Logspace Computability and Regressive Machines

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Outline

1. Preliminaries
2. Regressive Functions and Machines
3. New Characterizations of Logspace Functions and Predicates
4. Future Work
Introduction

- The set $L$ of languages decidable in logarithmic space is the set of languages recognized by:
  - read-only while programs [Jones, 1999];
  - the functions belonging to the closure with respect to substitution and simultaneous recursion on notation of the constant functions and the projection functions [Kristiansen, 2005].
Results

This work:

- introduces the class $E$ of number theoretic functions generated by the constant functions, the projection functions, the predecessor function, the substitution operator, and the recursion on notation operator;
- introduces *regressive machines*, i.e. register machines which have the division by 2 and the predecessor as basic operations;
- shows that $E$ is the class of functions computable by regressive machines and that the sharply bounded functions of $E$ coincide with the sharply bounded logspace computable functions.
Advantages

- Function algebra $E$ is defined without making use of ramified (safe) or bounded recursion schemes.
- Even if the present work is concerned with number theoretic functions, it can be considered an improvement of the characterization of $L$ given in [Kristiansen, 2005] because:
  - recursion on notation is used instead of simultaneous recursion.
  - not only the $0−1$ valued logspace computable functions, but also the sharply bounded logspace computable functions are characterized.\(^1\)
- Regressive machines are a simple computation model for $L$.

\(^1\)Sharply bounded logspace functions were characterized using safe recursion in [Bellantoni thesis].
Definitions

- Let $f, g, h$ be functions of finite arity on the set $\mathbb{N} = \{0, 1, \ldots\}$ of natural numbers.
  - $f$ is a **polynomial growth function** iff there is a polynomial $p$ such that $|f(x)| \leq p(|x|)$ for any $x$.\(^2\)
  - $f$ is **sharply bounded** iff there is a polynomial $p$ such that $f(x) \leq p(|x|)$ for any $x$.
  - $f$ is **regressive** iff there is some constant $k$ such that $f(x) \leq \max(x, k)$ for any $x$.
- For any $f$, we set $\text{bit}_f(x, i) = \text{bit}(f(x), i)$ and $\text{len}_f(x) = |f(x)|$.
- The characteristic function $ch_P$ of a predicate $P$ returns 1 if $P(x)$ is true, 0 otherwise.

\(^2\)|x_1, \ldots, x_n| = |x_1|, \ldots, |x_n| and |x| = \lceil \log_2(x + 1) \rceil is the number of bits of the binary representation of $x$.\]
Some basic functions

- the constant functions \( C_n : x \mapsto n \);
- the binary successor functions \( s_i : x \mapsto 2x + i \ (i \in \{0, 1\}) \);
- the bit function \( bit : x, y \mapsto \text{rem}(\lfloor x/2^y \rfloor, 2) \);
- the length function \( len : x \mapsto |x| = \lceil \log_2(x + 1) \rceil \);
- the conditional function \( cond : 0, y, z \mapsto y; x + 1, y, z \mapsto z \);
- the smash function \( smash : x, y \mapsto x \# y = 2^{|x|\cdot|y|} \);
- the most significant part function \( MSP : x, y \mapsto \lfloor x/2^y \rfloor \);
- the log most significant part function \( msp : x, y \mapsto \lfloor x/2^{|y|} \rfloor \);
- the predecessor function \( P : x + 1 \mapsto x; 0 \mapsto 0 \);
- the projection functions \( I^a[i] : x_1, \ldots, x_a \mapsto x_i \).
The recursion on notation operator $\text{RN}(g, h_0, h_1)$ transforms function $g : \mathbb{N}^a \rightarrow \mathbb{N}$ and functions $h_0, h_1 : \mathbb{N}^{a+1} \rightarrow \mathbb{N}$ into the function $f : \mathbb{N}^{a+1} \rightarrow \mathbb{N}$ such that

\[
\begin{align*}
    f(0, y) &= g(y), \\
    f(s_i(x), y) &= h_i(x, y, f(x, y))
\end{align*}
\]

where $i \in \{0, 1\}$ and $x > 0$ when $i = 0$.
Function algebras and class $E(F)$

