Binary 3-compressible automata

Alessandra Cherubini and Andrzej Kisielewicz

Politecnico di Milano, Dipartimento di Matematica

Department of Mathematics and Computer Science, University of Wrocław

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Preliminaries

- $\mathcal{A} = \langle Q, \Sigma, \delta \rangle$ deterministic finite complete automaton;
  binary: $|\Sigma| = 2$

transition function

- $\delta : \; Q \times \Sigma \to Q : (q, a) \to qa$ action of letters
- $Q \times \Sigma^* \to Q : (q, w) \to qw$ action of words

transformation monoid $\subseteq T(Q)$

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Definition

$\mathcal{A}$ *k-compressible* if $|Q| - |Qw| \geq k$ for some $w \in \Sigma^*$;
word w *k-compresses* $\mathcal{A}$.

Collapsing words

Theorem (Sauer, Stone, 1991)

For each alphabet Σ there exists a word v such that v k-compresses each k-compresible automaton over Σ.

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Examples (Ananichev, Petrov, Volkov, 2005)

aba^2b^2ab — 2-collapsing over $\{a, b\}$
$aba^2c^2bab^2acbabcacbcb$ — 2-collapsing over $\{a, b, c\}$

Results

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Natural question

What about 3-collapsing words over 2-element alphabet?

Main result

Theorem

The problem whether a given word $w \in \{\alpha, \beta\}^*$ *is 3-collapsing is co-NP-complete.*

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Main problem: no characterization of 3-collapsing words

Characterization of 2-collapsing words

Theorem (Cherubini, Gawrychowski, Kisielewicz, Piochi)

A word $w \in \Sigma^*$ is 2-collapsing if and only if it is 2-full and the following conditions holds:

1. $\Gamma_w(B_0, \ldots, B_r)$ has no nontrivial solution for any partition $(B_0, \ldots, B_r)$ of Σ;
2. $\Gamma'_w(B_0, B_1, B_2)$ has no nontrivial solution for any 3-partition (B_0, B_1, B_2) of Σ.

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Every k-collapsing word is k-full

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$\Gamma_w(B_0, \ldots, B_r)$ – system of permutation conditions:

To each factor of w of the form $\alpha v \beta$, $v \in B_0^+$, $\alpha \notin B_0$, and $\beta \in B_j$, we assign a condition of the form

$$1v \in \{1, j\},$$

(letters of B_0 are treated as permutation variables).

Binary 3-compressible automata

Theorem (Cherubini, Frigeri, Liu, 2014)

If $\mathcal{A}$ is a proper 3-compressible automaton over the alphabet $\Sigma = \{\alpha, \beta\}$ then each letter in Σ is either a permutation or is one of the following types:

1. $[x, y, z] \backslash x, y$;
2. $[x, y][z, t] \backslash x, z$;
3. $[x, y] \backslash x$;
4. $[x, y] \backslash z$ with $z\alpha \in \{x, y\}$.

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- Binary automata of type $(\mathbf{3}, \mathbf{p})$

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If $\mathcal{A}$ is a proper 3-compressible automaton over the alphabet $\Sigma = \{\alpha, \beta\}$ then each letter in Σ is either a permutation or is one of the following types:

1. $[1,2,3]\backslash 1,2$;
2. $[1,2][3,4]\backslash 1,3$;
3. $[1,2]\backslash 1$;
4. $[1,2]\backslash 3$ with $3\alpha \in \{1,2\}$.

- Binary automata of type $(\mathbf{3}, \mathbf{p})$, $Q = \{1, 2, \ldots, n\}$

Binary automata of type $(\mathbf{3}, \mathbf{p})$

Theorem

Let $\mathcal{A} = \langle \{1, 2, \ldots, n\}, \{\alpha, \beta\}, \delta \rangle$ be a proper 3-compressible automaton of type $(\mathbf{3}, \mathbf{p})$, and w a word over $\{\alpha, \beta\}$.
*Then, w **does not** 3-compresses $\mathcal{A}$ iff if for every factor of w of the form $\alpha u \alpha$ with u satisfying $(*)$ the condition $1u \in \{1, 2\}$ holds.*

$$(*) \quad \begin{cases} u = \beta^{k_1}\alpha^{m_1} \ldots \beta^{k_t}\alpha^{m_t}\beta^{k_{t+1}}, \\ 1\beta^{k_1} \notin \{1, 2\}; \\ 1\beta^{k_{t+1}} \notin \{1, 2\}; \\ 1\beta^{k} \in \{1, 2\}, \text{otherwise} \end{cases}$$

