Optimal placement of storage nodes in a wireless sensor network

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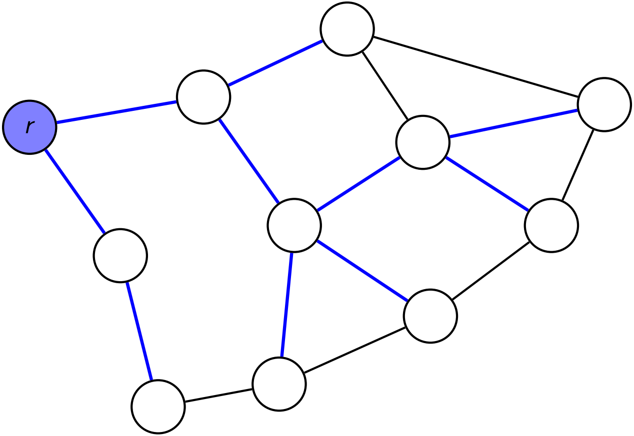
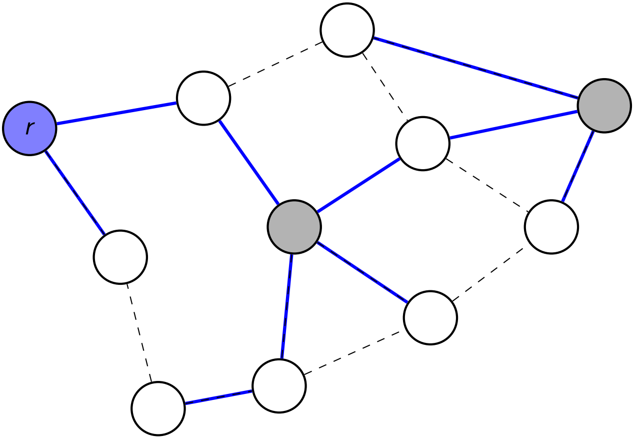
Scenario

- Given a wireless sensor network represented as a graph
- And a special *sink* node r
- All the sensors collect data with a regular frequency and send them to r along the shortest paths



- Alternatively the data can be forwarded to some *storage* nodes
- Storage nodes *compress* and *aggregate* the data, and then send them to the sink (reduced in size)



Given a fixed integer k, how to choose the "best" k storage nodes among the nodes of the network in order to minimize the energy consumption?

Outline

1. The Minimum k-Storage Problem
2. Polynomial-time exact algorithms
3. Hardness of approximation
4. Local search algorithm
5. Experimental analysis
6. Conclusions

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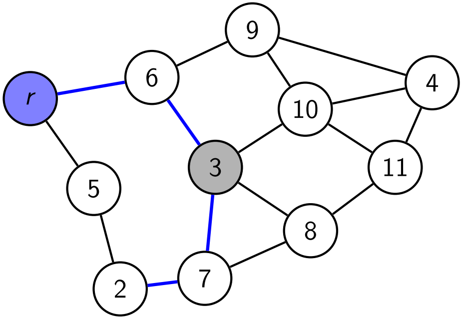
Given:

- a weighted connected graph $G = (V, E, w)$ representing a wireless sensor network where each $v \in V$ generates raw data with size $s_d(v)$
- an integer k.

We aim at finding a set $S \subseteq V$ of *storage nodes* such that $|S| \leq k$

- Each $v \in V$ is associated to a storage node, denoted as $\sigma(v) \in S$
- In $\sigma(v)$, the compressed size of the data produced by a node v becomes $\alpha s_d(v)$, with $\alpha \in [0, 1]$

Total cost: $cost(S) = \sum_{v \in V} s_d(v) \left(d(v, \sigma(v)) + \alpha d(\sigma(v), r) \right)$



For $v = 2$ the cost is:
$s_d(2) \cdot (w(2,7) + w(7,3)) + \alpha \cdot s_d(2) \cdot (w(3,6), w(6,r))$

Total cost: $cost(S) = \sum_{v \in V} s_d(v) \left(d(v, \sigma(v)) + \alpha d(\sigma(v), r) \right)$

The *minimum k-storage problem* (briefly, *MSP*) consists in finding a subset $S \subseteq V$, with $|S| \leq k$ that minimizes $cost(S)$

Related Work

- [Sheng et al. 2007] 10-approximation algorithm for the case
  - $s_d(v)$ is a constant for any v
  - The distances are given by Euclidean distances
- [Sheng et al. 2010] Optimal algorithms for trees
  - Either limited or unlimited k
  - They consider the cost of diffusing the query
  - The algorithms are polynomial only if the degree of the tree is bounded

