B_{r_n}} \beta(u_0^n)^p dz \quad (\text{see } H(f))$$

$$\leq \lambda \int_{B_{r_n}} \beta^+(u_0^n)^p dz$$

$$= \lambda \int_{B_{r_n}} (\beta_1 + \beta_2)(u_0^n)^p dz \quad (\text{see } H(\beta)). \quad (39)$$
We have
\[
\int_{B_{r_n}} \beta_1 (u^n_0)^p \, dz \leq \xi^p \int_{B_{r_n}} |\beta_1| \, dz \quad \text{(by (35))}
\]
\[
\leq \xi^p \int_{\mathbb{R}^N} |\beta_1| \, dz
\]
\[
= c_{9} = c_{9}(\xi, \|\beta\|_1) > 0 \quad \text{(see } H(\beta)) .
\]  

(40)

Moreover, from the estimates found for (13) in the proof of Proposition 1, we have
\[
\int_{B_{r_n}} \beta_2 (u^n_0)^p \, dz \leq c_{10} \varepsilon \int_{B_{r_n}} \| D u^n_0 \|^p \, dz
\]
for some $c_{10} > 0$.

We return to (39) and use (40), (41), obtaining
\[
\int_{B_{r_n}} \| D u^n_0 \|^p \, dz \leq c_{9} + c_{10} \varepsilon \int_{B_{r_n}} \| D u^n_0 \|^p \, dz \quad \text{for all } n \geq 1 .
\]  

(42)

Now choose $\varepsilon \in (0, 1)$ so small that $c_{10} \varepsilon < 1$. Then from (42) it follows that
\[
\int_{B_{r_n}} \| D u^n_0 \|^p \, dz \leq c_{11} \quad \text{for some } c_{11} > 0 \text{ and all } n \geq 1 .
\]  

(43)

But we know that
\[
D u^n_0 |_{B_{r_n}} = D u^n_0 \quad \text{and } D u^n_0 |_{\mathbb{R}^N \setminus B_{r_n}} = 0 \quad \text{for all } n \geq 1 ,
\]
see, for example, Kesavan [20, p. 69,70]. Therefore, from (43) we have
\[
\| D u^n_0 \|_{L^p(\mathbb{R}^N, \mathbb{R}^N)} \leq c_{11} \quad \text{for all } n \geq 1 .
\]  

(44)

We also have
\[
0 \leq \int_{\mathbb{R}^N} \| D u^n_0 \|^p \, dz = \lambda \int_{\mathbb{R}^N} (\beta(u^n_0))^p \, dz - \lambda \int_{\mathbb{R}^N} f(z, \bar{u}^n_0) \bar{u}^n_0 \, dz \leq \lambda \int_{\mathbb{R}^N} (\beta(u^n_0))^p \, dz
\]
by $H(f)$, so that
\[
\lambda \int_{\mathbb{R}^N} \beta^- (u^n_0)^p \, dz \leq \lambda \int_{\mathbb{R}^N} \beta^+ (u^n_0)^p \, dz .
\]  

(45)

From the estimates in the proof of Proposition 1 and (44), (45) it follows
\[
\lambda \int_{\mathbb{R}^N} \beta^+ (u^n_0)^p \, dz \leq c_{12} \quad \text{for some } c_{12} > 0 \text{ and all } n \geq 1 .
\]  

(46)

Then from (44) and (46), we infer that $\{ \bar{u}^n_0 \}_{n \geq 1} \subset V$ is bounded, and so we may assume that
\[
\bar{u}^n_0 \rightharpoonup u_0 \text{ in } V .
\]  

(47)
Let $\tilde{A} : V \to V^*$ be the nonlinear map defined by

$$\langle \tilde{A}(y), v \rangle_{V^*,V} = \int_{\mathbb{R}^N} \|Dy\|^{p-2}(Dy, Dv)_{\mathbb{R}^N} \, dz$$

for all $y, v \in V$.

From Szulkin–Willem [24] (see also Huang [18]) we know that $\tilde{A}$ is of type $(S)_+$ in $V$. By the Claim above, we know that

$$\tilde{A}(\bar{u}^n_0) = \lambda \beta(\bar{u}^n_0)^{p-1} - \lambda N_f(\bar{u}^n_0)$$

in $V^*$ for all $n \geq 1$. (48)

Acting on (48) with $\bar{r} > 0$ for

$$\tilde{A}(\bar{u}^n_0) = \beta(\bar{u}^n_0)^{p-1}  - \lambda N_f(\bar{u}^n_0)$$

in $V^*$ for all $n \geq 1$. (48)

