Graph Drawing Beyond Planarity: Some Results and Open Problems

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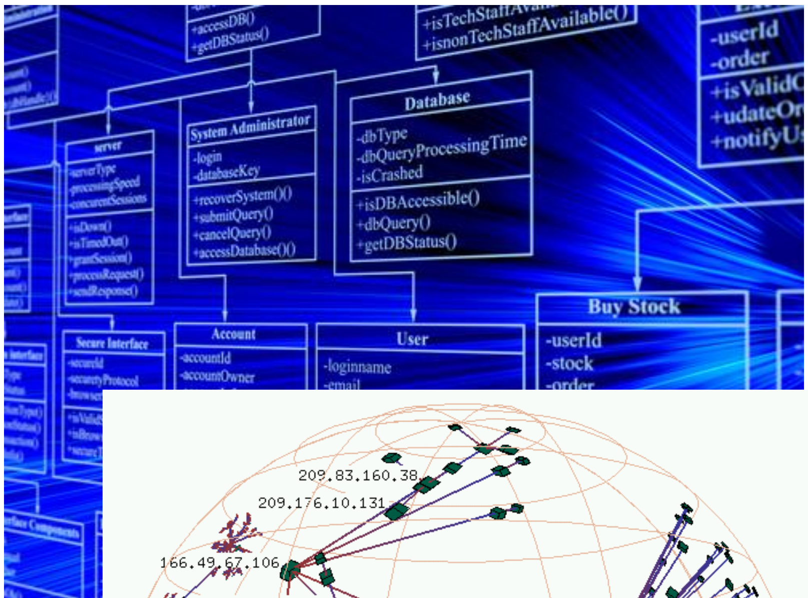
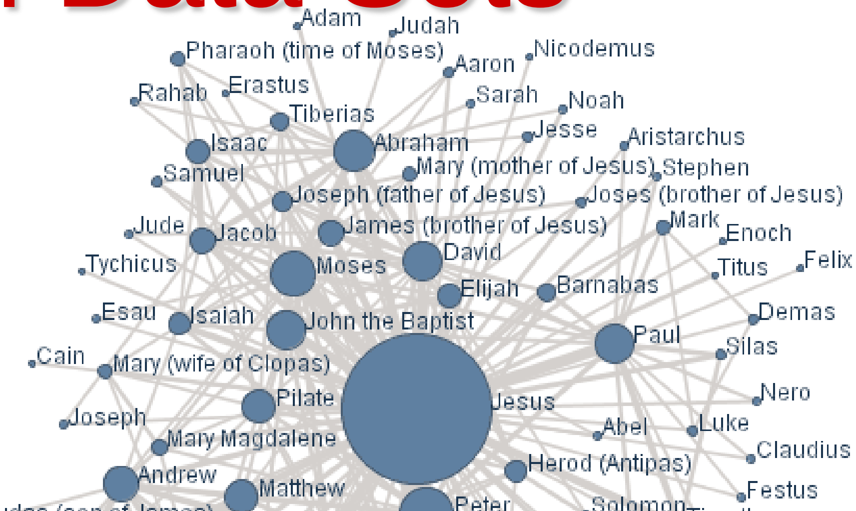
Outline

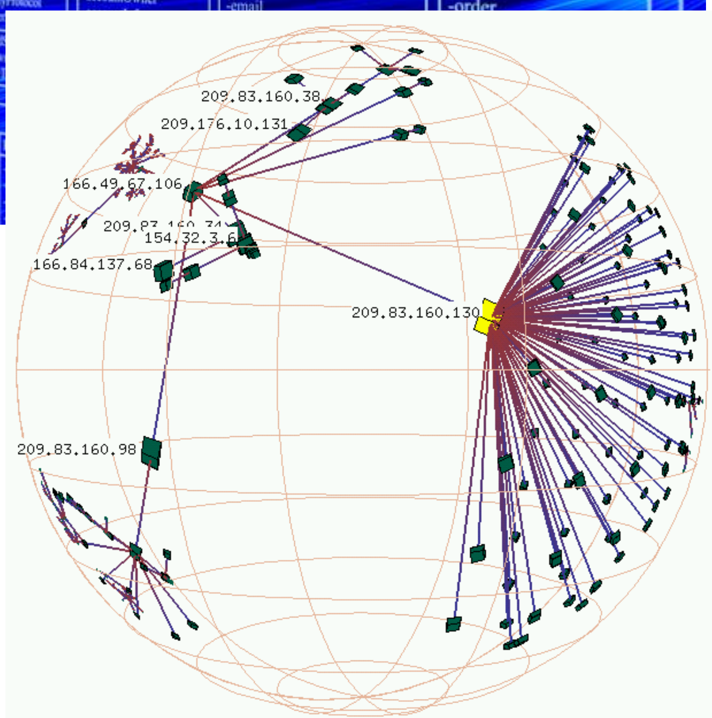
- Graph Drawing (GD) beyond planarity
- Combinatorial relationships
- Optimization trade-offs and algorithms
- Open problems

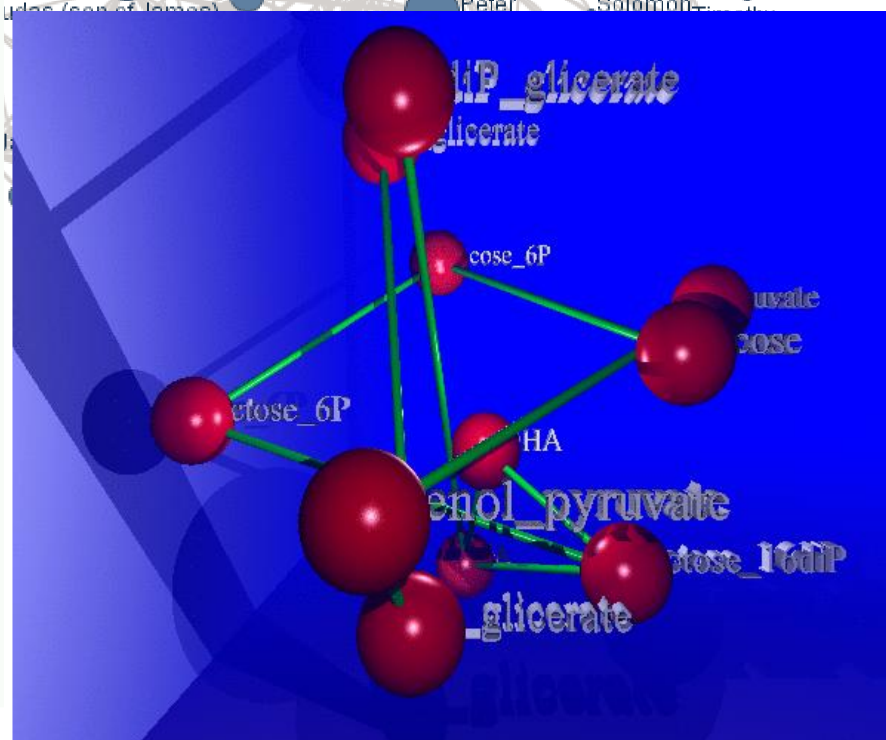
GD beyond planarity

Relational Data Sets









Graph Drawing

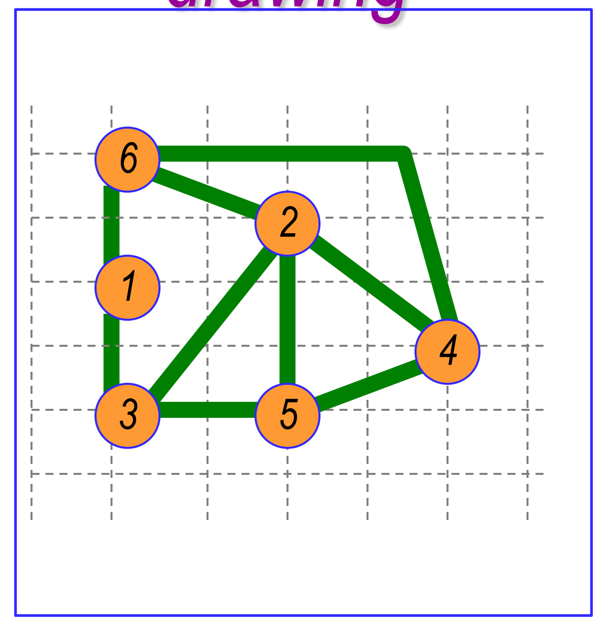
G = (V, E)
V = {1, 2, 3, 4, 5, 6}
E = { (1,3) (1,6) (2,3) (2,5) (2,4) (2,6) (3,5) (4,5) (4,6) }

in →

Graph Drawing System

out →

drawing



the drawing must be readable

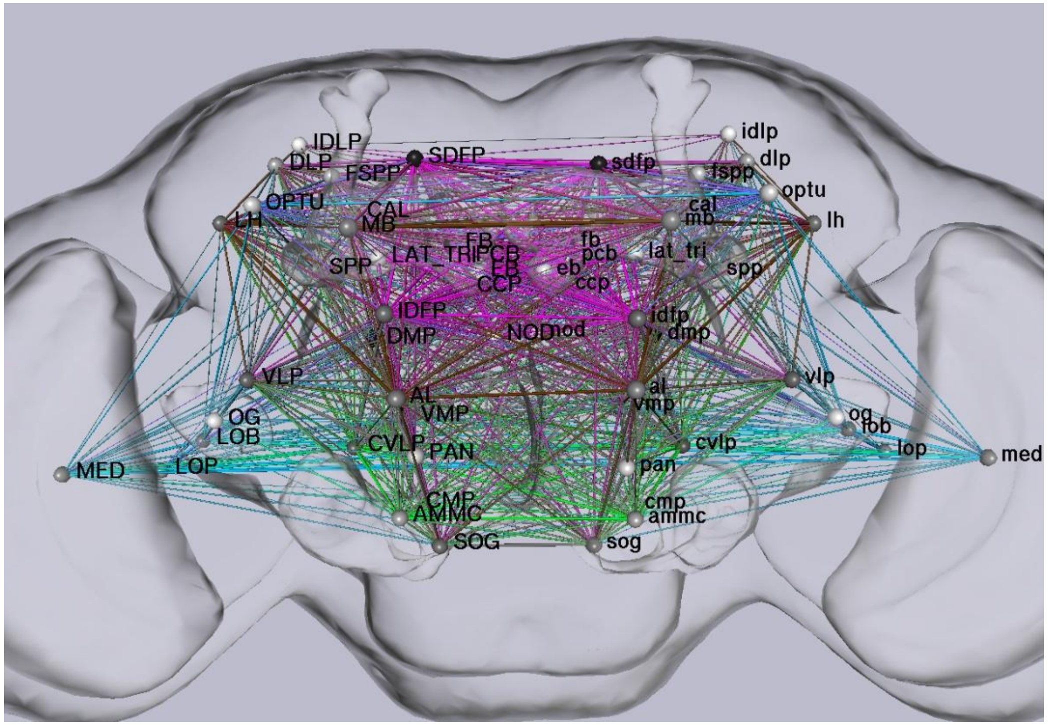
Readability and Crossings

edge crossings significantly affect the readability (see, e.g., *Sugiyama et al., Warshall, North et al., Batini et al., mid 80s*) - confirmed by cognitive experimental studies (*Purchase et al., 2000-2002*)

rich body of graph drawing techniques assume the input is a planar (planarized) graph and avoid edge crossings as much as possible

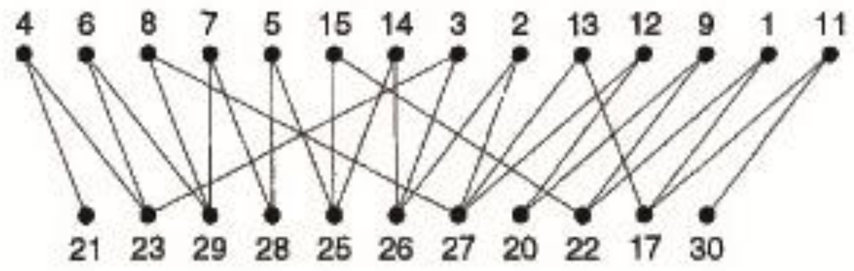
The planarization handicap

for dense enough or constrained enough drawings, many edge crossing are unavoidable



FlyCircuit Database, NTHU

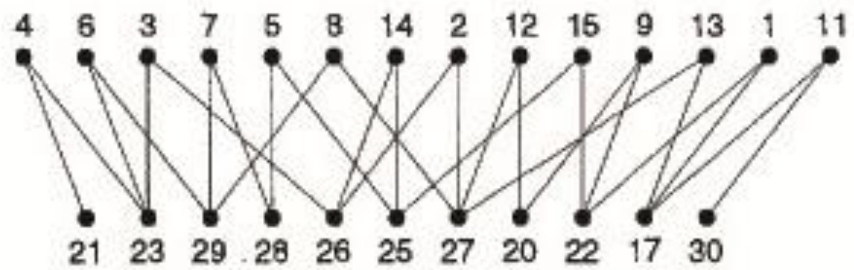
Mutzel's intuition about crossings



(a)

34 crossings:

minimum "skewness"
(number of edges whose deletion makes it planar)



(b)

24 crossings:

