A graph easy class of mute terms

A. Bucciarelli, A.Carraro, G.Favro, A.Salibra

ICTCS 2014, Perugia

Terms representing undefinedness.

A natural problem arising in λ-calculus is what terms should be considered as representative of undefined programs.

Terms representing undefinedness.

A natural problem arising in λ-calculus is what terms should be considered as representative of undefined programs.

$\Omega \equiv (\lambda x.xx)(\lambda x.xx)$ is the simplest term that embodies this intuitive idea.

Unsolvable terms.

Unsolvable terms.

Every λ-term has one of the following form:

- $\lambda x_1 \ldots x_m . y M_1 \ldots M_n$

Unsolvable terms.

Every λ-term has one of the following form:

- $\lambda x_1 \ldots x_m.yM_1 \ldots M_n$
- $\lambda x_1 \ldots x_m.(\lambda z.M)M_1 \ldots M_n$

Unsolvable terms.

Every λ-term has one of the following form:

- $\lambda x_1 \dots x_m . y M_1 \dots M_n$
- $\lambda x_1 \dots x_m . (\lambda z . M) M_1 \dots M_n$

If a term β-reduces to a term of the first kind, we say it has a *head normal form*.

Unsolvable terms.

Every λ-term has one of the following form:

- $\lambda x_1 \ldots x_m . y M_1 \ldots M_n$
- $\lambda x_1 \ldots x_m . (\lambda z . M) M_1 \ldots M_n$

If a term β-reduces to a term of the first kind, we say it has a *head normal form*.

Definition

A term is called **unsolvable** if it does not have an head normal form.

Unsolvable terms.

Every λ-term has one of the following form:

- $\lambda x_1 \ldots x_m . y M_1 \ldots M_n$
- $\lambda x_1 \ldots x_m . (\lambda z . M) M_1 \ldots M_n$

If a term β-reduces to a term of the first kind, we say it has a *head normal form*.

Definition

A term is called **unsolvable** if it does not have an head normal form.

Unsolvables can be considered as the terms representing the undefined (Barendregt, Wadsworth).

λ-theories and unsolvable terms.

Definition

A λ-theory is a theory of equations between λ-terms that contains $\lambda\beta$.

λ-theories and unsolvable terms.

Definition

A λ-theory is a theory of equations between λ-terms that contains $\lambda\beta$.

Theorem (Berarducci-Intrigila)

There exists a closed unsolvable t such that

$$\forall M \text{ s.t. } M \neq_\beta I,\ \lambda\beta + \{t = M\} \textit{ is a consistent theory,}$$

while

$$\forall M \text{ s.t. } M =_\beta I,\ \lambda\beta + \{t = M\} \textit{ is not consistent.}$$

Easy terms.

A closed unsolvable **term** t is called **easy** if for any closed term M the theory

$$\lambda\beta + \{t = M\}$$

is consistent.

Easy terms.

A closed unsolvable **term** t is called **easy** if for any closed term M the theory

$$\lambda\beta + \{t = M\}$$

is consistent.

Example

- Ω

Easy terms.

A closed unsolvable **term** t is called **easy** if for any closed term M the theory

$$\lambda\beta + \{t = M\}$$

is consistent.

Example

- Ω
- $\Omega_3 \mathbf{I}$, where $\Omega_3 \equiv (\lambda x.xxx)(\lambda x.xxx)$

Easy terms.

A closed unsolvable **term** t is called **easy** if for any closed term M the theory

$$\lambda\beta + \{t = M\}$$

is consistent.

Example

- Ω
- $\Omega_3\mathbf{I}$, where $\Omega_3 \equiv (\lambda x.xxx)(\lambda x.xxx)$

Ω_3 is unsolvable but not easy.

Easy sets.

Definition

A **set** A of closed unsolvable terms is an **easy set** if for any closed M the theory

$$\lambda\beta + \{t = M \mid t \in A\}$$

is consistent.

Easy sets.

Definition

A **set** A of closed unsolvable terms is an **easy set** if for any closed M the theory

$$\lambda\beta + \{t = M \mid t \in A\}$$

is consistent.