Our results

- Polynomial-time exact algorithms
  - For trees in directed graphs
  - For bounded-treewidth undirected graphs
- Approximation lower bounds
  - Not in *APX* in directed graphs
  - $1 + \frac{1}{e} > 1.367$ for undirected graph
- Local search algorithm for undirected graphs with constant approximation ratio
- Experimental evaluation of such algorithm on several graph topologies

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Directed trees

Idea: Transform the generic rooted tree into an equivalent binary tree
We devise a dynamic programming algorithm for binary trees

Theorem

Given a directed tree T, there exists an algorithm that optimally solves MSP in $O(\min\{kn^2, k^2P\})$, where P is the path-length *of T.*

Path-length: Sum over the whole tree of the number of arcs on the path from each tree node to the root

- Balanced binary tree: $P = \Theta(n \log n)$,
- Random tree: $P = \Theta(n\sqrt{n})$
- Worst case: $P = O(n^2)$

Undirected graph

We exploit the concept of tree decomposition to devise a dynamic programming algorithm

Theorem

Given an undirected graph G and a tree-decomposition of G with width w, there exists an algorithm that optimally solves MSP in $O(w \cdot k \cdot n^{w+3})$ time.

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We show that *MSP* in undirected graphs cannot be approximated within a factor of $1 + \frac{1}{e}$, unless $P = NP$

In detail,

- We show that the metric k-median problem cannot be approximated within a factor of $1 + \frac{1}{e}$, unless $P = NP$
- We show that *MSP* is at least as hard to approximate as the metric k-median problem

The metric k-median problem

Let

- $G = (V, E)$ be a complete graph
- $k \in \mathbb{N}$
- $dist(u, v) \in \mathbb{N}$ be the distance from u to v over the edge $(u, v) \in E$

A k-median set for G is a subset $V' \subseteq V$ with $|V'| \leq k$

The *minimum k-median problem* consists in finding a k-median set V' that minimizes

$$\sum_{u \in V} \min_{v \in V'} dist(u, v)$$

In the minimum *metric* k-median problem (briefly, *MMP*) the distance function is symmetric and satisfies the triangle inequality

Theorem

There is no approximation algorithm for the metric minimum k-median problem with approximation factor $\gamma < 1 + \frac{1}{e}$*, unless* $P = NP$*.*

Sketch of the proof:

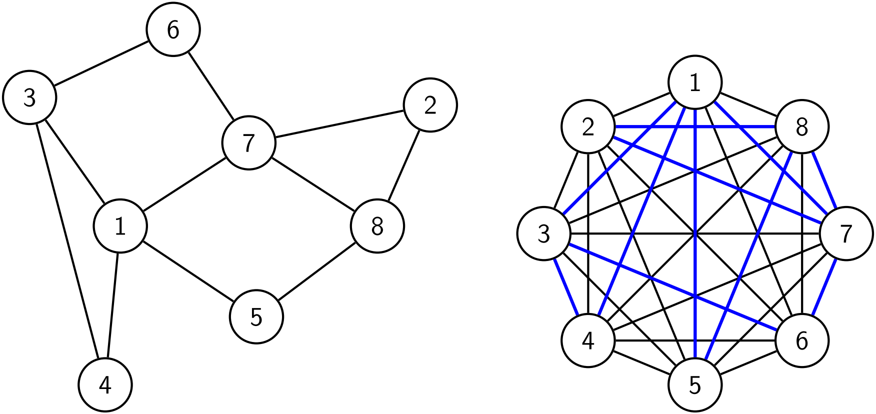
It is based on an approximation factor preserving reduction from the minimum dominating set problem

Let $G = (V, E)$ be an undirected graph, a dominating set for G is a subset $V' \subseteq V$ such that for each $u \in V \setminus V'$ there is a $v \in V'$ for which $\{u, v\} \in E$

The *minimum dominating set problem* consists in finding the minimum cardinality dominating set

Given an instance of the minimum dominating set problem, we define an instance of the minimum metric k-median problem with $G' = (V, E')$, $E' = V \times V$ and

$$dist(u, v) = \begin{cases} 1 & \text{if } \{u, v\} \in E \\ 2 & \text{otherwise.} \end{cases}$$