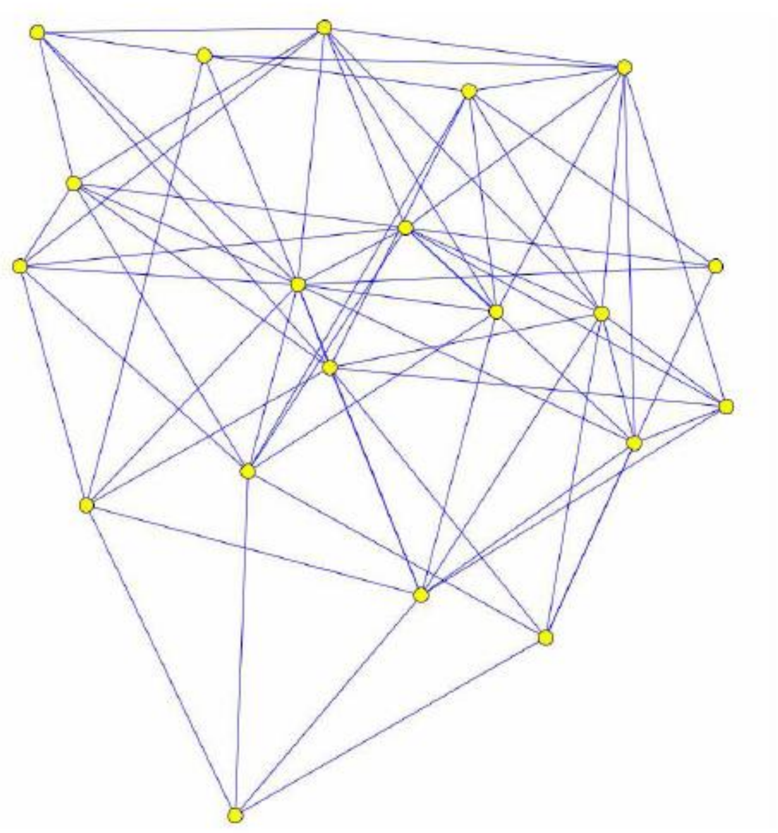
minimum number of crossings

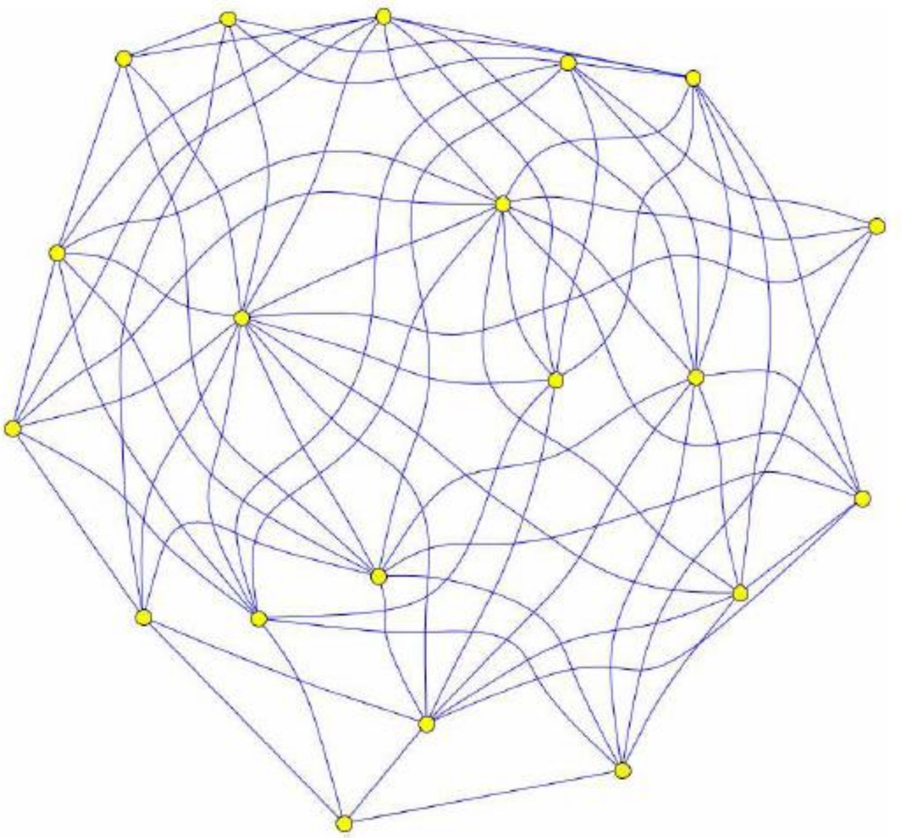
Experiments of Eades,Hong,Huang

Observations from eye tracking

- *No crossings*: eye movements were smooth and fast.
- *Large crossing angle*: eye movements were smooth, but a little slower.
- *Small crossing angle*: eye movements were very slow and no longer smooth (back-and-forth movements at crossing points).

Example





[Didimo, L., Romeo, "A Graph Drawing Application to Web Site Traffic Analysis", JGAA 2011]

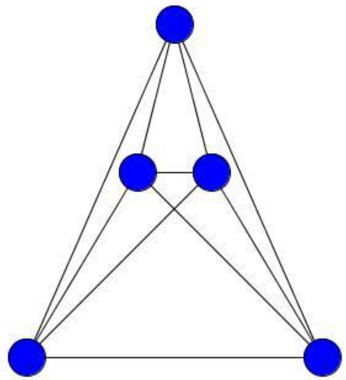
Beyond planarity

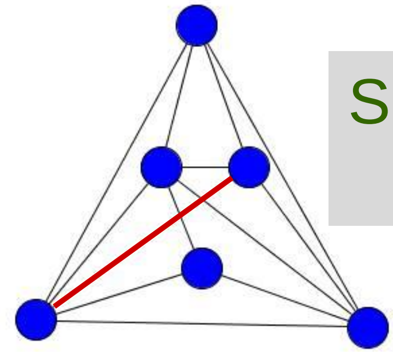
the visual complexity not only depends on the number of crossings but also on the type of crossings

challenge: compute drawings where some “bad” crossing configurations are forbidden (minimized)

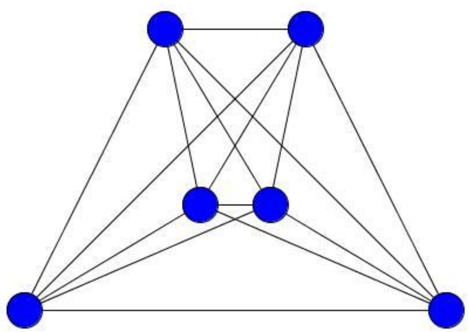
Drawings with forbidden crossing configurations

RAC

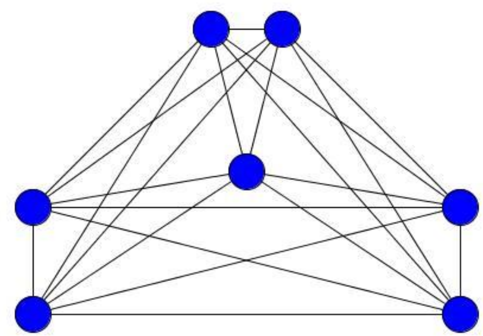




SKEWNESS-h
(h=1)

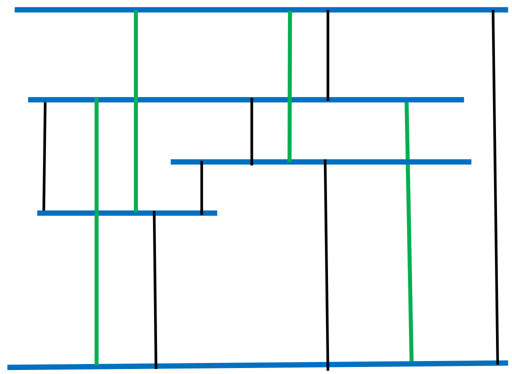


h-PLANAR
(h=3)

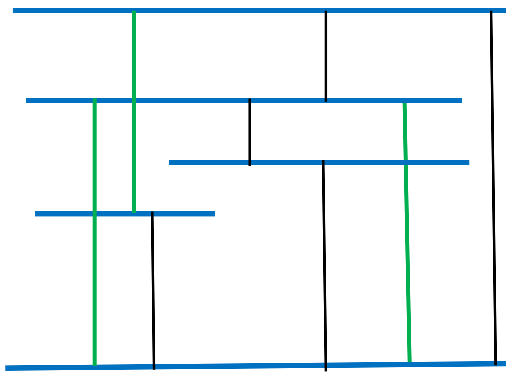


h-QUASI-PLANAR
(h=3)

Drawings with forbidden crossing configurations



Strong 1-visibility drawing



Weak 1-visibility drawing

Most explored research directions

Turán-type: find upper bounds on the edge density

Recognition: how hard is it to test whether a graph admits a drawing with a forbidden configuration?

Fáry-type: given a drawing (with jordan arcs), is there a straight-line drawing that preserves the given topology?

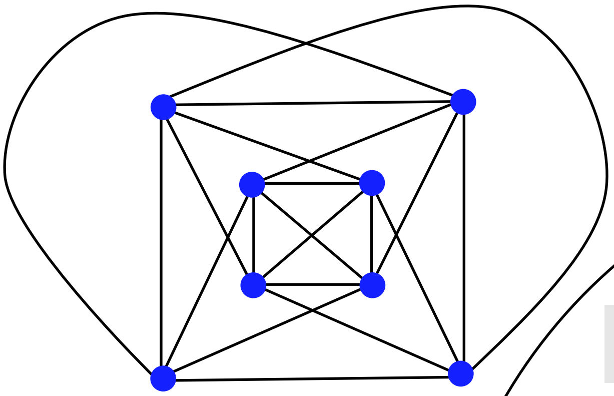
New research directions

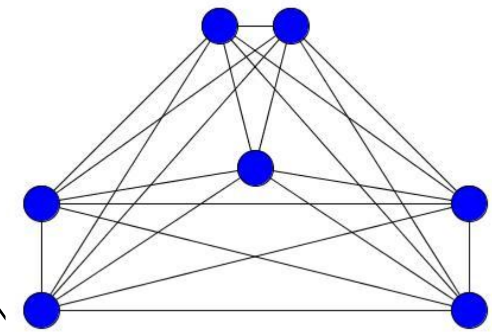
study the combinatorial relationships between different families of nearly planar graphs

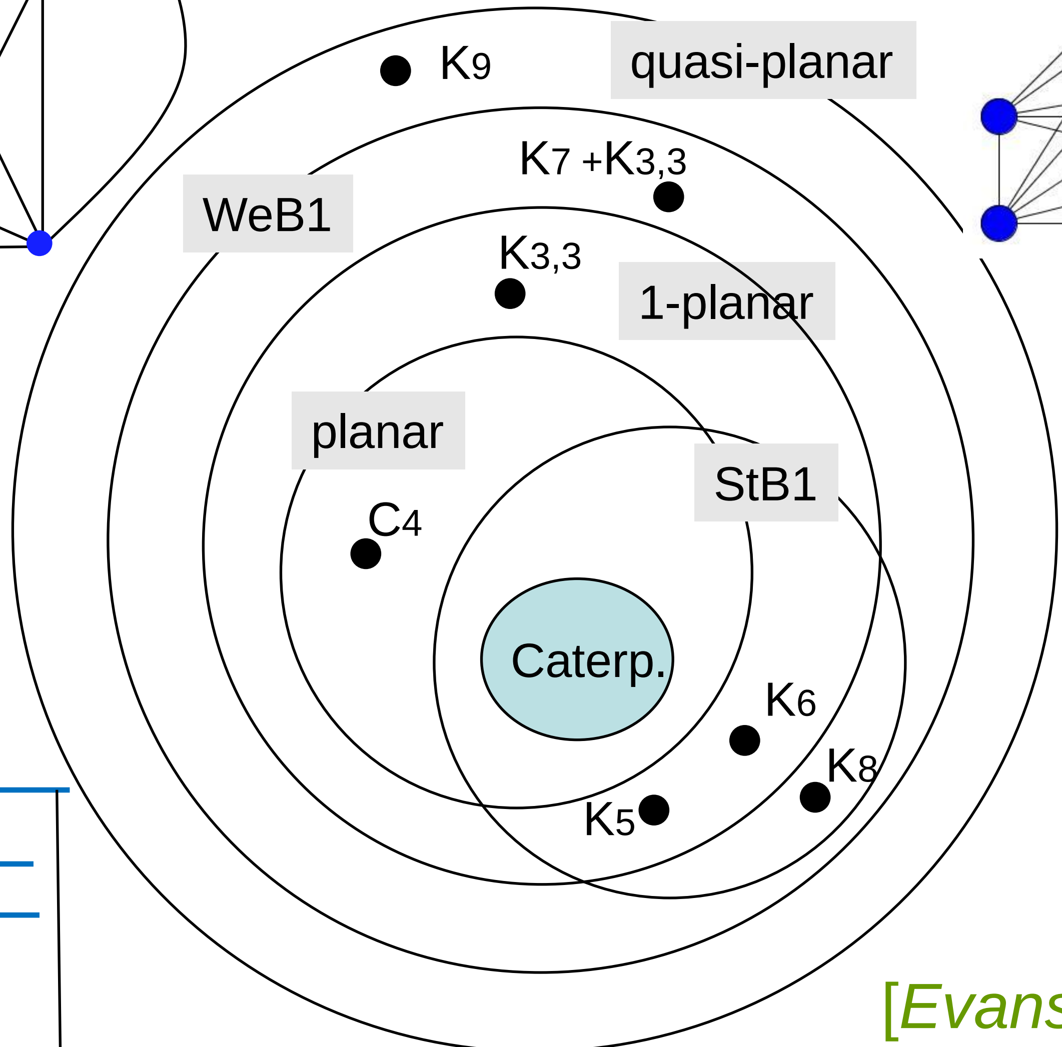
study trade-offs between crossing complexity and other aesthetic criteria

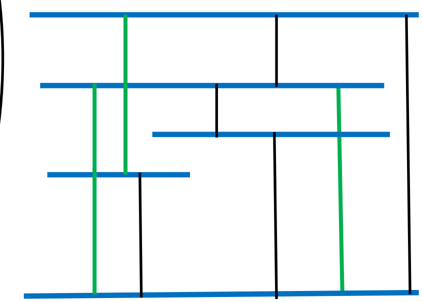
Combinatorial relationships between nearly planar graphs

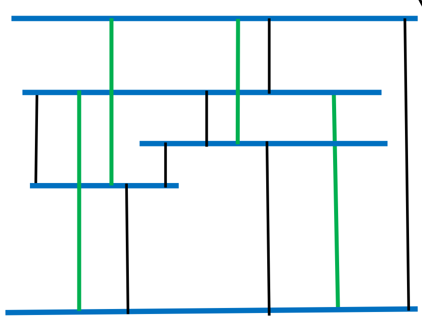
1-planarity, quasi-planarity and 1-visibility