Example

$\{\Omega(\lambda x_1 \dots x_{k+1}.x_{k+1}) \mid k \in \omega\}$

Easy sets.

Definition

A **set** A of closed unsolvable terms is an **easy set** if for any closed M the theory

$$\lambda\beta + \{t = M \mid t \in A\}$$

is consistent.

Example

$\{\Omega(\lambda x_1 \ldots x_{k+1}.x_{k+1}) \mid k \in \omega\}$

Theorem

The set of easy terms is not an easy set.

Mute terms.

Berarducci, “Infinite λ-calculus and non-sensible models”.

Mute terms.

Berarducci, “Infinite λ-calculus and non-sensible models”.

Definition

A term M is a **zero term** if it does not reduce to an abstraction.

Mute terms.

Berarducci, “Infinite λ-calculus and non-sensible models”.

Definition

A term M is a **zero term** if it does not reduce to an abstraction.

Definition

A zero term is **mute** if it does not reduce to a variable or to a term of the form

$$\textit{(Zero term)} \cdot \textit{Term}$$

Examples and properties of Mute terms.

Example

- Ω

Examples and properties of Mute terms.

Example

- Ω
- BB, where $B \equiv \lambda x.x(\lambda y.xy)$

Properties of the mute terms.

- The set of mute terms is an easy set.

Properties of the mute terms.

- The set of mute terms is an easy set.
- The set of mute terms is not recursively enumerable, as well as the set of easy sets.

Properties of the mute terms.

- The set of mute terms is an easy set.
- The set of mute terms is not recursively enumerable, as well as the set of easy sets.

Problem

Is $Y\Omega_3$, *where* $Y \equiv \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$, *easy?*

Regular mute terms

Hereditarily n-ary terms.

Definition

Let $n > 0$ and $\bar{x} \equiv x_1, \ldots x_k$ be distinct variables. The set of *hereditarily n-ary λ-terms over* $\bar{x}$, $H_n[\bar{x}]$, is the smallest set of terms such that:

Hereditarily n-ary terms.

Definition

Let $n > 0$ and $\bar{x} \equiv x_1, \ldots x_k$ be distinct variables. The set of *hereditarily n-ary* λ*-terms over* $\bar{x}$, $H_n[\bar{x}]$, is the smallest set of terms such that:

- For all $i = 1, \ldots, k$

$$\frac{}{x_i \in H_n[\bar{x}]}$$

Hereditarily n-ary terms.

Definition

Let $n > 0$ and $\bar{x} \equiv x_1, \ldots x_k$ be distinct variables. The set of *hereditarily n-ary λ-terms over $\bar{x}$*, $H_n[\bar{x}]$, is the smallest set of terms such that:

- For all $i = 1, \ldots, k$

$$\frac{}{x_i \in H_n[\bar{x}]}$$

- For all fresh distinct variables $\bar{y} \equiv y_1, \ldots, y_n$,

$$\frac{t_1 \in H_n[\bar{x}, \bar{y}], \ldots, t_n \in H_n[\bar{x}, \bar{y}]}{\lambda y_1 \ldots \lambda y_n . y_i t_1 \ldots t_n \in H_n[\bar{x}]}$$

Examples of hereditarily terms.

- $\lambda x.xx \in H_1 = H_1[]$

Examples of hereditarily terms.

- $\lambda x.xx \in H_1 = H_1[]$
- $\lambda y.yx \in H_1[x]$

Examples of hereditarily terms.

- $\lambda x.xx \in H_1 = H_1[]$
- $\lambda y.yx \in H_1[x]$

$\lambda x.xxx$ is not an hereditarily *n*-ary term.

A hierarchy of sets based on hereditarily terms.

Definition

Let $\bar{x} \equiv x_1, \ldots x_k$ and $\bar{y} \equiv y_1, \ldots, y_n$ be distinct variables.

- $H_n^0[\bar{x}] = H_n[\bar{x}]$

A hierarchy of sets based on hereditarily terms.