Let

$$\text{clos}(f_1, \ldots, f_n, F_1, \ldots, F_m; op_1, \ldots, op_b)$$

be the inductive closure of $\{f_1, \ldots, f_n\} \cup F_1 \cup \ldots \cup F_m \cup I$ with respect to operators $op_1, \ldots, op_b$ where $I$ is the set of the projection functions $I^a[i] : x_1, \ldots, x_a \mapsto x_i$ with $1 \leq i \leq a$. We define

$$E(F) = \text{clos}(P, \{C_n\}_n, F; SUBST, RN)$$

and set

$$E = E(\emptyset), E(f_1, \ldots, f_a) = E(\{f_1, \ldots, f_a\}).$$
Properties of class $E$

- $div_2, rem_2, cond, msp, MSP, bit, max, min \in E$
- $E$ contains the (characteristic functions of the) Boolean closure of $x < y, x \leq y, x > y, x \geq y, x = y, x \neq y$
- $E$ contains sharply bounded versions of the arithmetic operations $add, sub, mult, rem, div$ (e.g. $add_p(x, y, z) = y + z$ for $y, z \leq p(|x|)$)
- ($Polynomial\ iteration$) For any $g \in E$ and any polynomial $p$ there is a function $f_p \in E$ such that

$$f_p(x_1, \ldots, x_k, y, z) = f(2^p(|x_1|, \ldots, |x_k|) - 1, y, z)$$

where

$$f(0, y, z) = z,$$
$$f(s_i(x), y, z) = g(y, f(x, y, z)).$$
Properties of class $E - 2$

- $E$ is closed w.r.t. the *sharply bounded maximization* operator $\text{MAX}(g, h)$ transforming function $g : \mathbb{N}^{a+1} \to \mathbb{N}$ and s.b. function $h : \mathbb{N}^a \to \mathbb{N}$ into the function $f : \mathbb{N}^a \to \mathbb{N}$ such that

  $$f(x) = \max\{i \leq h(x)|g(x, i) \neq 0\}$$

  if $\{i \leq h(x)|g(x, i) \neq 0\} \neq \emptyset$, otherwise $f(x) = 0$.

  (CP 1) $\text{len}_g \in E(\text{bit}_g)$ for any polynomial growth function $g$;

  (CP 2) $ch_{g_1 < g_2} \in E(\text{bit}_{g_1}, \text{bit}_{g_2})$ for any pol. growth functions $g_1, g_2$;

  (CP 3) $g \in E(\text{bit}_g)$ for any sharply bounded function $g$. 
Regressive Machines

*Regressive machines* are register machines that operate on a finite number of variables (the registers) $X_1, \ldots, X_b$. Programs are built up according to the following grammar:

$$P ::= X_i := e | \text{pred}(X_i) | \text{half}(X_i) | P_1; P_2 | \text{loop } X_i \text{ do } P \text{ end}.$$  

Expression $e$ can be any natural number constant, any register, or the least significant bit $\text{lsb}(X_j)$ of $X_j$. Instructions $\text{pred}(X_i)$ and $\text{half}(X_i)$ compute the predecessor and (the quotient of) the division by 2 of $X_i$, respectively. The program $\text{loop } X_i \text{ do } P \text{ end}$ executes $|x|$ times program $P$, where $x$ is the value of $X_i$.  

S. Mazzanti
Logspace Computability and Regressive Machines
Regressive Machines - 2

- Regressive machines compute regressive functions and operate in polynomial time.

- A program $P$ with $b$ registers computes a function $f : \mathbb{N}^a \rightarrow \mathbb{N}$ w.r.t. inputs $X_1, \ldots, X_a$ and output $X_j$ iff for any $n_1, \ldots, n_a$ the value $f(n_1, \ldots, n_a)$ is returned in register $X_j$ when $P$ is executed with $X_i$ having initial value $n_i$ for $1 \leq i \leq a$ and all the other registers are initialized to zero.
The Main Theorem ...