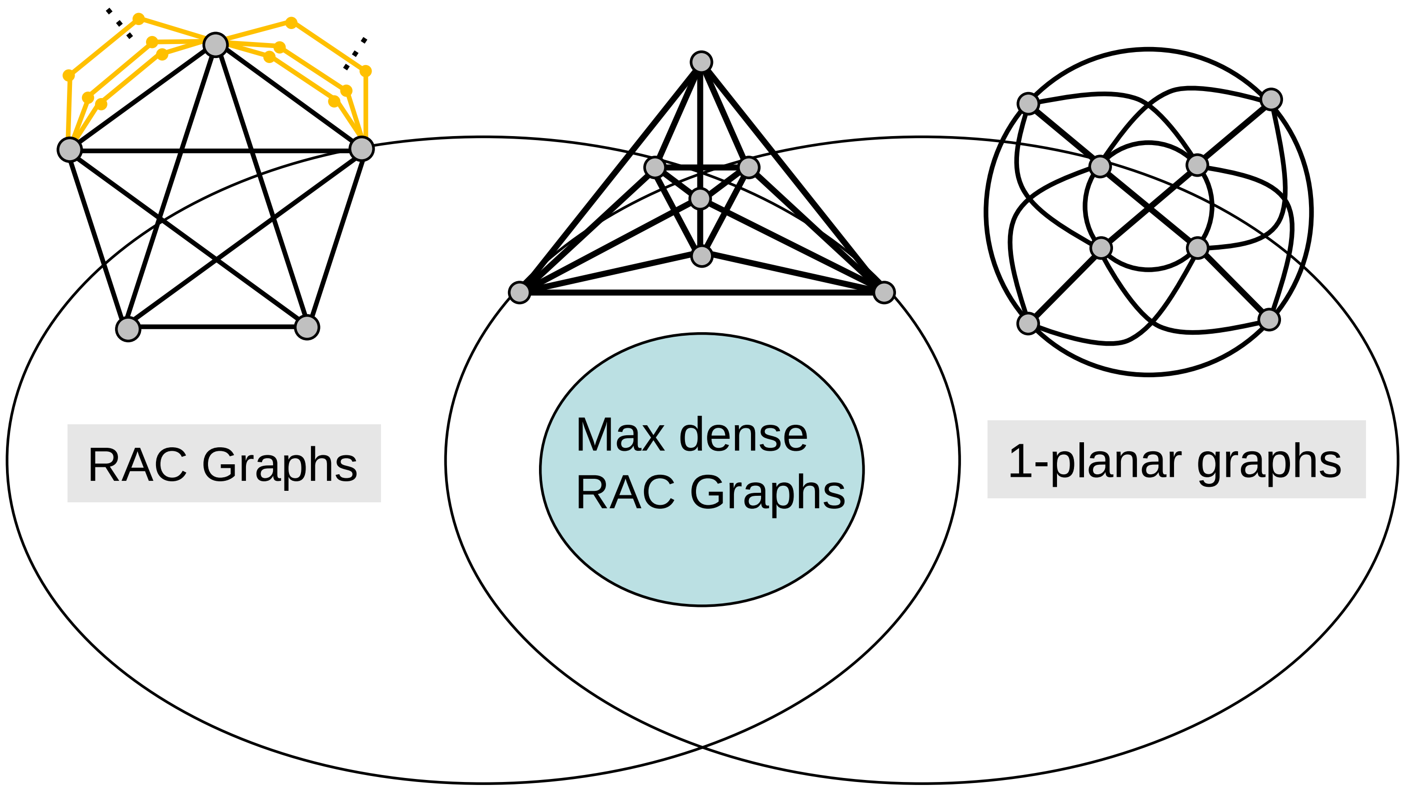






[*Evans et al.*, 2014]

RAC and 1-planarity



[*Eades,L.*, 2013]

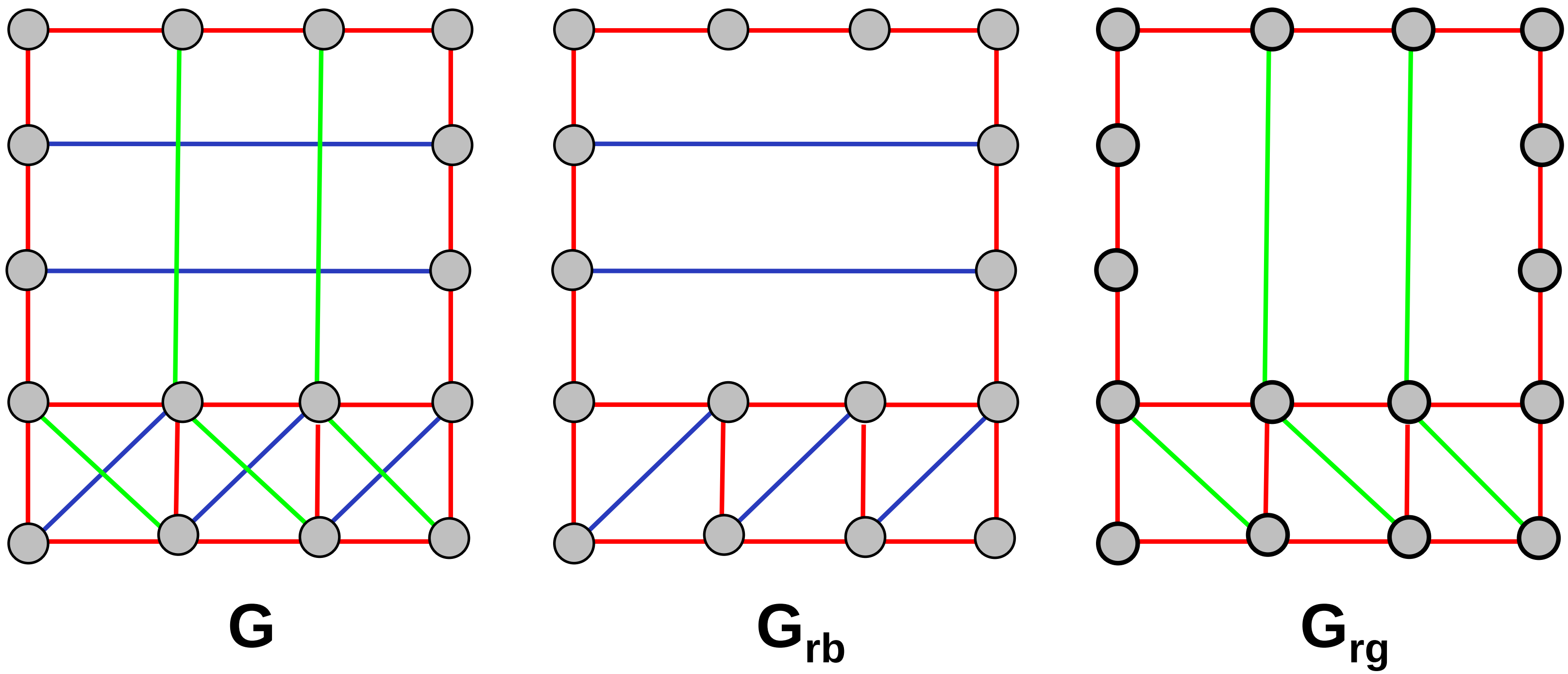
RAC graphs and 1-planarity

Theorem

A maximally dense RAC graph is 1-planar. Also, for every integer **i** such that **$i \geq 0$** there exists a 1-planar graph with **$n=8+4i$** vertices and **$4n-10$** edges that is not a RAC graph. Finally, for every integer **$n > 85$**, there exists a RAC graph with **n** vertices that is not 1-planar. [*Eades,L.,* 2013]

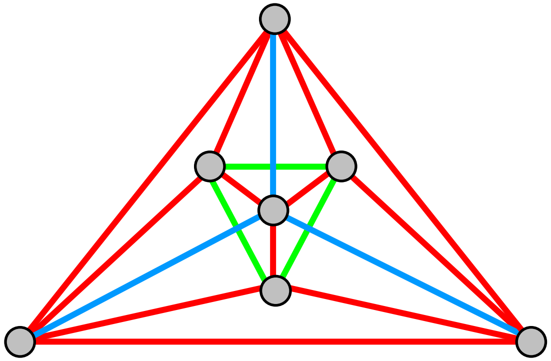
Some details about the proof

Preliminaries: edge coloring

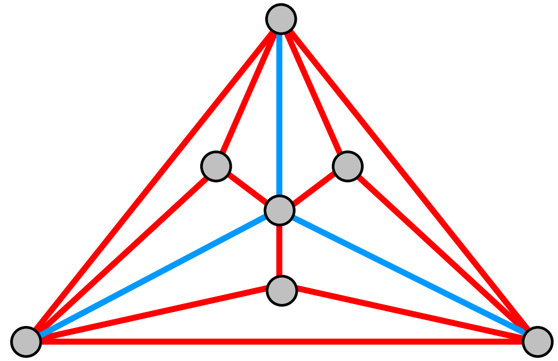


- **red** edges do not cross
- each **green** edge crosses with a **blue** edge
  - red-blue (embedded planar) graph = **red** + **blue** edges
  - red-green (embedded planar) graph = **red** + **green** edges

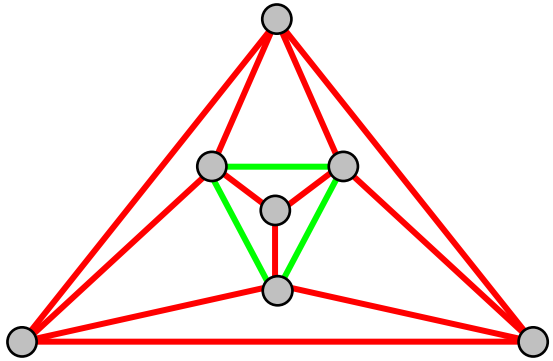
Preliminaries: $\mathbf{G_{rb}}$ and $\mathbf{G_{rg}}$ in a maximally dense RAC graph



G



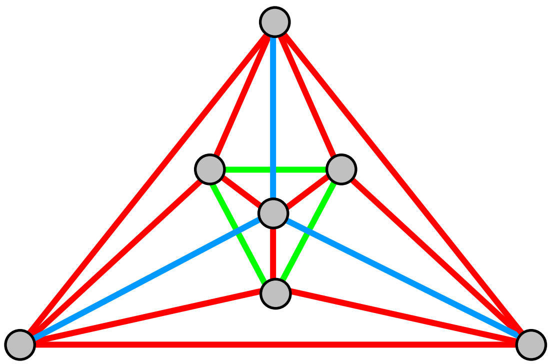
G_{rb}



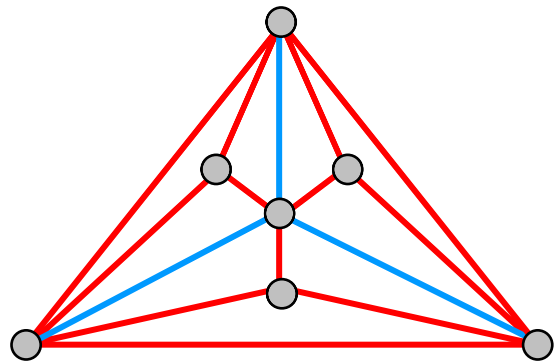
G_{rg}

each internal face of G_{rb} (G_{rg}) has at least two red edges [*Didimo, Eades, L.*, 2011]

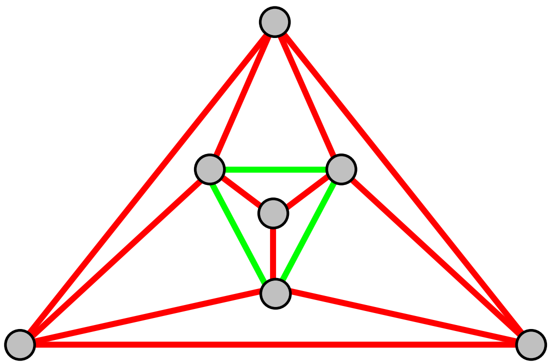
Preliminaries: $\mathbf{G_{rb}}$ and $\mathbf{G_{rg}}$ in a maximal RAC graph



G



G_{rb}



G_{rg}

Notation:

- $\mathbf{m_r}$, $\mathbf{m_b}$, $\mathbf{m_g}$ = number of red, blue, and green edges
- $\mathbf{f_{rb}}$ = number of faces of the red-blue graph $\mathbf{G}_{rb}$

Assumption:

- $\mathbf{m_g} \leq \mathbf{m_b}$

Maximally dense RAC graphs are 1-planar

Approach:

• suppose we can show that G_{rb} and G_{rg} are both maximal planar graphs; then:

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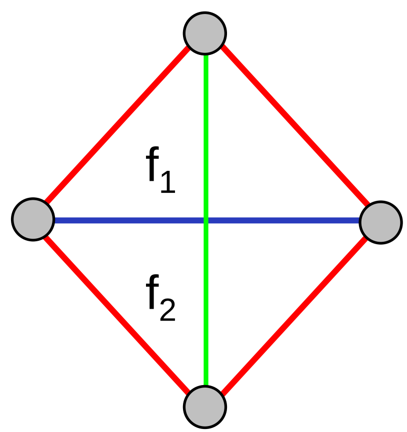
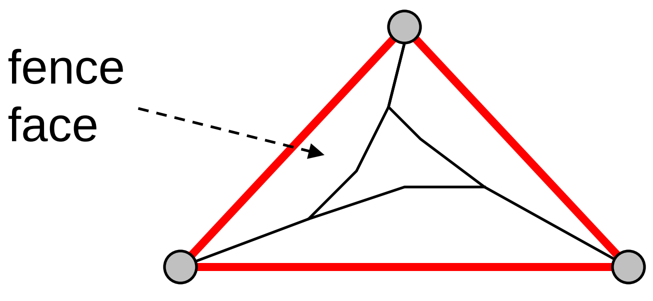


G_{rb} and G_{rg} are maximal planar graphs (1)

• the following is proven first:

Claim 1: the external face of G_{rb} and G_{rg} is a 3-cycle

• then, we consider the internal faces of G_{rb} that share at least one edge with the external face (fence faces)



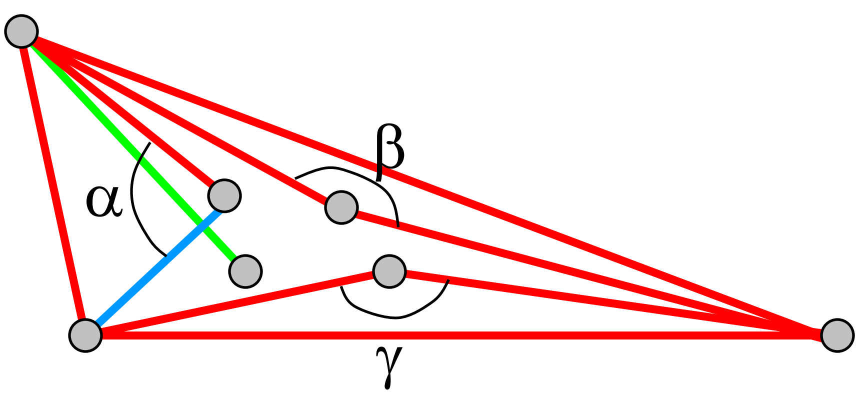
there are at least 1 and at most 3 fence faces

G_{rb} and G_{rg} are maximal planar graphs (2)

- ...and prove the following

Claim 2: If G is maximal, G_{rb} has three fence faces and each fence face is a 3-cycle