Definition

Let $\bar{x} \equiv x_1, \dots x_k$ and $\bar{y} \equiv y_1, \dots, y_n$ be distinct variables.

- $H_n^0[\bar{x}] = H_n[\bar{x}]$
- $H_n^{m+1}[\bar{x}] = \{s[\overline{u}/\overline{y}] : s \in H_n^m[\bar{x}, \bar{y}], \bar{u} \equiv u_1, \dots, u_n \in H_n^m[\bar{x}]\}$

A hierarchy of sets based on hereditarily terms.

Definition

Let $\bar{x} \equiv x_1, \dots x_k$ and $\bar{y} \equiv y_1, \dots, y_n$ be distinct variables.

- $H_n^0[\bar{x}] = H_n[\bar{x}]$
- $H_n^{m+1}[\bar{x}] = \{s[\bar{u}/\bar{y}] : s \in H_n^m[\bar{x}, \bar{y}], \bar{u} \equiv u_1, \dots, u_n \in H_n^m[\bar{x}]\}$
- $S_n[\bar{x}] = \bigcup_m H_n^m[\bar{x}]$.

A new class of mute terms.

Theorem

Given $s_0, \ldots, s_n \in S_n$, *the term* $s_0 \ldots s_n$ *is mute.*

A new class of mute terms.

Theorem

Given $s_0, \dots, s_n \in S_n$, the term $s_0 \dots s_n$ is mute.

Proof.

Sketch: the key point of the proof is that every reduction path can be seen as starting from a term of this form:

$$(\underbrace{\lambda y_1 \dots \lambda y_n}_{n \text{ abstractions}} .y_i \underbrace{t_1 \dots t_n}_{n \text{ terms}}) \underbrace{M_1 \dots M_n}_{n \text{ terms}}$$

with $t_j, M_j \in S_n$

A new class of mute terms.

Theorem

Given $s_0, \ldots, s_n \in S_n$, the term $s_0 \ldots s_n$ is mute.

Proof.

Sketch: the key point of the proof is that every reduction path can be seen as starting from a term of this form:

$$(\underbrace{\lambda y_1 \ldots \lambda y_n}_{n \text{ abstractions}} .y_i \underbrace{t_1 \ldots t_n}_{n \text{ terms}}) \underbrace{M_1 \ldots M_n}_{n \text{ terms}}$$

with $t_j, M_j \in S_n$

This means that at each step the whole term has a shape among those who are allowed for mute terms.

□

Definition

Terms of the form $s_0 \ldots s_n \in S_n$ where s_i belongs to S_n, are called **Regular mute terms.**

Definition

Terms of the form $s_0 \ldots s_n \in S_n$ where s_i belongs to S_n, are called **Regular mute terms.**

$\mathcal{M}_n$ is the set of regular mute terms of the form $s_0 \ldots s_n$.

Definition

Terms of the form $s_0 \ldots s_n \in S_n$ where s_i belongs to S_n, are called **Regular mute terms.**

$\mathcal{M}_n$ is the set of regular mute terms of the form $s_0 \ldots s_n$.

$\mathcal{M}$ is the set of all regular mute.

Definition

Terms of the form $s_0 \dots s_n \in S_n$ where s_i belongs to S_n, are called **Regular mute terms.**

$\mathcal{M}_n$ is the set of regular mute terms of the form $s_0 \dots s_n$.

$\mathcal{M}$ is the set of all regular mute.

Example

- $\Omega \in \mathcal{M}_1$

Definition

Terms of the form $s_0 \dots s_n \in S_n$ where s_i belongs to S_n, are called **Regular mute terms.**

$\mathcal{M}_n$ is the set of regular mute terms of the form $s_0 \dots s_n$.

$\mathcal{M}$ is the set of all regular mute.

Example

- $\Omega \in \mathcal{M}_1$

Definition

Terms of the form $s_0 \ldots s_n \in S_n$ where s_i belongs to S_n, are called **Regular mute terms.**

$\mathcal{M}_n$ is the set of regular mute terms of the form $s_0 \ldots s_n$.