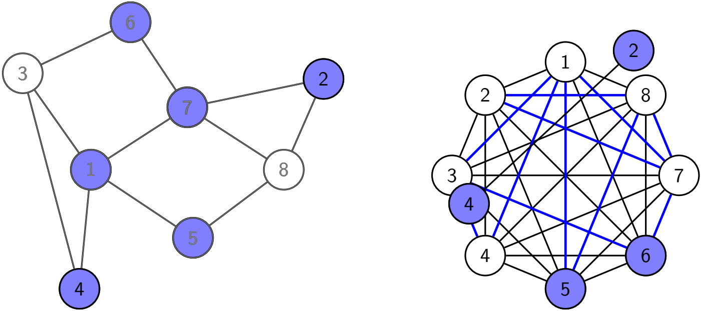
Let us assume that there exists an approximation algorithm γ-*MMP* with approximation factor γ for *MMP*

Let us suppose that the size k of an optimal dominating set is known

We devise an algorithm for the minimum dominating set

- Select a set of size k by applying γ-*MMP* with parameter k
- Remove the nodes in the graph corresponding to the chosen set and their neighbors
- Repeat until all the nodes are covered

$k = 2$



Let λ be the number of iterations (the number of times that we apply γ-*MMP*)

At each iteration we selected k nodes

We selected $k \cdot \lambda$ nodes

As k is the value of the optimal solution, λ is the approximation ratio of the algorithm for the minimum dominating set problem

We give an upper bound for λ:

After the first iteration,

- there are k selected nodes, d_1 nodes covered *directly* (with weight 1), i_1 nodes covered *indirectly* (with weight 2), $k + d_1 + i_1 = |V| = n$
- The cost for *MMP* is $d_1 + 2i_1 \leq \gamma OPT \leq \gamma(n-k)$
- Therefore, $i_1 \leq (n-k)(\gamma - 1) \leq n(\gamma - 1)$

After $\lambda - 1$ iterations there are at most $n(\gamma-1)^{\lambda-1} = \eta$ uncovered nodes, for some $1 \leq \eta \leq n$, and then, $\lambda - 1 = \log_{(\gamma-1)} \frac{\eta}{n} \leq \log_{(\gamma-1)} \frac{1}{n} = \frac{\ln n}{\ln \frac{1}{\gamma-1}}$

Cannot exists a $(c \ln n)$-approximation algorithm for the minimum dominating set for each $c < 1$, unless $P = NP$

Therefore, $\frac{1}{\ln \frac{1}{\gamma-1}} \geq 1$ which implies $\frac{1}{\gamma-1} \leq e$, and hence $\gamma \geq 1 + \frac{1}{e}$

Theorem

MSP is at least as hard to approximate as the metric k-median problem.

Corollary

There is no approximation algorithm for MSP with approximation factor $\gamma < 1 + \frac{1}{e}$, unless $P = NP$.

Theorem

For directed graphs, MSP does not belong to APX, unless $P = NP$.

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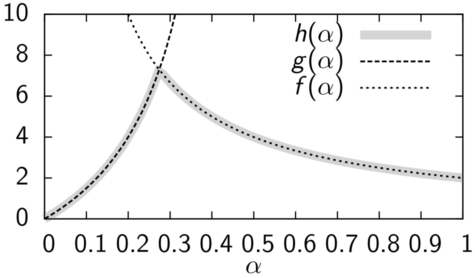
We define a local search algorithm as follows

- Given any initial solution S_0
- *Swap operation* of $t \leq |S|$ nodes:
  - remove t nodes from S and add t nodes in $V \setminus S$ to S
- If any swap move yields a solution of lower cost the resulting solution is set to be the new current solution
- Repeat until from the current solution no swap operation decreases the cost
- The solution found represents a local optimum

We first analyze the case of $t = 1$: a swap is defined between two nodes $s \in S$ and $s' \in V \setminus S$ and consists in adding s' and removing s

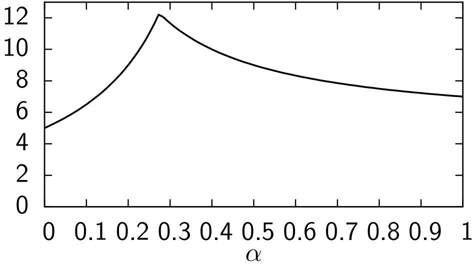
Let us define
$f : (0, 1] \to \mathbb{R}$, $f(\alpha) = \frac{2}{\alpha}$
$g : [0, \frac{1}{2}) \to \mathbb{R}$, $g(\alpha) = \frac{12\alpha}{1-2\alpha}$
$h : [0, 1] \to \mathbb{R}$

$$h(\alpha) = \begin{cases} g(\alpha) & \text{if } \alpha = 0 \\ \min\{f(\alpha), g(\alpha)\} & \text{if } \alpha \in (0, \frac{1}{2}) \\ f(\alpha) & \text{if } \alpha \in [\frac{1}{2}, 1] \end{cases}$$