- obs: at least two fence faces consist of red edges



$\alpha + \beta + \gamma \geq 360°$

$\alpha < 90°$

$\Rightarrow \beta \geq 90°$ and $\gamma \geq 90°$

G_{rb} and G_{rb} are maximal planar graphs (3)

since: (1) each internal face of G_{rb} has at least 2 red edges; (2) the external face of G_{rb} is a red 3-cycle; (3) at least two fence faces are red 3-cycles $\Rightarrow$ $2m_r \geq 2(f_{rb} - 3) + 3 + 3 + 3$

since m_r and f_{rb} are integers, we obtain $m_r \geq f_{rb} + 2$

By Euler's formula for planar graphs $\Rightarrow$ $m_r + m_b \leq n + f_{rb} - 2$

$m_b \leq n - 4$

G_{rb} and G_{rb} are maximal planar graphs (4)

$m_b \leq n - 4$

G is a maximally dense RAC graph $\Rightarrow$ $m_b + m_r + m_g = 4n - 10$

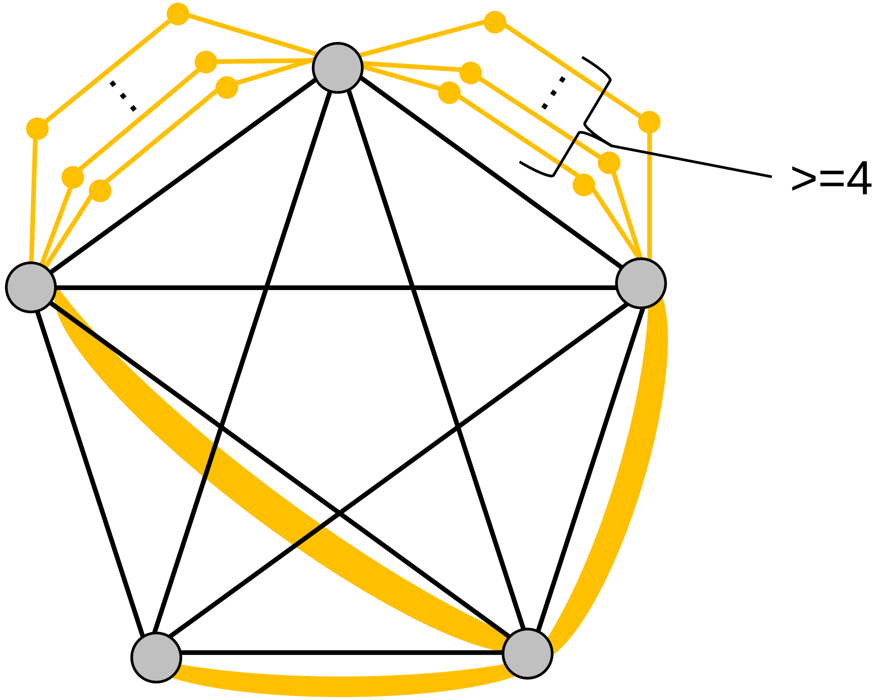
$m_r + m_g \geq 3n-6$

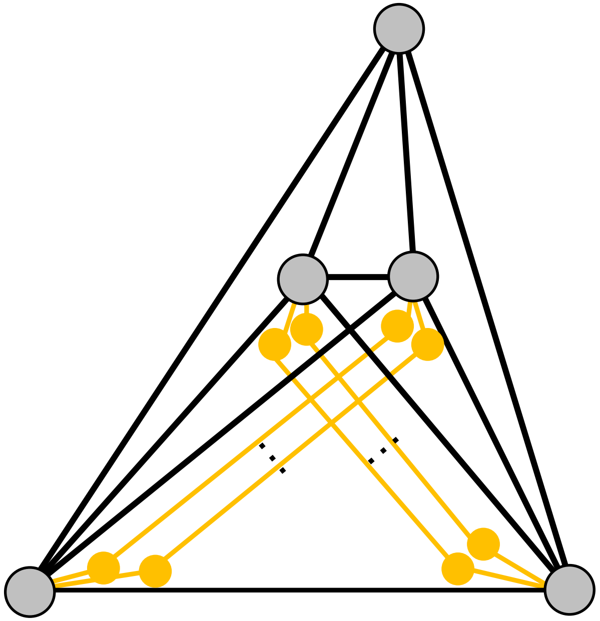
since by assumption $m_g \leq m_b$ and since both G_{rg} and G_{rb} are planar $\Rightarrow$ G_{rg} and G_{rb} are both maximal planar graphs

Therefore a maximal RAC graph is 1-planar

RAC Graphs that are not 1-planar

There exists a graph **G** with less than **4n-10** such that **G** is a RAC graph but is not 1-planar





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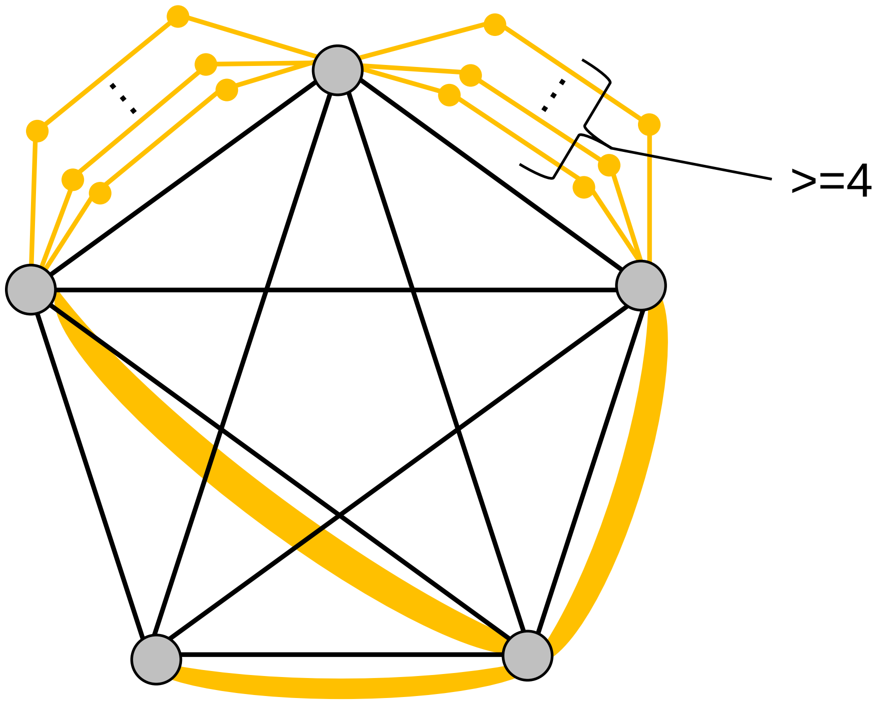
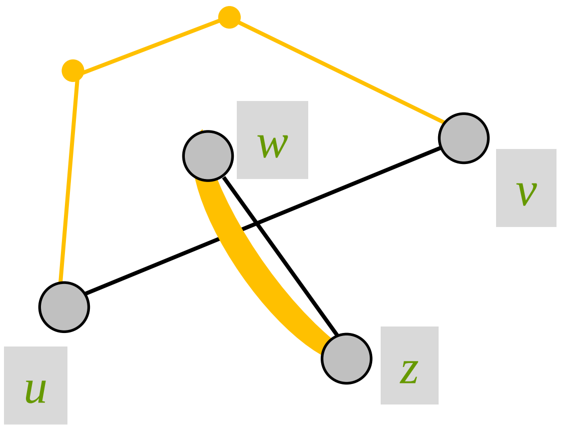
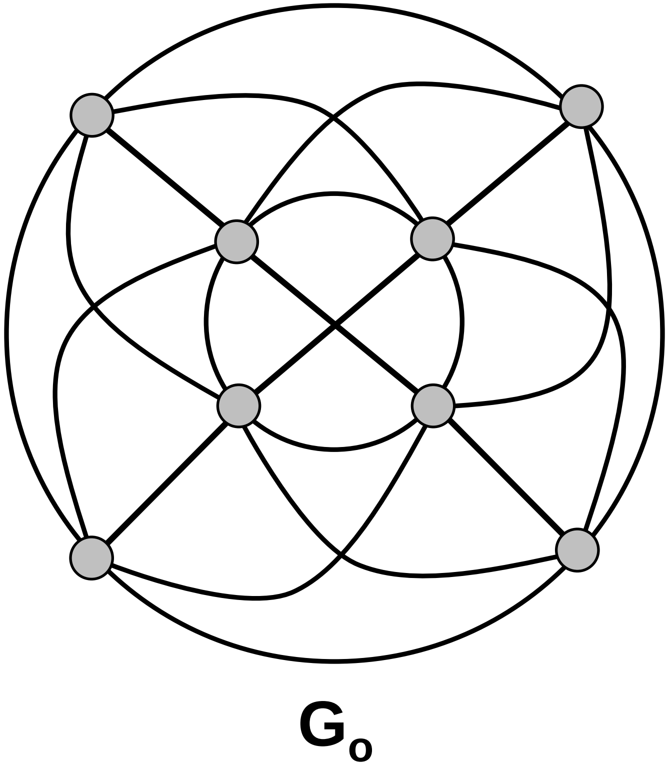
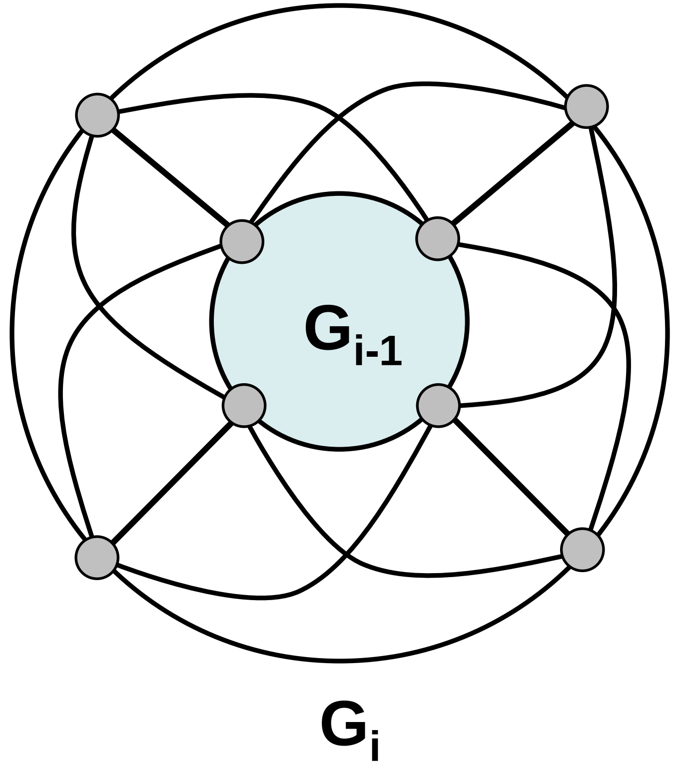




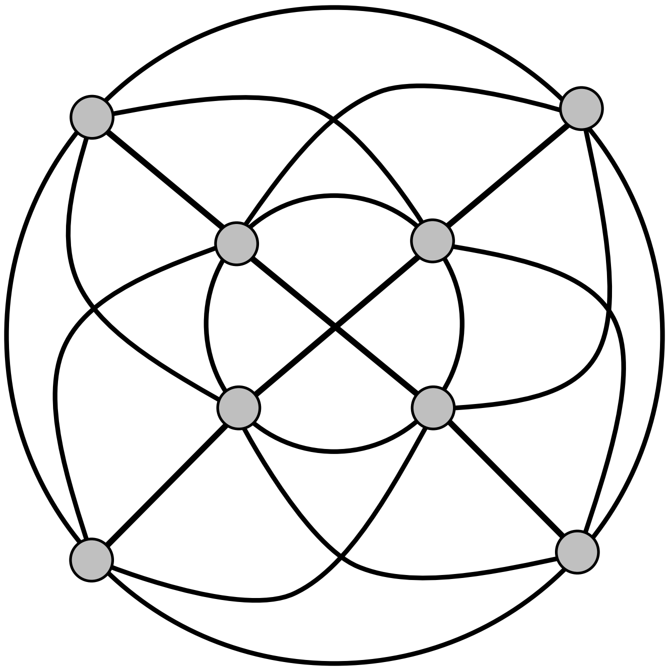
Not all 1-planar graphs with 4n-10 edges are maximal RAC



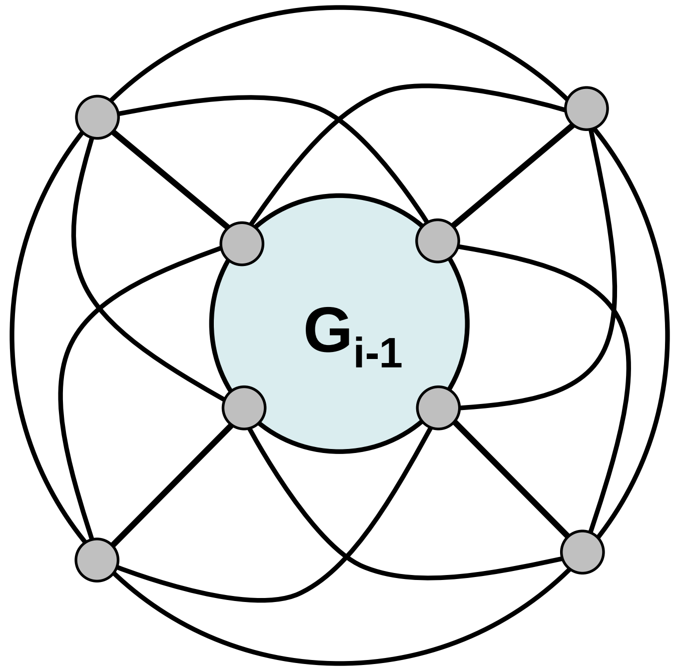


G_o has **n=8** vertices and **4n-10=22** edges; for **i≥0**, **G_i** has **n=8+4i** vertices and **4n-10** edges