$\mathcal{M}$ is the set of all regular mute.

Example

- $\Omega \in \mathcal{M}_1$
- $(\lambda x.x(\lambda y.yx))(\lambda x.xx) \in \mathcal{M}_1$

Definition

Terms of the form $s_0 \dots s_n \in S_n$ where s_i belongs to S_n, are called **Regular mute terms.**

$\mathcal{M}_n$ is the set of regular mute terms of the form $s_0 \dots s_n$.

$\mathcal{M}$ is the set of all regular mute.

Example

- $\Omega \in \mathcal{M}_1$
- $(\lambda x.x(\lambda y.yx))(\lambda x.xx) \in \mathcal{M}_1$

Definition

Terms of the form $s_0 \ldots s_n \in S_n$ where s_i belongs to S_n, are called **Regular mute terms.**

$\mathcal{M}_n$ is the set of regular mute terms of the form $s_0 \ldots s_n$.

$\mathcal{M}$ is the set of all regular mute.

Example

- $\Omega \in \mathcal{M}_1$
- $(\lambda x.x(\lambda y.yx))(\lambda x.xx) \in \mathcal{M}_1$
- $AAA \in \mathcal{M}_2$, where $A := \lambda xy.x(\lambda zt.tzx)y$.

Definition

Terms of the form $s_0 \dots s_n \in S_n$ where s_i belongs to S_n, are called **Regular mute terms.**

$\mathcal{M}_n$ is the set of regular mute terms of the form $s_0 \dots s_n$.

$\mathcal{M}$ is the set of all regular mute.

Example

- $\Omega \in \mathcal{M}_1$
- $(\lambda x.x(\lambda y.yx))(\lambda x.xx) \in \mathcal{M}_1$
- $AAA \in \mathcal{M}_2$, where $A := \lambda xy.x(\lambda zt.tzx)y$.

BB, where $B := \lambda x.x(\lambda y.xy)$, is a mute term that is not regular.

Regular mute
and
graph models

Semantic of λ-calculus.

Definition

A model of λ-calculus is a reflexive object in a cartesian closed cathegory.

Semantic of λ-calculus.

Definition

A model of λ-calculus is a reflexive object in a cartesian closed cathegory.

Problem

Graph easiness of $\mathcal{M}$:

> *is it possible to find, for every closed term M, a graph models that equates M to every $t \in \mathcal{M}$?*

This is part of a general problem, the analysis of the expressive power of λ-models:

This is part of a general problem, the analysis of the expressive power of λ-models:

> *given a class of* λ*-models, which theories can they express?*

This is part of a general problem, the analysis of the expressive power of λ-models:

given a class of λ-models, which theories can they express?

Graph easiness proves that graph models can express the theory

$$\lambda\beta + \{t = M | t \in \mathcal{M}\}$$

for all closed M.

Graph models.

Definition

A **graph model** is a pair (D, p), where D is an infinite set and $p : \mathcal{P}_{fin}(D) \times D \to D$ is an injective total function.

Graph models.

Definition

A **graph model** is a pair (D, p), where D is an infinite set and $p : \mathcal{P}_{fin}(D) \times D \rightarrow D$ is an injective total function.

Using such pair (D, p) it is possible to define a λ-**model** whose universe is $\mathcal{P}(D)$.

Graph models.

Definition

A **graph model** is a pair (D, p), where D is an infinite set and $p : \mathcal{P}_{fin}(D) \times D \to D$ is an injective total function.

Using such pair (D, p) it is possible to define a λ-**model** whose universe is $\mathcal{P}(D)$.

Interpretation of terms is defined as follows:

- $|x|^p_\rho = \rho(x)$, where $\rho : Var \to \mathcal{P}(D)$ evaluates free variables.
- $|tu|^p_\rho = \{\alpha : (\exists a \subseteq |u|^p_\rho)\;\; p(a, \alpha) \in |t|^p_\rho\}$
- $|\lambda x.t|^p_\rho = \{\, a \to \alpha : \alpha \in |t|^p_{\rho[x:=a]}\}$

Main theorem.