- Let $SB$ be the set of sharply bounded functions,
- let $RM$ be the set of functions computable by regressive machines,
- let $FL$ be the set of logspace computable functions.

**Theorem (Main Theorem)**

$$FL \cap SB \subseteq E \subseteq RM \subseteq FL \cap E.$$
Corollary (E is the class of functions computable by regressive machines, $E \subseteq FL$)

$$E = RM \subseteq FL.$$ 

Corollary (The s.b. functions in $E$ are the s.b. logspace functions)

$$FL \cap SB = E \cap SB.$$ 

Corollary (New characterization of $L$)

*The characteristic functions of logspace predicates coincide with the $\{0, 1\}$-valued functions in $E$.***
Clote-Takeuti’s characterization of FL

Clote and Takeuti ([Clo-Ta 1995]) have shown that

\[ FL = \text{clos}(C_0, s_0, s_1, \text{len}, \text{bit}, \text{smash}; \text{SUBST}, \text{CRN}, \text{SBRN}) \]

where:

- \( \text{CRN}(g, h_0, h_1) \) is the *concatenation recursion on notation* of \( g, h_0, h_1 \) (\( h_0, h_1 \) are 0–1 valued functions), i.e. the function

  \[
  f(0, y) = g(y), \\
  f(s_i(x), y) = s_{h_i(x,y)}(f(x, y));
  \]

- \( \text{SBRN}(g, h_0, h_1, l) \) is the *sharply bounded recursion on notation* of \( g, h_0, h_1, l \), i.e. the function \( f \) s.t.

  \[ f = \text{RN}(g, h_0, h_1) \]

provided that \( f(x, y) \leq |l(x, y)| \).
Theorem

For any \( f \in FL \) and any polynomial growth functions \( g_1, \ldots, g_a \),

\[
\text{bit}_f(g_1, \ldots, g_a) \in E(\text{bit}_{g_1}, \ldots, \text{bit}_{g_a}).
\]

Proof.

The proof is carried out by induction on the characterization of \( FL \) given by Clote and Takeuti.

Induction Basis (\( f = \text{bit} \))

\[
\text{bit}_{\text{bit}(g_1, g_2)}(x, i) = \begin{cases} 
\text{bit}_{g_1}(x, g_2(x)) & \text{if } (i = 0) \land (g_2(x) < |g_1(x)|) \\
0 & \text{otherwise}
\end{cases}
\]

Induction Step: \( SUBST \) (trivial); \( CRN, SBRN \) (difficult)
Step 1: $\text{FL} \cap \text{SB} \subseteq E$ (end)

Corollary

$\text{bit}_f \in E$ for any $f \in \text{FL}$.

Proof.

$\text{bit}_f = \text{bit}_f(I^a[1],...,I^a[a])$ where $a$ is the arity of $f$.

Then, $\text{FL} \cap \text{SB} \subseteq E$ because for any $g \in \text{FL} \cap \text{SB}$ we have

$g \in E(\text{bit}_g) = E$

by the corollary above and CP 3.
Step 2: $E \subseteq R\text{M}$

- We show by induction on $E$ that for any $f \in E$ there is a regressive machine computing $f$.
- The induction basis is trivial, as well as the induction step concerning function substitution.
- The case of recursion on notation is shown by using the $\text{LOOP}$ construct.
Step 2: \( E \subseteq \text{RM} \)

We show the case of recursion on notation. By ind. hyp. assume that there are \( P, Q_0 \) and \( Q_1 \) s.t.

\[
\{ V_1 = y_1, \ldots V_a = y_a \} P \{ Z = g(y) \},
\]

\[
\{ U = x, V_1 = y_1, \ldots V_a = y_a, W = z \} Q_i \{ Z_i = h_i(x, y, z) \} (i = 0, 1).
\]

Let

\[
f(0, y) = g(y),
\]

\[
f(s_i(x), y) = h_i(x, y, f(x, y)).
\]
Step 2: \( E \subseteq RM \)