Not all 1-planar graphs with 4n-10 edges are maximal RAC



G_o

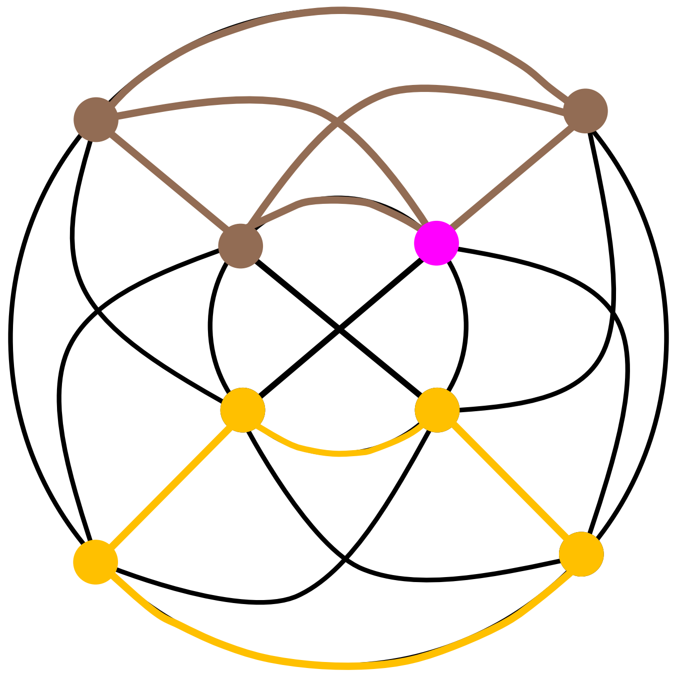


G_i

G_o has $n=8$ vertices and $4n-10=22$ edges; for $i \geq 0$, G_i has $n=8+4i$ vertices and $4n-10$ edges; they are 1-planar graphs

we show that G_i cannot be realized as a RAC graph (by induction on i)

G_o is not RAC realizable



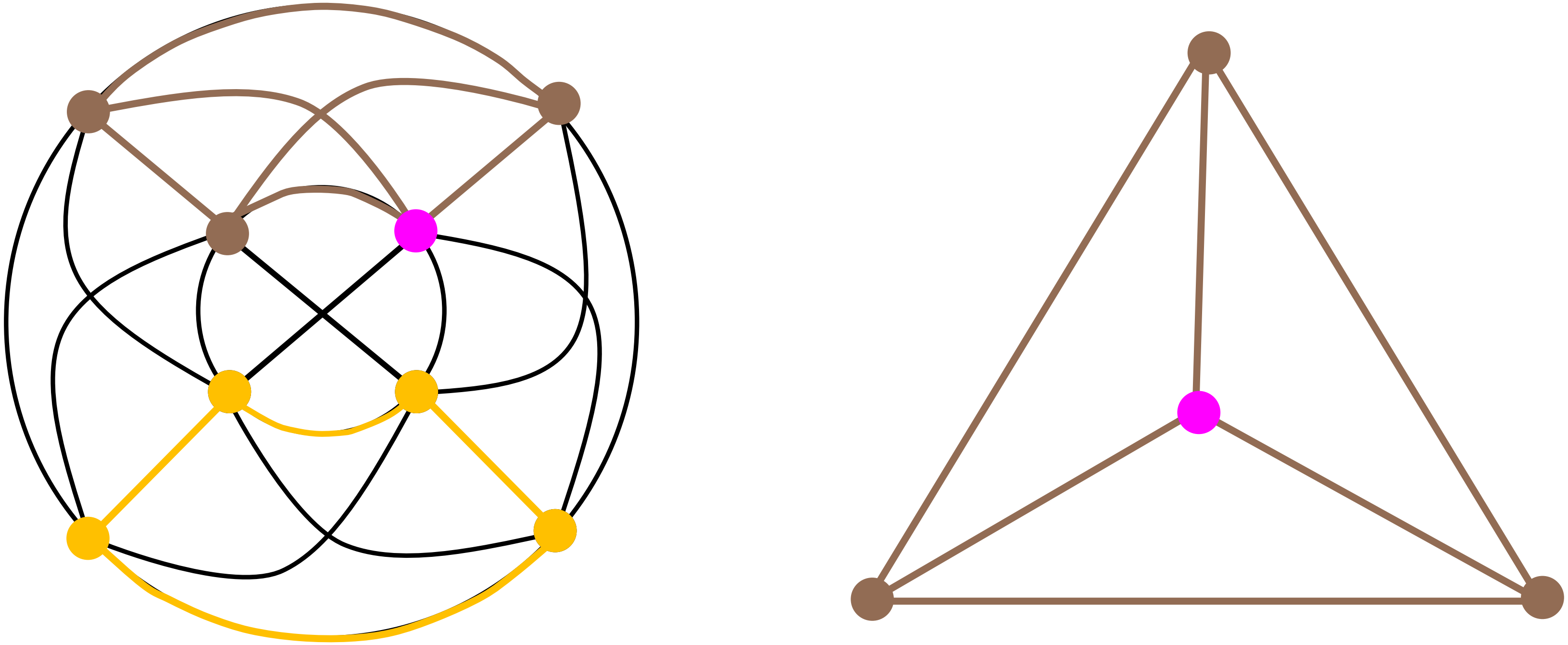
every vertex has degree 5 or 6.

for every 3-cycle there is a K4

for every K4, there is a 4-cycle through the other vertices

G_o is not RAC realizable

if G_o were RAC realizable, the external face of the realization would be a 3-cycle



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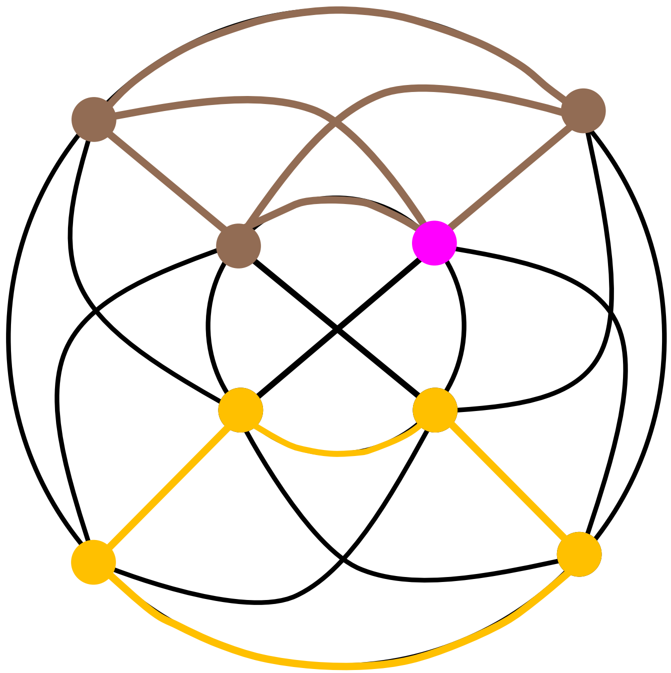

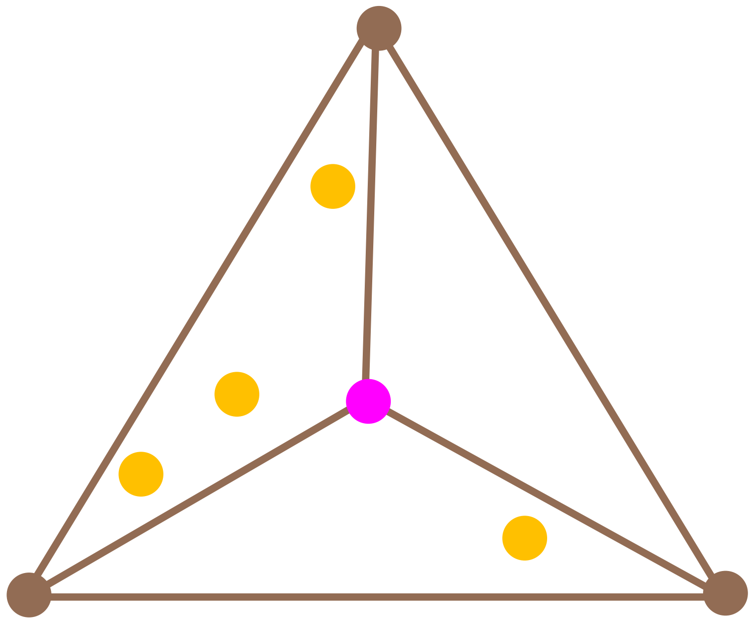

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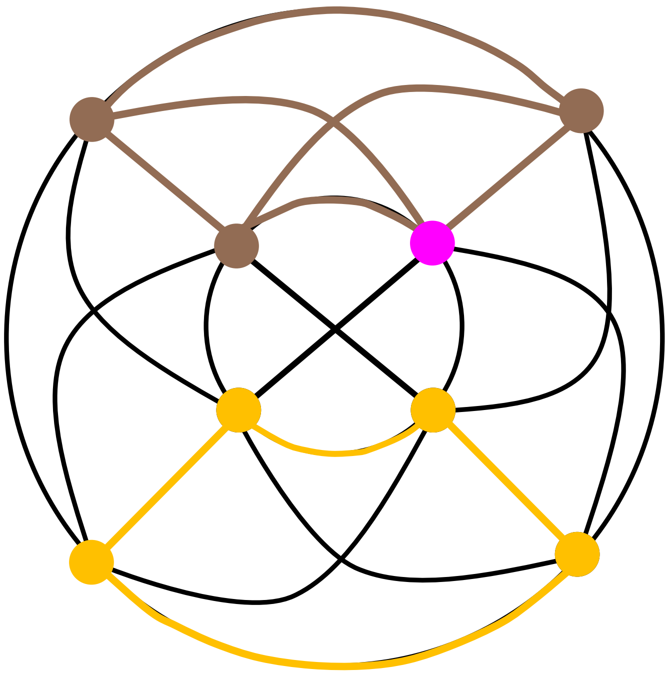

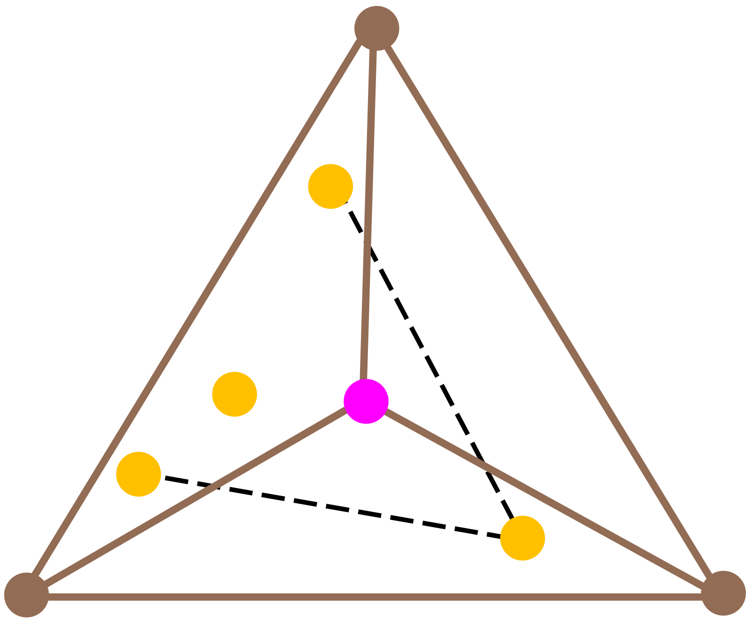

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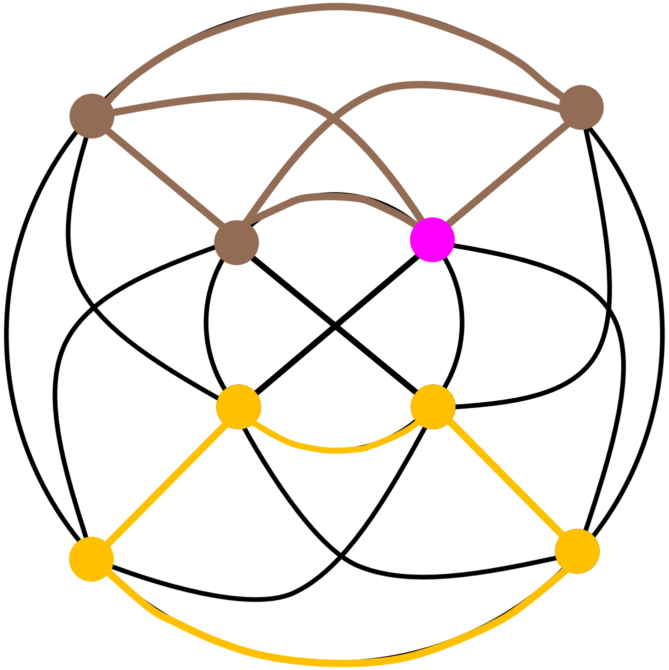

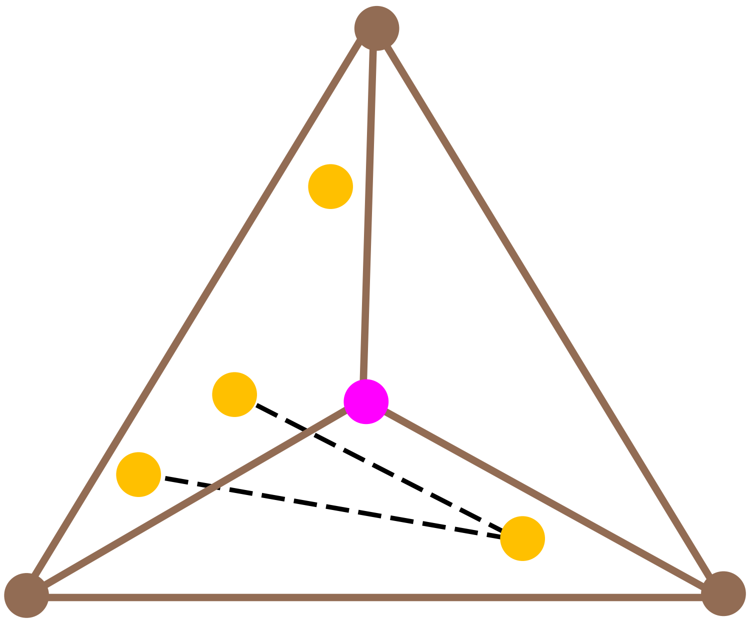

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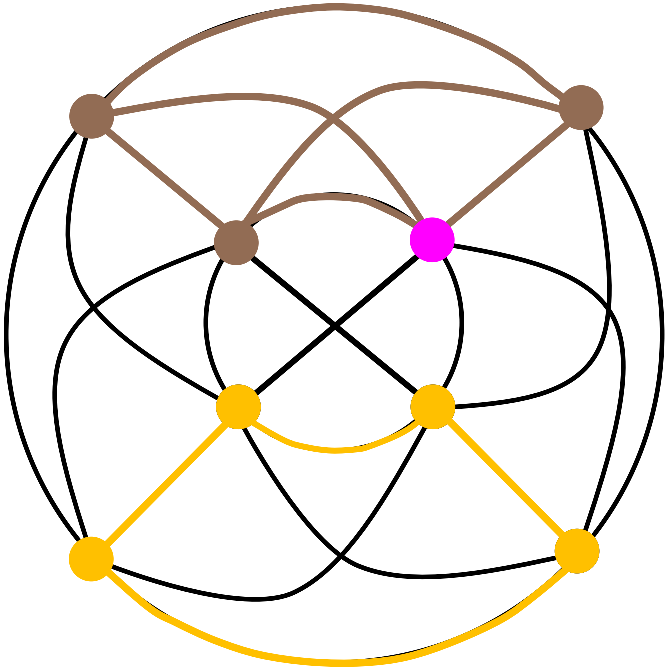