Main theorem.

Theorem

Let M be a closed term. Then, for every $e \subseteq_{\text{fin}} \mathbb{N} \setminus 0$ there exists a graph model (D, I) such that

$$(D, I) \models t = M \text{ for all } t \in \mathcal{M}_e,$$

where $\mathcal{M}_e = \bigcup_{n \in e} \mathcal{M}_n$, the set of n-regular mute terms for $n \in e$.

Forcing.

Definition

(Forcing) For a closed term M, a partial pair (D, q) and $\alpha \in D$, the abbreviation $q \Vdash \alpha \in M$ means that for all total injections $p \supseteq q$ we have that $(D, p) \models \alpha \in |M|^p$.

Forcing.

Definition

(Forcing) For a closed term M, a partial pair (D, q) and $\alpha \in D$, the abbreviation $q \Vdash \alpha \in M$ means that for all total injections $p \supseteq q$ we have that $(D, p) \models \alpha \in |M|^p$.

Lemma

For every closed term M, the function $F_M : \mathcal{I}(D) \to \mathcal{P}(D)$ defined by $F_M(q) = \{\, \alpha \in D : q \Vdash \alpha \in M \}$ is weakly continuous, and we have

$$F_M(p) = |M|^p \text{ for each total } p.$$

Main lemma on mute terms and graph models.

Lemma

Let $F : \mathcal{I}(D) \to \mathcal{P}(D)$ *be a weakly continuous function and let* $e \subseteq_{\mathrm{fin}} \mathbb{N} \setminus 0$*. Then there exists a total* $I : \mathcal{P}_{fin}(D) \times D \to D$ *such that*

$$(D, I) \models t = F(I) \text{ for all terms } t \in \mathcal{M}_e.$$

Proof.

- Given a closed M, using the forcing lemma we get a weakly continuous function F.

Proof.

- Given a closed M, using the forcing lemma we get a weakly continuous function F.
- Using F in the other theorem, we get a total I such that

$$(D, I) \models t = F(I)$$

for all $t \in \mathcal{M}_e$.

Proof.

- Given a closed M, using the forcing lemma we get a weakly continuous function F.
- Using F in the other theorem, we get a total I such that

$$(D, I) \models t = F(I)$$

for all $t \in \mathcal{M}_e$.
- By the forcing lemma, $F(p) = |M|^p$ for all total p. So

$$(D, I) \models t = M$$

□

Ultraproducts

λ-models are first order structures, so we can use the theory of ultraproducts to prove graph easiness of regular mute.

Ultraproducts

λ-models are first order structures, so we can use the theory of ultraproducts to prove graph easiness of regular mute.

- Łoś theorem

Ultraproducts

λ-models are first order structures, so we can use the theory of ultraproducts to prove graph easiness of regular mute.

- Łoś theorem
- (Bucciarelli,Carraro,Salibra)
  Let $(D_j, p_j)_{j \in J}$ be a family of total pairs, $\mathbf{A} = (\mathbf{A}_j : j \in J)$ be the corresponding family of graph λ-models, where $\mathbf{A}_j = (\mathcal{P}(D_j), \cdot, \mathbf{k}, \mathbf{s})$, and let $\mathcal{F}$ be an ultrafilter on J. Then there exists a graph model (E, q) such that the ultraproduct $(\Pi_{j \in J} \mathbf{A}_j)/\mathcal{F}$ can be embedded into the graph λ-model determined by (E, q).

Final theorem.

Theorem

Let M be a closed term and $\mathcal{M} = \bigcup_{n>0} \mathcal{M}_n$ be the set of all regular mute λ-terms. Then there exists a graph model (E, q) such that

$$(E, q) \models M = t \quad \text{for every } t \in \mathcal{M}.$$

Final comments.

Our result is a first step on the investigation of subclasses of mute terms.

Open questions.

- Are regular mute terms easy with respect to other kind of models?

Open questions.

- Are regular mute terms easy with respect to other kind of models?
- Is the set of regular mute a maximal graph easy class?