By Szulkin–Willem [24], Lemma 4.2, we have

$$\int_{\mathbb{R}^N} \beta(\bar{u}^n_0)^{p-1}(\bar{u}^n_0 - u_0) \, dz$$

(49)

as $n \to \infty$. Moreover, for any $r > 0$

$$\int_{\mathbb{R}^N} f(\bar{z}, \bar{u}^n_0)(\bar{u}^n_0 - u_0) \, dz \leq \int_{\mathbb{R}^N} a_{\epsilon} |\bar{u}^n_0 - u_0| \, dz$$

(51)

Evidently, by $H(f)(i)$ and the compact embedding of $V$ in $L^1_{\text{loc}}(\mathbb{R}^N)$, we have

$$\lim_{n \to \infty} \int_{B_r} a_{\epsilon} |\bar{u}^n_0 - u_0| \, dz = 0.$$  

Moreover, given $\epsilon > 0$, using Hölder’s inequality, we get

$$\int_{\mathbb{R}^N \setminus B_r} a_{\epsilon} |\bar{u}^n_0 - u_0| \, dz \leq \|\bar{u}^n_0 - u_0\|_{L^{p'}(\mathbb{R}^N \setminus B_r)} \left( \int_{\mathbb{R}^N \setminus B_r} a_{\epsilon}^{(p')} \, dz \right)^{1/(p')}$$

for $r > 0$ large enough and some $c_{13} > 0$.

Therefore, if in (51) we pass to the limit as $n \to \infty$ and use (52) and (53), we obtain

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} f(\bar{z}, \bar{u}^n_0)(\bar{u}^n_0 - u_0) \, dz$$

(54)
We now return to (49), pass to the limit as \( n \to \infty \) and use (50) and (54), obtaining

\[
\lim_{n \to \infty} \langle \tilde{A}(\bar{u}_0^n), \bar{u}_0^n - u_0 \rangle_{V^*, V} = 0,
\]

and since \( \tilde{A} \) is of type \((S)_+\), we finally get

\[
\bar{u}_0^n \to u_0 \text{ in } V. 
\] (55)

Note that for \( h \in V \) we have

\[
|\langle N_f(\bar{u}_0^n) - N_f(u_0), h \rangle_{V^*, V}| \leq \left| \int_{B_r} [f(z, \bar{u}_0^n) - f(z, u_0)]h \, dz \right| + \left| \int_{\mathbb{R}^N \setminus B_r} [f(z, \bar{u}_0^n) - f(z, u_0)]h \, dz \right|. \] (56)

From the continuity of the Nemitsky operator on the bounded domain \( B_r \), we have

\[
\lim_{n \to \infty} \int_{B_r} [f(z, \bar{u}_0^n) - f(z, u_0)]h \, dz = 0. \] (57)

On the other hand, via Hölder’s inequality, as in (53), we obtain

\[
\left| \int_{\mathbb{R}^N \setminus B_r} [f(z, \bar{u}_0^n) - f(z, u_0)]h \, dz \right| \leq c_{14} \varepsilon \] (58)

for \( r > 0 \) large enough, for all \( n \geq 1 \) and some \( c_{14} > 0 \).

Thus, if in (56) we let \( n \to \infty \) and we use (57), (58) and the fact that \( \varepsilon > 0 \) was arbitrary, we conclude that

\[
\lim_{n \to \infty} \langle N_f(\bar{u}_0^n) - N_f(u_0), h \rangle_{V^*, V} = 0 \text{ for all } h \in V;
\]

that is

\[
N_f(\bar{u}_0^n) \rightharpoonup N_f(u_0) \text{ in } V^*. \] (59)

From (55) it follows that

\[
\int_{\mathbb{R}^N} \beta^-(\bar{u}_0^n)^{p-1}h \, dz \to \int_{\mathbb{R}^N} \beta^- u_0^{p-1}h \, dz \text{ for all } h \in V,
\]

while, invoking [24, Lemma 4.2], we have

\[
\int_{\mathbb{R}^N} \beta^+(\bar{u}_0^n)^{p-1}h \, dz \to \int_{\mathbb{R}^N} \beta^+ u_0^{p-1}h \, dz \text{ for all } h \in V,
\]

so that

\[
\int_{\mathbb{R}^N} \beta(\bar{u}_0^n)^{p-1}h \, dz \to \int_{\mathbb{R}^N} \beta u_0^{p-1}h \, dz \text{ for all } h \in V. \] (60)

Therefore, if in (48) we pass to the limit as \( n \to \infty \) and we use (55), (59) and (60), obtaining

\[
\tilde{A}(u_0) = \lambda \beta u_0^{p-1} - \lambda N_f(u_0) \text{ in } V^*,
\]

and so \( u_0 \in C^1(\mathbb{R}^N) \cap V \) is a solution of \((P_\lambda)\) and
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\[ u_0 > 0 \text{ in } \mathbb{R}^N. \]