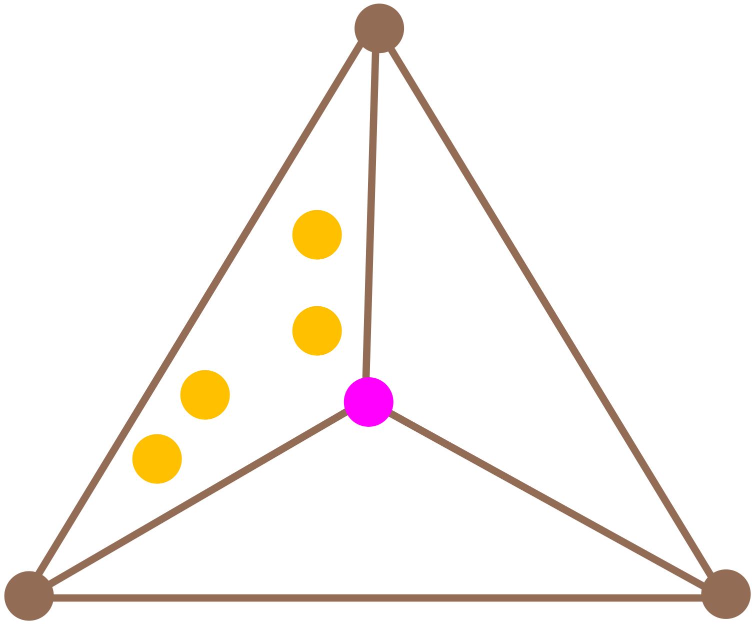

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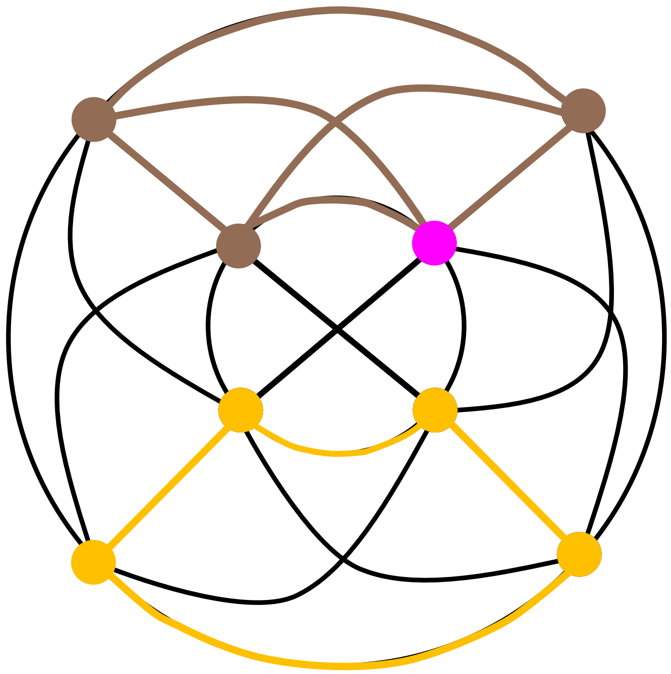

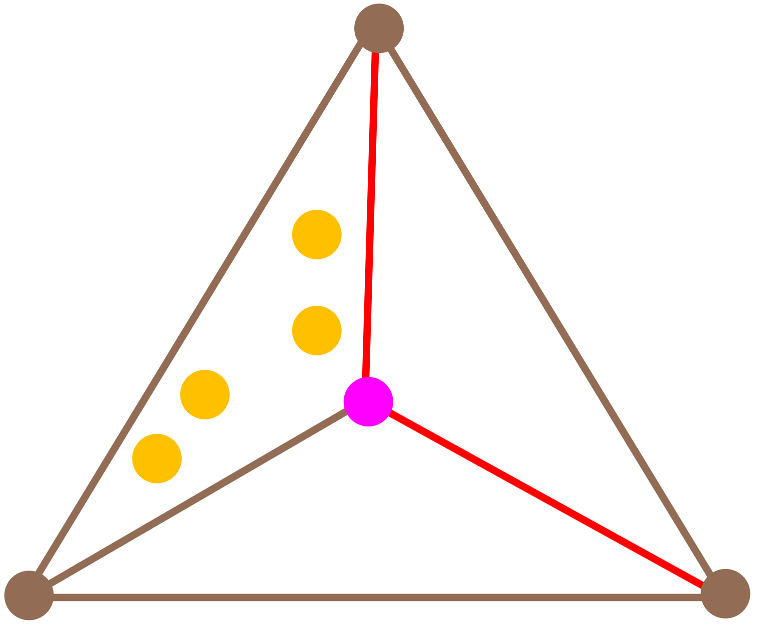

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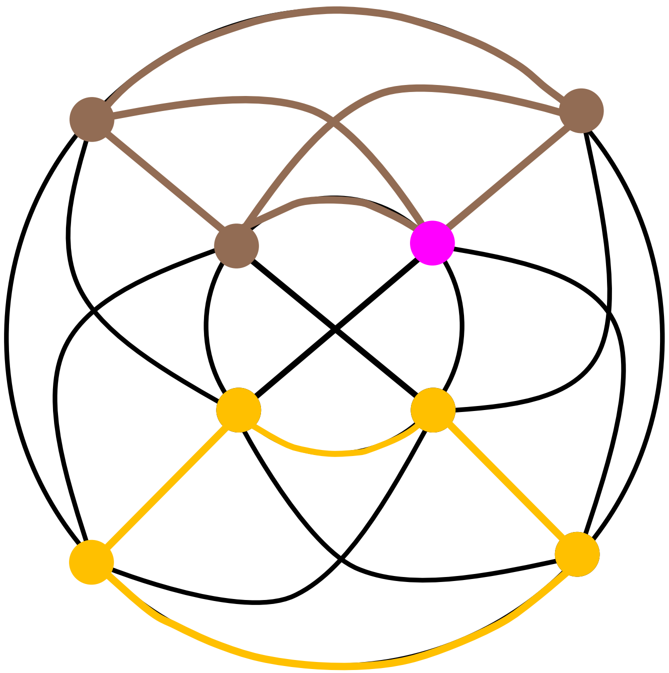

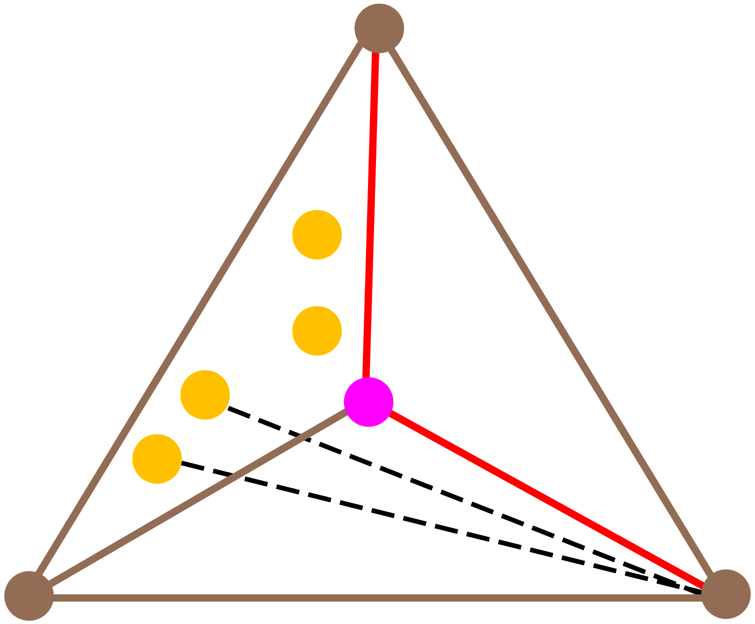

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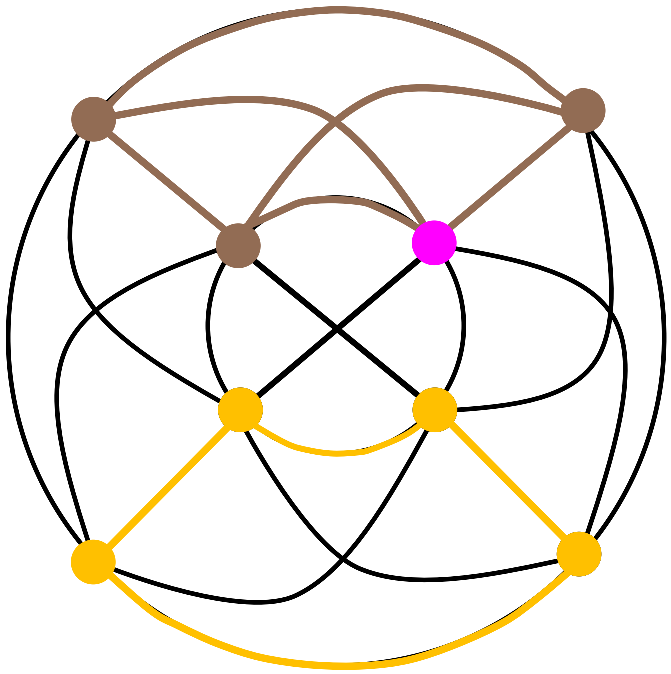

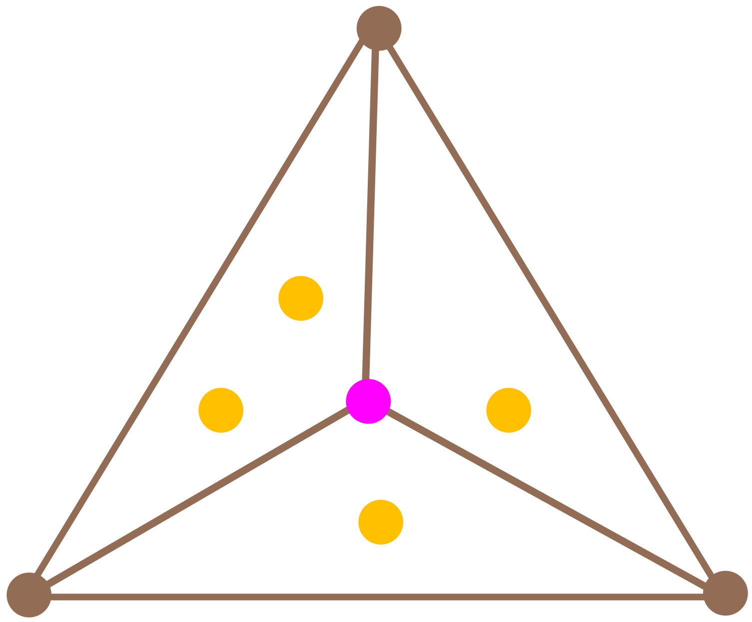

G_o is not RAC realizable

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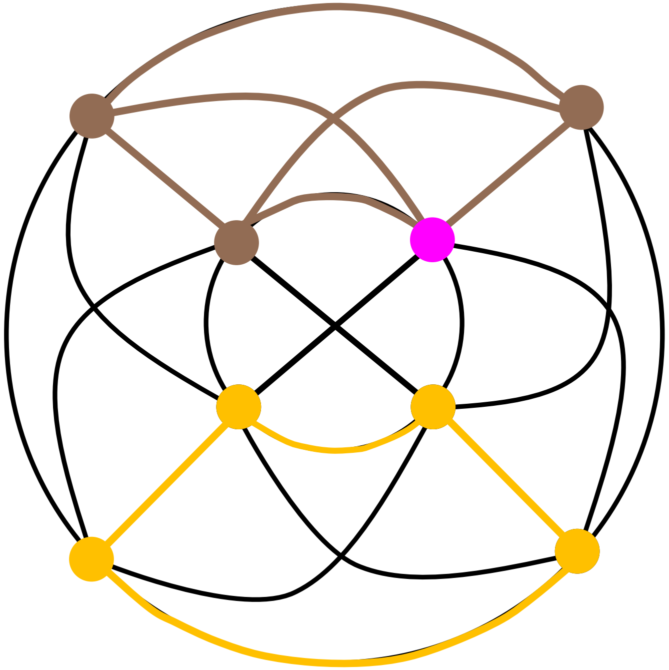

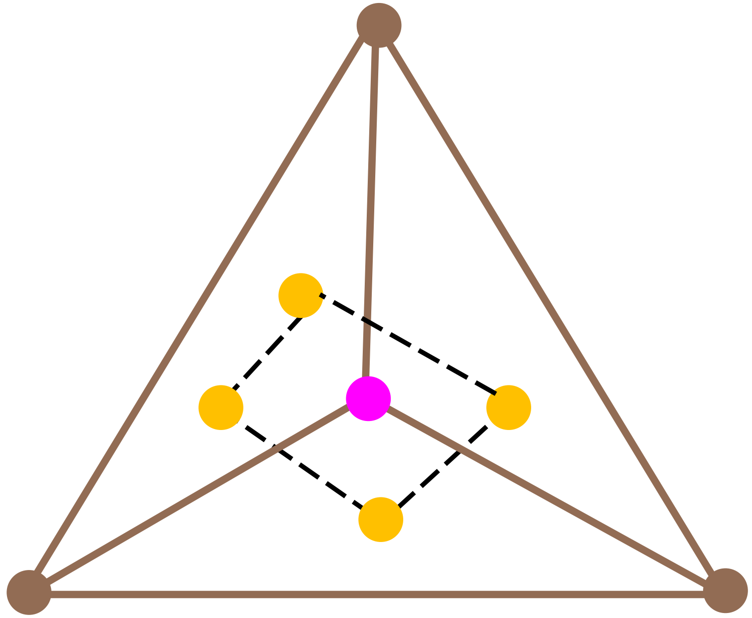

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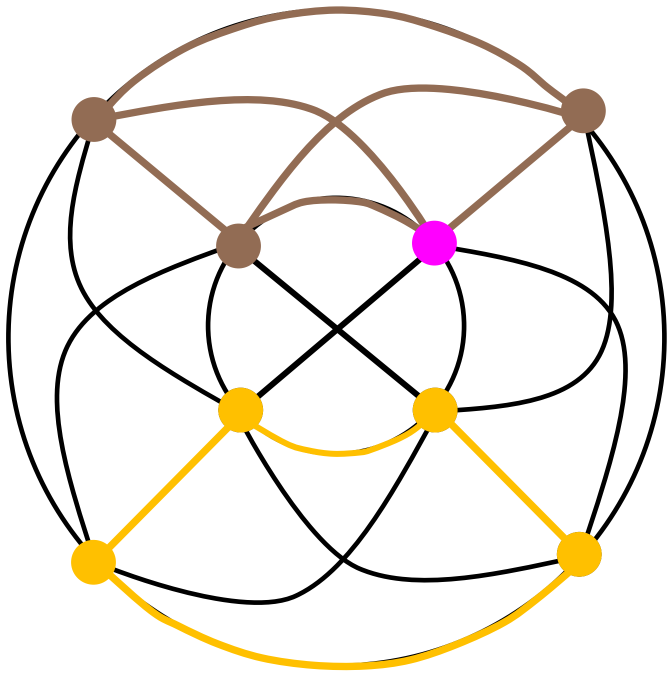