Theorem

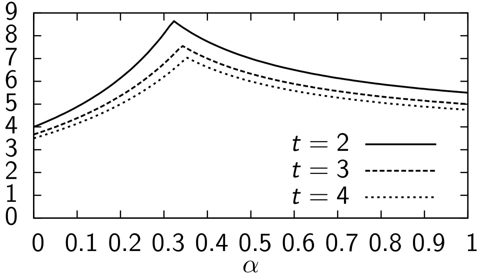
The local search algorithm for MSP exhibits a locality gap of at most $5 + h(\alpha)$.



Maximum: ≈ 12.3 for $\alpha \approx 0.274$ where $f(\alpha) = g(\alpha) \approx 7.3$

We can generalize for $t \geq 1$: the locality gap is $h'(\alpha)$, where
$f' : (0,1] \to \mathbb{R}$, $f'(\alpha) = 1 + \frac{t+1}{t}\frac{1+2\alpha}{\alpha}$
$g' : [0, \frac{t}{t+1}] \to \mathbb{R}$, $g'(\alpha) = \frac{(3+\alpha)t+2+\alpha}{(1-\alpha)t-\alpha}$
$h' : [0,1] \to \mathbb{R}$

$$h'(\alpha) = \begin{cases} g'(\alpha) & \text{if } \alpha = 0 \\ \min\{f'(\alpha), g'(\alpha)\} & \text{if } \alpha \in (0, \frac{t}{t+1}) \\ f'(\alpha) & \text{if } \alpha \in [\frac{t}{t+1}, 1] \end{cases}$$



Maximum: ≈ 8.67, 7.78 and 7.05

The algorithm can have a superpolynomial number of iterations

We change the stopping condition: it finishes as soon as it finds a solution S is such that every neighboring solution S' of S has $cost(S') > (1-\epsilon)cost(S)$, for some $\epsilon > 0$

The number of iterations is at most $\frac{\log(\frac{cost(S_0)}{cost(S^*)})}{\log(\frac{1}{1-\epsilon})}$, where S_0 is the initial solution

Corollary

There exists an $\frac{1}{1-\epsilon}h'(\alpha)$-approximation algorithm MSP for any $\epsilon \in (0,1)$.

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Environment

We implemented the local search algorithm in C++ (gcc)
We compared the solution found with the optimal one obtained by an IP formulation (GLPK solver)

$$
\begin{array}{lll}
\min & \sum_{v,s\in V} x_{vs}\cdot s_d(v)(d(v,s)+\alpha d(s,r)) & \\
s.t. & \sum_{s\in V} x_{vs} = 1 & \text{for each } v\in V \\
 & x_{vs} \leq y_s & \text{for each } v,s\in V \\
 & \sum_{s\in V} y_s \leq k, & \\
 & y_r = 1 & \\
 & y_s, x_{vs} \in \{0,1\} & \text{for each } v,s\in V
\end{array}
$$

Input instances

Types of graphs:

- **Random geometric graphs** (RGG) $n \in \{100, 300, 1000\}$
- **Barabasi-Albert graphs** (BA) $n \in \{100, 300, 1000\}$
- **OR Library** (PMED) $100 \le n \le 900$
- **Erdős-Rényi random graphs** (ER) $n \in \{100, 150\}$

Other parameters:

- The sink node is chosen uniformly at random
- $\alpha \in \{0.0, 0.1, \ldots, 1\}$ (11 values)
- $k \in \{1, \ldots, n\}$ (30 values with step $\lfloor n/30 \rfloor$)
- $s_d(v)$ uniformly at random in the interval $[1, 10]$, independently for each $v \in V$
- ϵ in $\{0.005, 0.01, 0.1\}$
- $t = 1$ (worst case for the algorithm's approximation ratio)