Then, the program

\[
P; \quad \text{W:=Z; } X_0, X_1, X_2 := X; \\
\text{loop } X \text{ do} \\
\quad \text{half}(X_0); \text{loop } X_0 \text{ do } \text{half}(X_1); \text{ end;} \\
\quad R := \text{lsb}(X_1); \text{half}(X_1); U := X_1; \\
\quad \text{if (R=0) then } Q_0; \text{ W:=Z}_0 \text{ else } Q_1; \text{ W:=Z}_1; \\
\quad X_1 = X_2;
\]

\text{end}

computes \( f \) with respect to inputs \( X, V_1, \ldots, V_a \) and output \( W \).
In order to show that $\text{RM} \subseteq \text{FL} \cap \text{E}$, we need to simulate the computations of regressive machines by storing the registers’ contents with at most $O(\log(\max(|x|)))$ bits where $x$ is the sequence of input values (in other words, we can store values bounded by $p(|x|)$ for some polynomial $p$).

Since regressive machines compute regressive functions, registers are bounded by $\max(x, c))$. So, if we encoded a memory state as usual, the encoding would exceed the logarithmic bound on memory space and we could not compute $\text{RM}$ functions in logarithmic space.

To overcome the memory space bound, we introduce *counter machines* and show that they simulate regressive machines using only a logarithmic amount of memory space.
Counter Machines

- A counter machine operates on read-only input registers $Y_1, \ldots, Y_a$ and read/write registers $Z_1, \ldots, Z_b$ called *counters*.
- A counter machine program is defined as follows:

  $$P ::= Z_i := e | \text{succ}(Z_i) | \text{half}(Z_i) | P_1; P_2$$
  $$\text{if } (e_1 = n) \text{ then } P_1 \text{ else } P_2 \text{ loop } E_i \text{ do } P \text{ end}$$

where $e$ is a constant, a counter or $\text{lsb}(E_j)$ and $e_1$ is a counter, $\text{bit}(Y_{Z_i}, Z_j)$ or $\text{lsb}(Z_i)$. $E_i$ has value

$$e_i^b(y, z) = \begin{cases} z_i + 2 & \text{if } z_i = 0, \\ \text{MSP}(y_{z_i}, z_{i+1}) - z_{i+2} & \text{otherwise} \end{cases} (1 \leq i \leq b - 2)$$

where $y = y_1, \ldots, y_a$ are the input values and $z = z_1, \ldots, z_b$ are the values of the counters.
Every regressive machine program $P$ with $b$ registers is simulated by a counter machine program $Q$ with $3b$ counters.

The value of register $X_i$ of program $P$ is represented by counters $Z_{3i-2}, Z_{3i-1}, Z_{3i}$ of $Q$ so that $e_{3i-2}(y, z) = x_i$.

If $X_i$ has been set to a constant value, then $z_{3i-2} = 0$ and $z_{3i}$ is the value of $X_i$.

Otherwise, an input value has been assigned (or copied) to $X_i$ and decrement or division instructions have been performed on it. In that case, the value of $X_i$ is $MSP(y_{z_{3i-2}}, z_{3i-1}) z_{3i}$. 
At start, assume that $X_j = n_j$. Then, we set $Y_j = n_j$ and $Z_{3j-2} = j, Z_{3j-1} = 0, Z_{3j} = 0$.

<table>
<thead>
<tr>
<th>OPERATION</th>
<th>$X_i$</th>
<th>$Y_i$</th>
<th>$Z_{3i-2}$</th>
<th>$Z_{3i-1}$</th>
<th>$Z_{3i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Init</td>
<td>$n_i$</td>
<td>$n_i$</td>
<td>$i$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$X_i := k$</td>
<td>$k$</td>
<td>$n_i$</td>
<td>0</td>
<td>0</td>
<td>$k$</td>
</tr>
<tr>
<td>$X_i := X_j$</td>
<td>$n_j$</td>
<td>$n_i$</td>
<td>$j$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>pred($X_i$)</td>
<td>$n_j - 1$</td>
<td>$n_i$</td>
<td>$j$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
<td>...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>pred($X_i$)</td>
<td>$n_j - n$</td>
<td>$n_i$</td>
<td>$j$</td>
<td>0</td>
<td>$n$</td>
</tr>
<tr>
<td>half($X_i$)</td>
<td>$\lfloor \frac{n_j - n}{2} \rfloor$</td>
<td>$n_i$</td>
<td>$j$</td>
<td>1</td>
<td>$\left\lfloor \frac{n}{2} \right\rfloor + r$</td>
</tr>
</tbody>
</table>