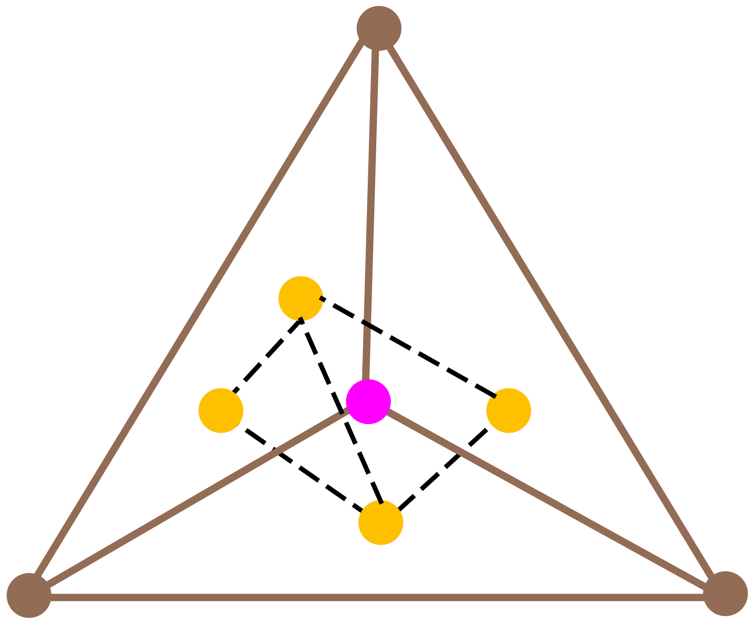

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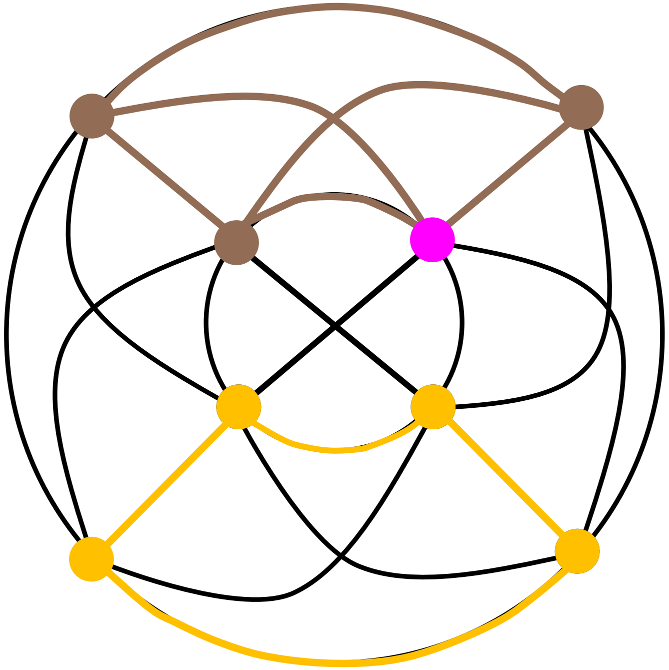
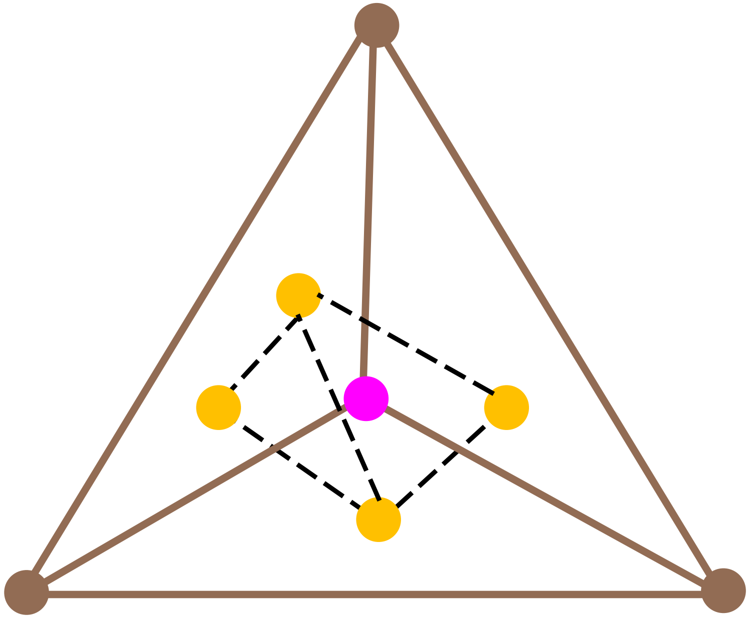
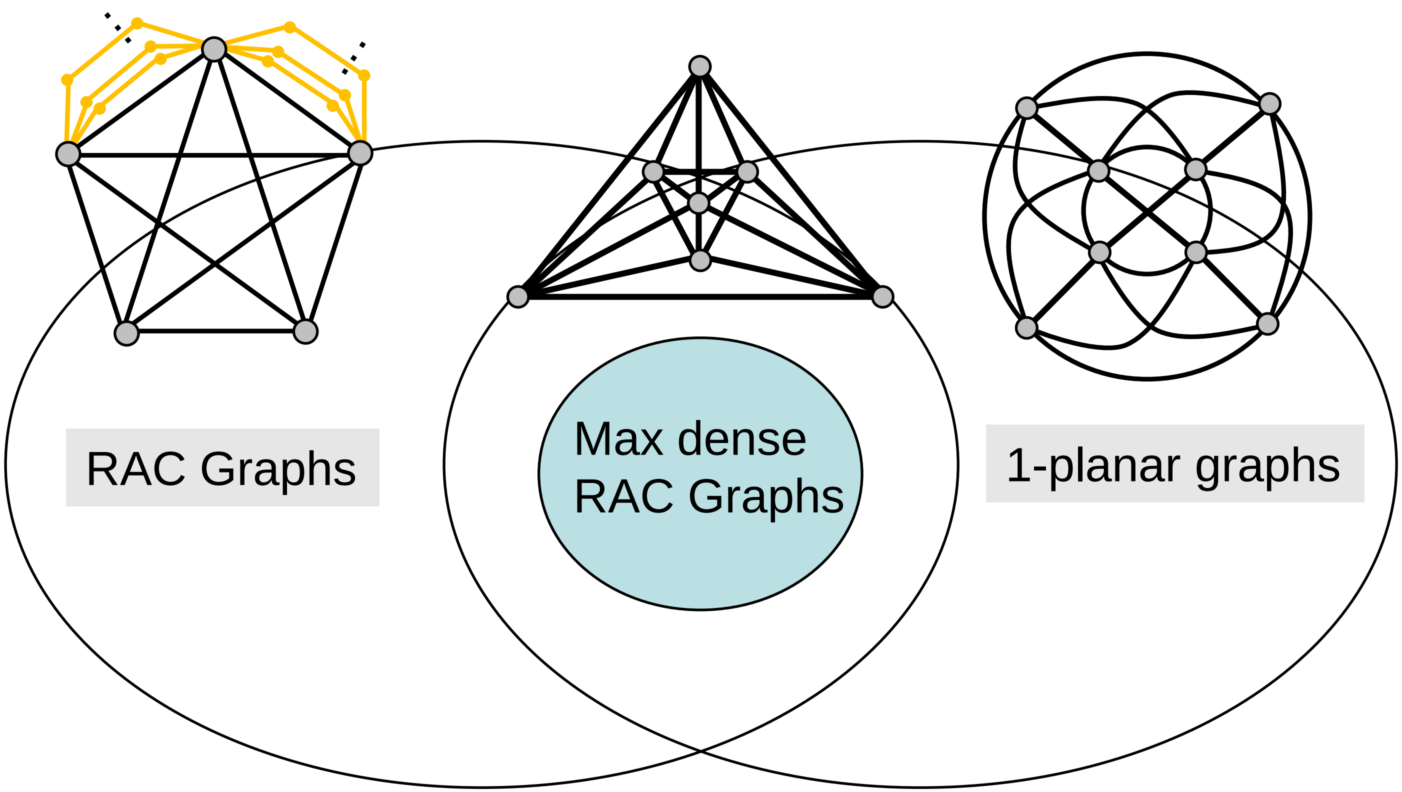
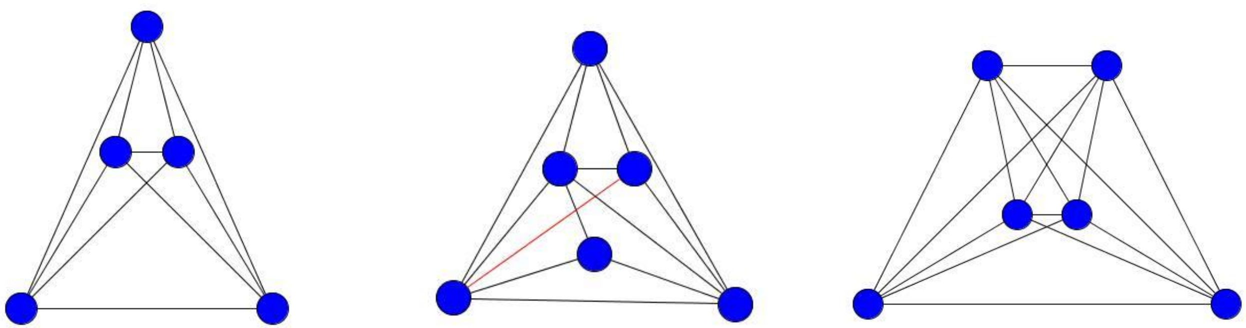




…summarizing….



Area Requirement Beyond Planarity



RAC skewness-h h-planar

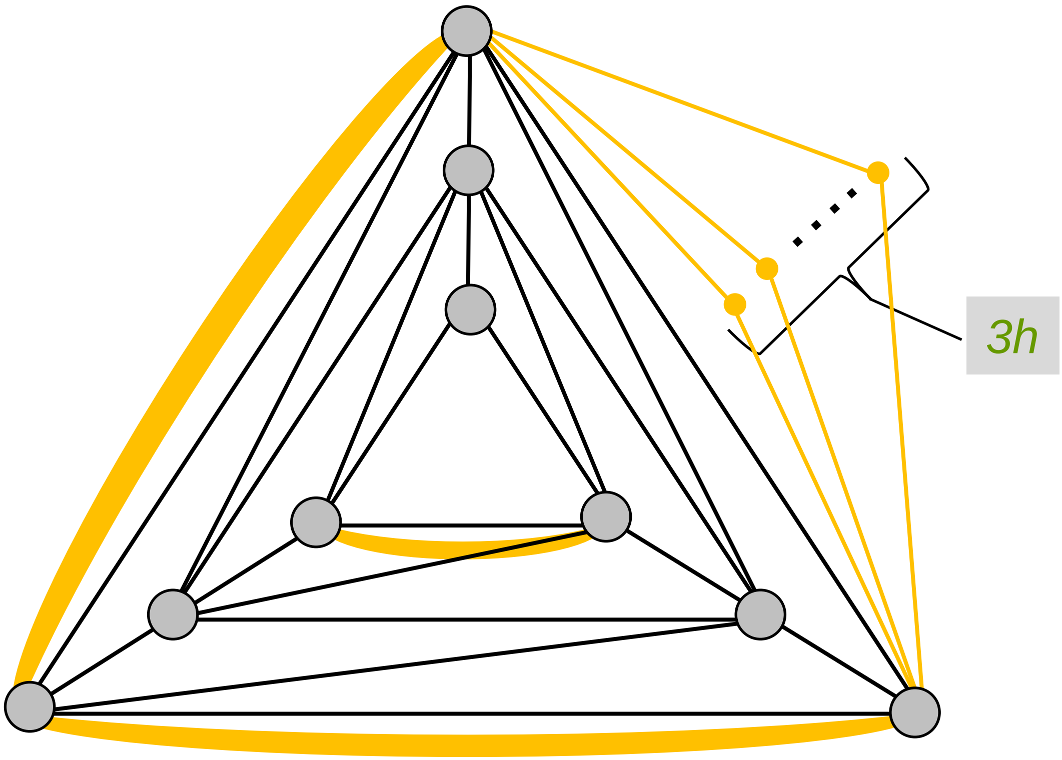
A result by Angelini et al.