Finally, we show that $\lim_{\|z\| \to \infty} u_0(z) = 0$. For this, it is enough to prove that the problem

\[
\begin{aligned}
-\Delta_p v &= \lambda \beta^+ u_0^{p-1} \in L^\infty_{\text{loc}}(\mathbb{R}^N) \quad \text{in } \mathbb{R}^N, \\
v &> 0 \text{ and } v(z) \to 0 \text{ as } \|z\| \to \infty
\end{aligned}
\]

(61)

has a bounded solution. Indeed, if this is the case, we find

\[-\Delta_p v = \lambda \beta^+ u_0^{p-1} \geq \lambda \beta^+(\bar{u}_n^0)^{p-1} \geq \lambda \beta(\bar{u}_n^0)^{p-1} - f(z, \bar{u}_n^0) = -\Delta_p \bar{u}_n^0.\]

Moreover, by (36), we know that $0 \leq \bar{u}_n^0 \leq \xi$ and $\bar{u}_n^0 = 0$ at infinity for every $n \in \mathbb{N}$, so that $\bar{u}_n^0 \leq v$ by Theorem 3.5.1 and the following Remark in [22]; hence the claim follows by (55). Thus we are reduced to consider problem (61) and to show that it admits a solution vanishing at infinity. But finding a bounded solution of (61) is equivalent to finding a bounded one for

\[
\begin{aligned}
-\Delta_p w &= \lambda \beta^+ u_0^{p-1} w^{q-1} \in \mathbb{R}^N, \\
w &> 0 \text{ and } w(z) \to 0 \text{ as } \|z\| \to \infty
\end{aligned}
\]

(62)

with $q < p$, see Abdellaoui-Peral [1, Theorem 4.6]. Hence, let us prove that (62) actually admits a $C^1$ solution vanishing at infinity, as required. For this, it is enough to apply a result of Santos [23, Theorem 1.1], whose assumptions $(H_1)$ and $(H_2)$ are satisfied with $a = \beta^+ u_0^{p-1}$ and $b = 0$, while hypothesis $(H_3)$ is a consequence of $H(\beta)_2$ and of the fact that $u_0$ is bounded.

Next, we prove that for $\lambda \in [0, \hat{\lambda}^*)$ problem $(P_\lambda)$ has no positive solutions.

**Proposition 6.** If hypotheses $H(\beta)$ and $H(f)$ hold and $\lambda \in (0, \hat{\lambda}^*)$, then problem $(P_\lambda)$ has no positive solutions.

**Proof.** Suppose that $(P_\lambda)$ has a positive solution $u$. Then, as usual, $u \in C^1(\mathbb{R}^N) \cap V$ and

\[-\Delta_p u(z) = \lambda (\beta(z) u(z)^{p-1} - f(z, u(z))) \text{ in } \mathbb{R}^N.\]

By definition of solution and by assumption, we have

\[
\int_{\mathbb{R}^N} \|Du\|^p dz = \lambda \int_{\mathbb{R}^N} [\beta u^p - f(z, u)u] dz < \lambda \int_{\mathbb{R}^N} \beta u^p dz \quad \text{ (see } H(f)).
\]

(63)

Now, choose $\zeta \in C^1_0(\mathbb{R}^N)$ such that

\[ 0 \leq \zeta \leq 1, \quad \zeta|_{\{|z| \leq 1\}} = 1 \quad \text{and} \quad \zeta|_{\{|z| \geq 2\}} = 0. \]

Let $\zeta_n(z) = \zeta \left( \frac{z}{n} \right)$ and define $u_n = \zeta_n u$. Then $u_n(z) \to u(z)$ for all $z \in \mathbb{R}^N$. Since $u \in D^{1,p}(\mathbb{R}^N)$, from the Sobolev Embedding theorem, we have that $u \in
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Since $0 \leq u_n \leq u$ for any $n \geq 1$, from the Lebesgue Dominated Convergence Theorem we infer that

$$u_n \to u \text{ in } L^p (\mathbb{R}^N).$$

(64)

Consider the ring $R_n = \left\{ z \in \mathbb{R}^N : n < \|z\| < 2n \right\}$. We have

$$\|D u_n - D u\|_{L^p(\mathbb{R}^N, \mathbb{R}^N)} \leq \| (\zeta_n - 1) D u\|_{L^p(\mathbb{R}^N, \mathbb{R}^N)} + \| u D \zeta_n\|_{L^p(\mathbb{R}^N, \mathbb{R}^N)}$$

$$\leq \| (\zeta_n - 1) D u\|_{L^p(\mathbb{R}^N, \mathbb{R}^N)} + \| u\|_{L^p(\mathbb{R}^N, \mathbb{R}^N)} \| D \zeta_n\|_{L^N(\mathbb{R}^N)}$$

(65)

by Hölder’s inequality.