$$\text{div}_2(MSP(u, v) \cdot w) = MSP(u, v + 1) \cdot (\text{div}_2(w) + ch\{\text{bit}(u, v) < \text{rem}_2(w)\}).$$
Simulation - formal definition

- Let $m_P : \mathbb{N}^b \rightarrow \mathbb{N}^b$ be the function such that $m_P(x)$ is the memory state after the computation of a R.M. program $P$ starting from state $x$.

- Let $M_Q : \mathbb{N}^{a+b} \rightarrow \mathbb{N}^b$ be the function such that $(y, M_Q(y, z))$ is the memory state after the computation of a C.M. program $Q$ starting from the state $(y, z)$.

- For any $x \in \mathbb{N}^b$, $y \in \mathbb{N}^a$ and $z \in \mathbb{N}^{3b}$, if $l^b[i](x) = e_{3i-2}(y, z)$ for any $1 \leq i \leq b$, then $l^b[i](m_P(x)) = e_{3i-2}(y, M_Q(y, z))$ for any $1 \leq i \leq b$. 
Step 3: \( \text{RM} \subseteq \text{FL} \cap \text{E} \)

- A function \( f : \mathbb{N}^a \rightarrow \mathbb{N} \) is computed by a R.M. program \( P \) with \( b \) registers iff
  \[
  f(x) = I^b[j](m_P(x, 0, \ldots, 0))
  \]
  where \( j \) is the index of the output register of \( P \).

- Then, there is a C.M. program \( Q \) with \( 3b \) counters such that
  \[
  f(x) = e_{3j-2}(x, M_Q(x, 1, 0, 0, \ldots, a, 0, 0, \ldots, 0)).
  \]
  and the counters of \( Q \) are less than \( p(|x|) \) for some \( p \).
Step 3: $RM \subseteq FL \cap E$ (end)

- Therefore, we encode the counters with a single number
  \[c_p(x, z) = z_1 p(|x|)^{(3^b - 1)} + \ldots + z_{3^b - 1} p(|x|) + z_{3^b} < p(|x|)^{3^b}\]
  and define functions $\tilde{e}_{p,i}, \tilde{M}_{p, Q} : \mathbb{N}^{a+1} \rightarrow \mathbb{N}$ in $FL \cap E$ s.t.
  \[\tilde{e}_{p,i}(x, c_p(x, z)) = e_i(x, z), \quad \tilde{M}_{p, Q}(x, c_p(x, z)) = c_p(x, M_Q(x, z))\]
  and
  \[f(x) = \tilde{e}_{p,3j-2}(x, \tilde{M}_{p, Q}(x, c_p(x, 1, 0, 0, \ldots, a, 0, 0, \ldots, 0))).\]

- Since $c_p \in FL \cap E$, we obtain that $f \in FL \cap E$. 

Open Questions

- $\text{FL} \cap \text{SB} \subseteq \text{clos}(\{C_n\}_n; \text{SUBST}, \text{RN})$?
- does $E$ equal the set of regressive logspace computable functions?

A function $f$ is non-size-increasing iff there is some constant $k$ such that $|f(x)| \leq \max(|x|, k)$ for any $x$.

For $i \in \{0, 1\}$, consider the bounded successor functions $bs_i : x, y \mapsto 2x + i$ if $|x| < |y|$; $x, y \mapsto x$ otherwise.

- It is easy to see that $E' = \text{clos}(bs_0, bs_1, \{C_n\}_n; \text{SUBST}, \text{RN})$ contains the non-size-increasing logspace computable functions.
- $E' \subseteq \text{FL}$? (Conjecture: $E'$ recognize $\text{P} \cap \text{LINSPACE}$).
References