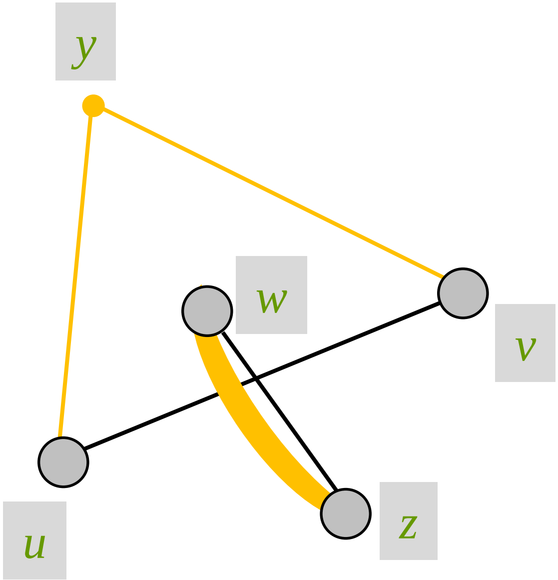
RAC straight-line drawings of planar graphs may require quadratic area

(Angelini et al., JGAA 2011)

Area req. of h-planar drawings

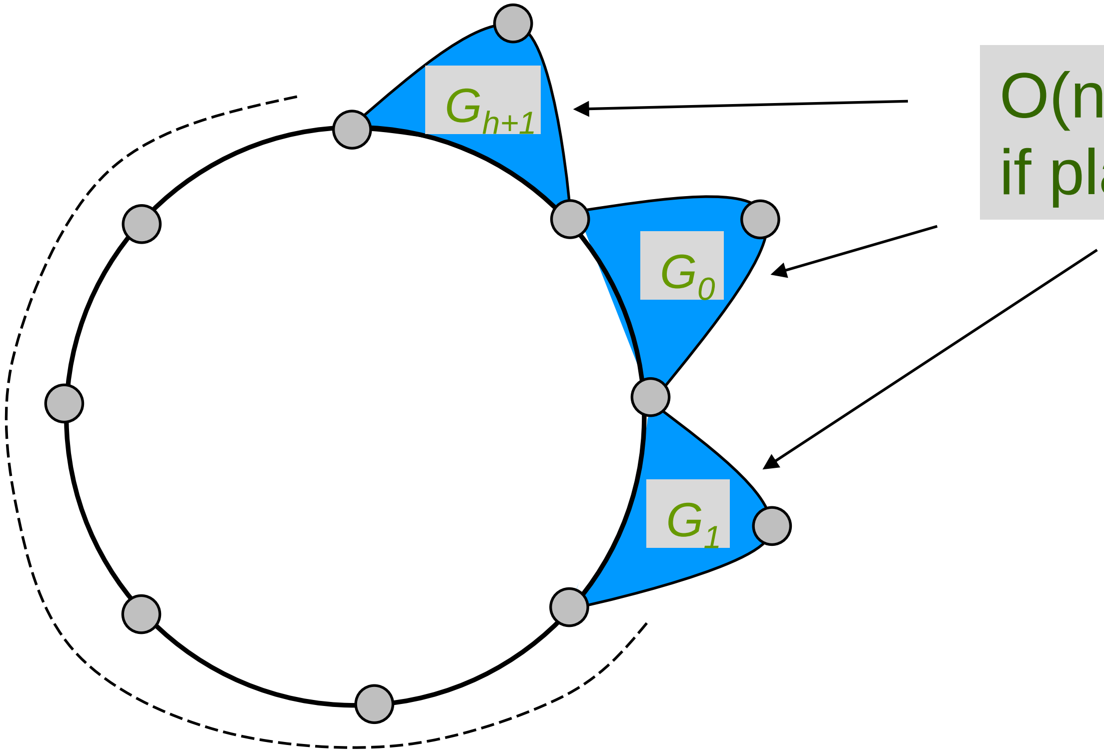
h-planar (constant *h*) straight-line drawings (and RAC straight-line drawings) of planar graphs may require quadratic area [Di Giacomo et al., 2012]



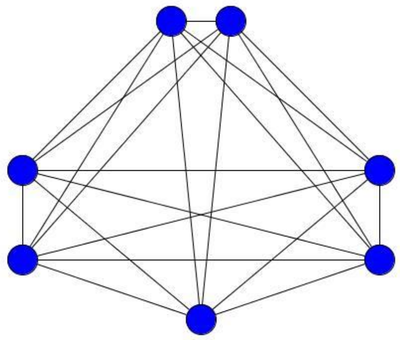


Area req. of skewness-h drawings

skewness-*h* (constant *h*) straight-line drawings of planar graphs may require quadratic area

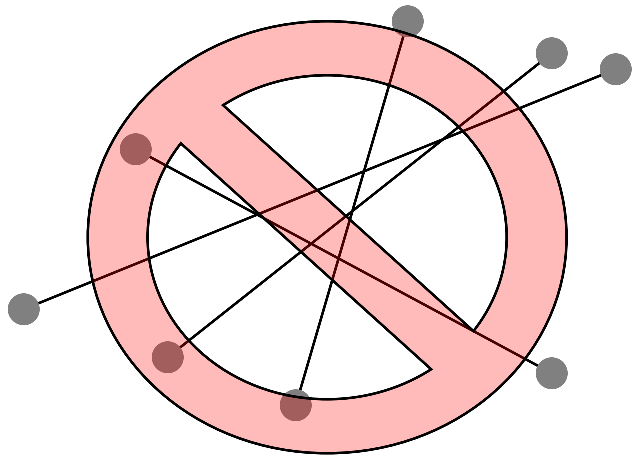


$O(n^2)$ area, if planar



4-quasi-planar

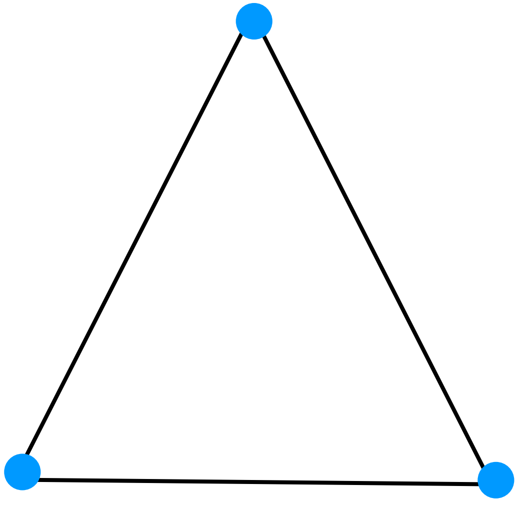
linear area upper bound



h-quasi-planar drawings

Bounded treewidth

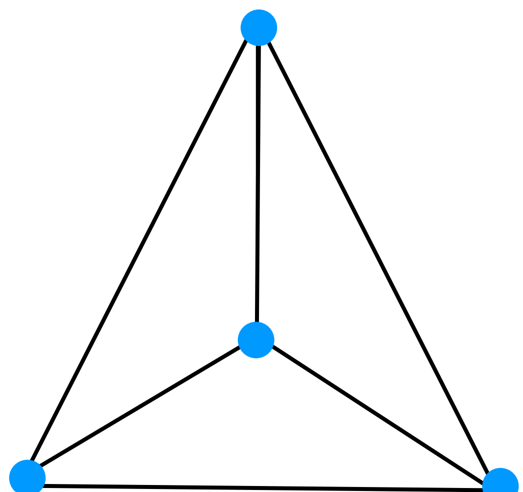
G has treewidth ≤ *k* ⇔ G is a partial *k*-tree



3-tree

Bounded treewidth

G has treewidth ≤ *k* ⇔ G is a partial *k*-tree



3-tree

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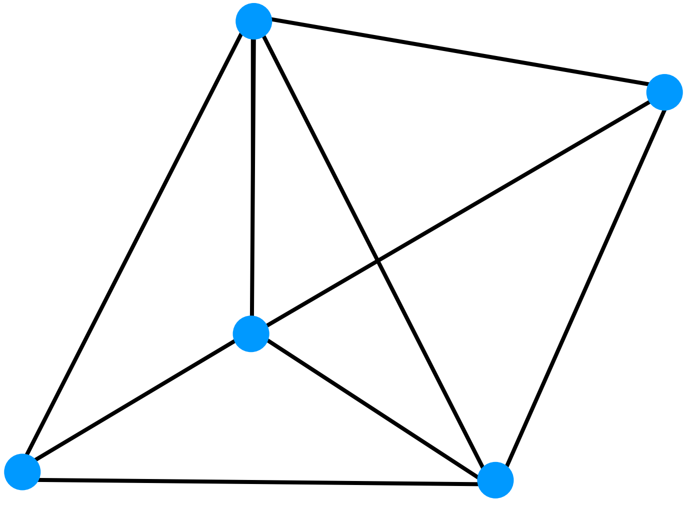

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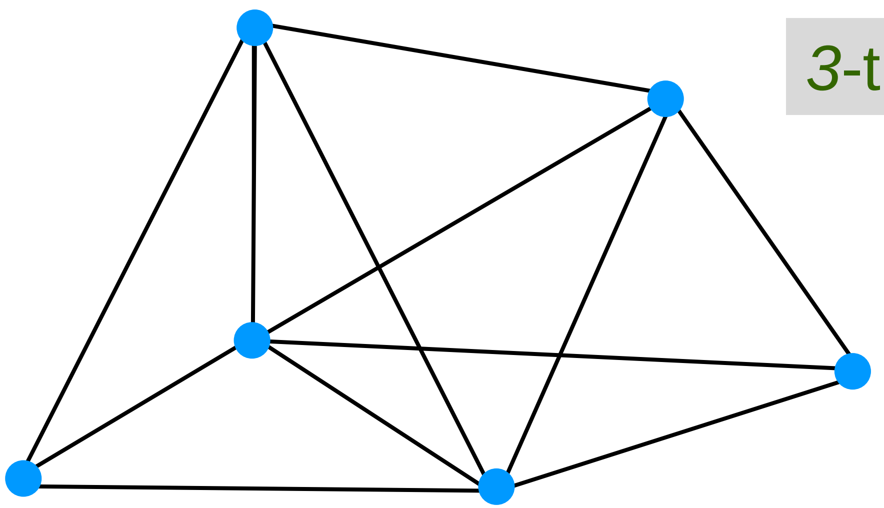

3-tree

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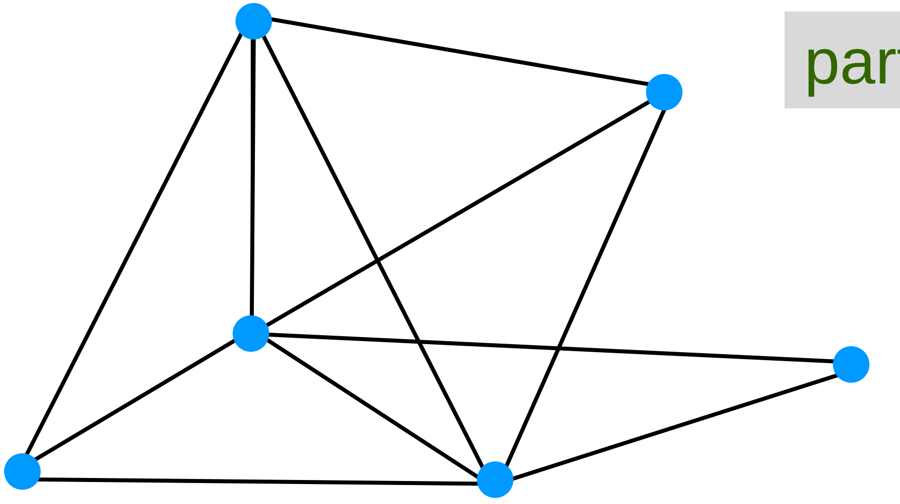


partial *3*-tree

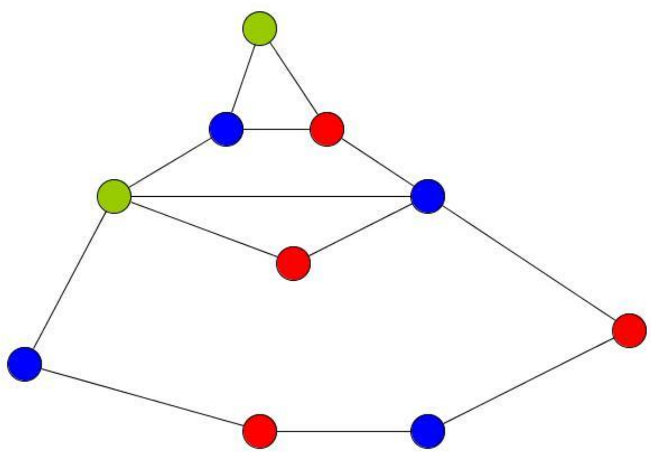
The good news

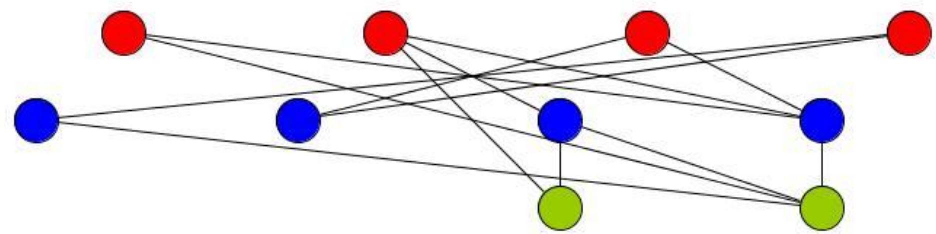
every n-vertex graph with bounded treewidth admits an *h*-quasi planar straight-line drawing in linear area such that the value of *h* does not depend on *n*

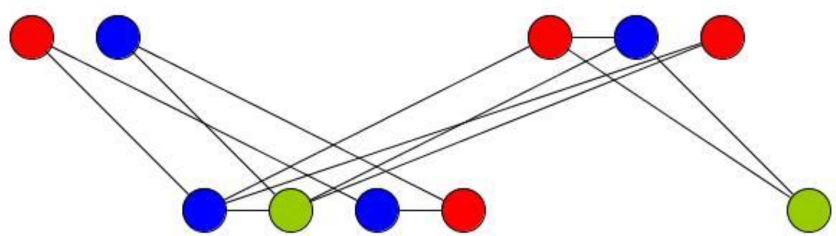
[Di Giacomo, Didimo, L., Montecchiani, 2013]

Applying the result

every *h*-colorable graph has a linear area s.l.drawing [Wood, CGTA, 2005]







Di Giacomo et al., 2013

Ingredients

study the relationship between (c,t)-track layouts and h-quasi planar straight-line drawings

new technique to compute a (2,t)-track layout of a partial k-tree

Open problems

Inclusion properties and RAC graphs

Characterize those 1-planar graphs that have a RAC drawing

Recognizing those graphs that have a RAC drawing is NP-hard. Does this problem remain NP-hard for those graphs with n vertices and 4n-10 edges?

Area-crossing complexity trade-offs

Do partial *k*-trees admit a *O*(*1*)-quasi planar straight line drawing in linear area and constant aspect ratio?

For, example, do outerplanar graphs admit a *3*-quasi planar straight line drawing in linear area and constant aspect ratio?

Do all planar graphs have a sub-quadratic area *h*-quasi planar straight-line drawing with constant h?

Other problem categories

	Turan-type	Recognition	Fary-type
RAC	O(n)	NP-hard (linear-time for 2-layer)	-
1-planar	O(n)	NP-hard (linear time for given rot. syst.)	charact. test, drawing
3-quasi-planar	O(n)	??	??
skewness-1	O(n)	polynomial	charact. test, drawing