Since $D u \in L^p(\mathbb{R}^N, \mathbb{R}^N)$ and $\zeta_n(z) \to 1$ for all $z \in \mathbb{R}^N$ as $n \to \infty$, from the Lebesgue Dominated Convergence Theorem we have

$$\| (\zeta_n - 1) D u\|_{L^p(\mathbb{R}^N, \mathbb{R}^N)} \to 0 \text{ as } n \to \infty.$$  

(66)

Moreover, we have

$$\| D \zeta_n\|_{L^N(\mathbb{R}_n)} = \| D \zeta_n\|_{L^N(\mathbb{R}^N, \mathbb{R}^N)} = \| D \zeta\|_{L^N(\mathbb{R}^N, \mathbb{R}^N)} < \infty$$

(67)

and

$$\| u\|_{L^{p^*(\mathbb{R}_n)}} \to 0 \text{ as } n \to \infty, \text{ since } u \in L^{p^*}(\mathbb{R}^N).$$

(68)

From (67) and (68) it follows that

$$\| u\|_{L^{p^*(\mathbb{R}_n)}} \| D \zeta_n\|_{L^N(\mathbb{R}_n)} \to 0 \text{ as } n \to \infty.$$  

(69)

We return to (65), pass to limit as $n \to \infty$ and use (66) and (69), obtaining

$$D u_n \to D u \text{ in } L^p(\mathbb{R}^N, \mathbb{R}^N).$$

(70)

We know that $\beta^- u^p \in L^1(\mathbb{R}^N)$ (recall that $u \in V$), while from the proof of Proposition 1 we have $\beta^+ u^p \in L^1(\mathbb{R}^N)$. Moreover, $0 \leq \beta^\pm u^p_n \leq \beta^\pm u^p$. So, again via the Lebesgue Dominated Convergence Theorem we have

$$\int_{\mathbb{R}^N} \beta^\pm u^p_n \, dz \to \int_{\mathbb{R}^N} \beta^\pm u^p \, dz \text{ as } n \to \infty,$$

and so

$$\int_{\mathbb{R}^N} \beta u^p_n \, dz \to \int_{\mathbb{R}^N} \beta u^p \, dz \text{ as } n \to \infty.$$  

(71)

From (63), (70) and (71) it follows that we can find $n_0 \geq 1$ such that

$$\int_{\mathbb{R}^N} \| D u_n\|^p \, dz < \lambda \int_{\mathbb{R}^N} \beta u^p_n \, dz \text{ for all } n \geq n_0.$$  

(72)

But from the definition of $\hat{\lambda}^*$, we have

$$\int_{\mathbb{R}^N} \| D u_n\|^p \, dz \geq \hat{\lambda}^* \int_{\mathbb{R}^N} \beta u^p_n \, dz \geq \lambda \int_{\mathbb{R}^N} \beta u^p_n \, dz,$$

(73)
since $0 < \lambda \leq \hat{\lambda}^*$ and $\int_{\mathbb{R}^N} \beta u^p dz > 0$ by (72). Comparing (72) and (73) we reach a contradiction and this proves that for $\lambda \in (0, \hat{\lambda}^*)$ there is no positive solution to problem $(P_\lambda)$.

In conclusion, summarizing the situation for problem $(P_\lambda)$, by Propositions 2 and 6, we can state the following bifurcation theorem:

**Theorem 7.** If hypotheses $H(\beta)$, $H(\beta)_2$ and $H(f)$ hold, then there exists $\hat{\lambda}^* > 0$ such that

(a) for every $\lambda > \hat{\lambda}^*$ problem $(P_\lambda)$ has at least one positive solution $u \in C^1(\mathbb{R}^N) \cap V$ vanishing at infinity;

(b) for all $\lambda \in (0, \hat{\lambda}^*)$ problem $(P_\lambda)$ has no positive solutions.

**Remark 3.** We emphasize the fact that the quantities introduced in (1) and (2) provide a complete variational description of the bifurcation point $\hat{\lambda}^*$.