Smaller class $\mathcal{D}$

Definition

Class $\mathcal{D}$ — all proper 3-compressible automata of type $(\mathbf{3}, \mathbf{p})$ (i.e. α is a transformation of type $[1, 2]\backslash 1$, and β a permutation) such that β is of the form $\beta = (12y)\ldots$

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Class $\mathcal{D}$ — all proper 3-compressible automata of type $(\mathbf{3}, \mathbf{p})$ (i.e. α is a transformation of type $[1,2]\backslash 1$, and β a permutation) such that β is of the form $\beta = (12y)\ldots$

Then:

$$1\beta^k \notin \{1,2\} \quad \text{iif} \quad k \equiv 2 \pmod 3$$

Smaller class $\mathcal{D}$

Theorem

Let $\mathcal{A} \in \mathcal{D}$, and w a word over $\{\alpha, \beta\}$.
*Then, w **does not** 3-compresses $\mathcal{A}$ iff if*
for every factor of w of the form $\alpha u \alpha$ with u satisfying $(*)$
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$$(*) \quad \begin{cases} u = \beta^{k_1}\alpha^{m_1}\ldots\beta^{k_t}\alpha^{m_t}\beta^{k_{t+1}}, \\ k_1 \equiv 2 \pmod 3; \\ k_{t+1} \equiv 2 \pmod 3; \\ k \equiv 0, 1 \pmod 3, \text{otherwise} \end{cases}$$

Corollary from Cherubini, Frigeri, Liu, 2014

Each word W containing as factors the following words

(I) $\alpha^2\beta\alpha\beta^2\alpha, \beta\alpha^2\beta^2\alpha^2\beta, \alpha\beta^4\alpha\beta^2\alpha\beta^4\alpha,$

(II) $\alpha\beta^3\alpha\beta^3\alpha, \beta\alpha^3\beta\alpha^3\beta, \beta^2\alpha\beta\alpha^2\beta,$ $\beta\alpha\beta\alpha^2\beta\alpha\beta,$ $\beta\alpha^2\beta\alpha^2\beta,$ $\beta\alpha^2\beta\alpha\beta\alpha^2\beta,$ $\beta\alpha^3\beta\alpha\beta,$ $\beta\alpha\beta\alpha^3\beta, \beta\alpha^3\beta\alpha^3\beta,$ $\beta^2\alpha^2\beta^2,$ $\beta^2\alpha^3\beta^2,$ $\beta^2\alpha\beta\alpha\beta^2, \alpha^2\beta^3\alpha^2,$

(III) $\alpha\beta^i\alpha^k\beta^j\alpha$, where $i, j \in \{1, 3, 4\}, k \in \{1, 2, 3\}$.

3-compresses all 3-compressible automata except those in the $\mathcal{D}$.

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(I) $\alpha^2\beta\alpha\beta^2\alpha, \beta\alpha^2\beta^2\alpha^2\beta, \alpha\beta^4\alpha\beta^2\alpha\beta^4\alpha,$

(II) $\alpha\beta^3\alpha\beta^3\alpha, \beta\alpha^3\beta\alpha^3\beta, \beta^2\alpha\beta\alpha^2\beta,\ \beta\alpha\beta\alpha^2\beta\alpha\beta,\ \beta\alpha^2\beta\alpha^2\beta,$ $\beta\alpha^2\beta\alpha\beta\alpha^2\beta,\ \beta\alpha^3\beta\alpha\beta,\ \beta\alpha\beta\alpha^3\beta, \beta\alpha^3\beta\alpha^3\beta,\ \beta^2\alpha^2\beta^2,\ \beta^2\alpha^3\beta^2,$ $\beta^2\alpha\beta\alpha\beta^2, \alpha^2\beta^3\alpha^2,$

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A word of the form Ww

- is 3-collapsing iff
- it 3-commpresses all automata in $\mathcal{D}$ iff (by Theorem above)
- a suitable system of transformation conditions $1u_i \in \{1, 2\}$ has a nontrivial solution