In the semilinear case (i.e. $p = 2$) we can improve this theorem; so from now on we consider the following problem:

$$(S_\lambda) \begin{cases} -\Delta u(z) = \lambda(\beta(z)u - f(z, u(z))) & \text{in } \Omega, \\
 u(z) \to 0 & \text{as } \|z\| \to \infty, \ u > 0, \lambda > 0 
\end{cases}$$

and $N > 2$. By strengthening hypotheses $H(f)$ we will prove uniqueness of the positive solution for problem $(S_\lambda)$ when $\lambda > \hat{\lambda}^*$. Indeed, the assumptions on the perturbation $f$ are now the following:

$H(f)'$: $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.e. $z \in \mathbb{R}^N$ $f(z,0) = 0$ and $f(z, x) > 0$ for all $x > 0$, hypotheses $H(f)'(i), (ii), (iii)$ are the same as the corresponding hypotheses $H(f)(i), (ii), (iii)$ with $p = 2$ and (iv) for a.e. $z \in \mathbb{R}^N$ the function $x \mapsto \frac{f(z,x)}{x}$ is strictly increasing in $(0, \infty)$.

**Proposition 8.** If hypotheses $H(\beta)$, $H(\beta)_2$ and $H(f)'$ hold and $\lambda > \hat{\lambda}^*$, then the positive solution of $(S_\lambda)$ is unique.

**Proof.** The existence of a positive solution in $C^1(\mathbb{R}^N) \cap V$ follows from Proposition 2. Now, suppose $u, v$ are two solutions of $(S_\lambda)$. As before, using Lemma 4.3 of [9] and the truncation technique introduced in the proof of Proposition 2, we see that we may assume $u \leq v$. Then, for any $r > 0$, using Green’s identity, we have

$$\int_{B_r} (Du, Dv)_{\mathbb{R}^N} dz - \int_{\partial B_r} \frac{\partial u}{\partial n} v d\sigma = \lambda \int_{B_r} [\beta u - f(z, u)]v dz \quad (74)$$

and

$$- \int_{B_r} (Dv, Du)_{\mathbb{R}^N} dz + \int_{\partial B_r} \frac{\partial v}{\partial n} u d\sigma = -\lambda \int_{B_r} [\beta v - f(z, v)]u dz. \quad (75)$$
Adding (74) and (75) we obtain
\[ \int_{\partial B_r} \left( \frac{\partial v}{\partial n} u - \frac{\partial u}{\partial n} v \right) d\sigma = \lambda \int_{B_r} \left( \frac{f(z, v)}{v} - \frac{f(z, u)}{u} \right) uv \, dz. \] (76)

If we show that
\[ \lim_{r \to \infty} \int_{\partial B_r} \left( \frac{\partial v}{\partial n} u - \frac{\partial u}{\partial n} v \right) d\sigma = 0, \] (77)
then, if in (76) we pass to the limit as \( r \to \infty \) and use (77), we have
\[ \lambda \int_{\mathbb{R}^N} \left( \frac{f(z, v)}{v} - \frac{f(z, u)}{u} \right) uv \, dz = 0, \]
which implies \( u \equiv v \) by \( H(f)'(iv) \).

Thus, we conclude proving (77), which does not hold \textit{a priori}, since \( u, v \) are functions in \( V \) and not in \( W^{1,p}(\mathbb{R}^N) \). However, if in (74) and (75) we pass to the limit, we get
\[ \int_{\mathbb{R}^N} (Du, Dv)_{\mathbb{R}^N} \, dz - \lambda \int_{\mathbb{R}^N} [\beta u - f(z, u)]v \, dz = \lim_{r \to \infty} \int_{\partial B_r} \frac{\partial u}{\partial n} v \, d\sigma \]
and
\[ \int_{\mathbb{R}^N} (Dv, Du)_{\mathbb{R}^N} \, dz - \lambda \int_{\mathbb{R}^N} [\beta v - f(z, v)]u \, dz = \lim_{r \to \infty} \int_{\partial B_r} \frac{\partial v}{\partial n} u \, d\sigma. \]

Since both \( u \) and \( v \) solve problem \((S_\lambda)\), we get that both limits in the previous two identities equal 0, and so (77) holds. \( \square \)

Therefore, for problem \((S_\lambda)\) we can state the following bifurcation theorem:

**Theorem 9.** If hypotheses \( H(\beta), H(\beta)_2 \) and \( H(f)' \) hold, then there exists \( \lambda^* > 0 \) such that

(a) for every \( \lambda > \lambda^* \) problem \((S_\lambda)\) has a unique positive solution \( u \in C^1(\mathbb{R}^N) \cap V \) vanishing at infinity;

(b) for all \( \lambda \in (0, \lambda^*] \) problem \((S_\lambda)\) has no positive solutions.

**References**


Diffusive logistic equations in $\mathbb{R}^N$