Random geometric graphs – approximation ratio

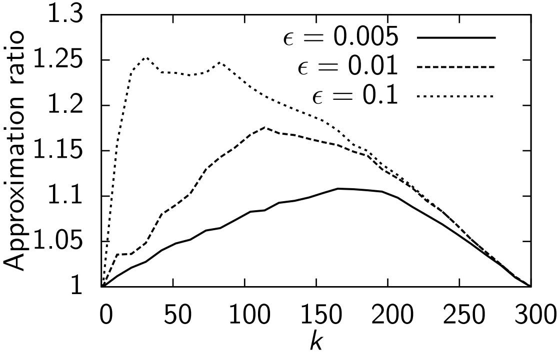

Figure : Random Geometric Graphs $n = 300$, $\alpha = 0.1$

- The ratio decreases with ϵ, for $\epsilon = 0.005$ it is < 1.108
- When k is small, the approximation ratio is reduced, for $k < 100$, it is < 1.07
- When k is big, the approximation ratio is reduced, for $k > 250$, it is < 1.05

Random geometric graphs – number of iterations

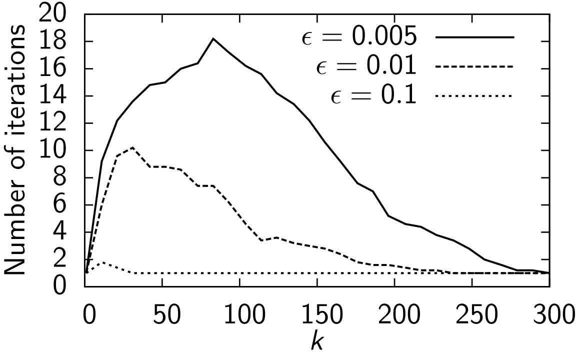

Figure : Random Geometric Graphs $n = 300$, $\alpha = 0.1$

- Decreasing ϵ increases the number of iterations
- The good values for small k require up to 18 iterations
- The good values for big k require up to 2 iterations

Random geometric graphs – approximation ratio

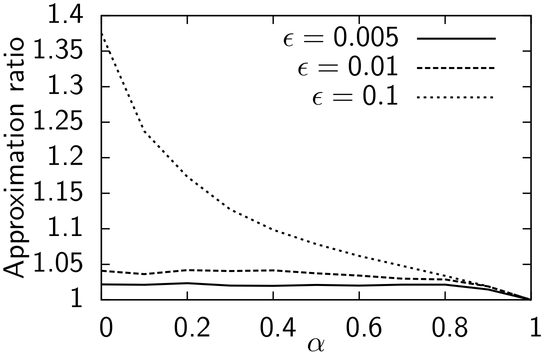

Figure : Random Geometric Graphs $n = 300$, $k = 21$

- For small values of ϵ the ratio is very small, < 1.023 for $\epsilon = 0.005$ and < 1.042 for $\epsilon = 0.01$
- Maximum value: 1.38, obtained when $\alpha = 0$

Random geometric graphs – number of iterations

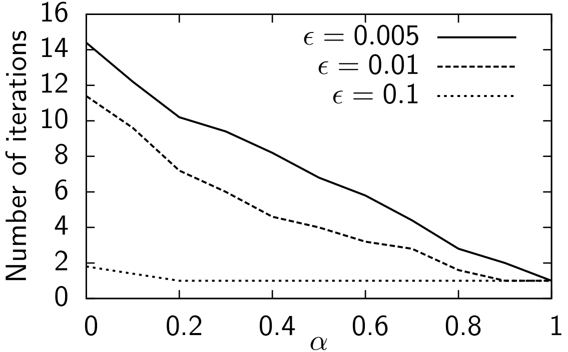

Figure : Random Geometric Graphs $n = 300$, $k = 21$

- The good values of the approximation ratio required more iterations if α is small
- When α approaches 1, then the usage of storage nodes does not significantly decrease the objective function and hence the first feasible solution already has a good approximation ratio

Other topologies

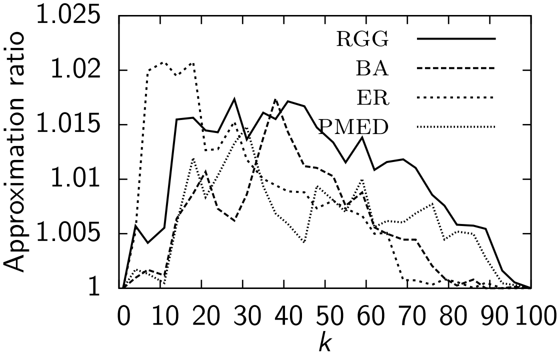

Figure : Graph type comparison $n = 100$, $\alpha = 0.1$, $\epsilon = 0.005$

- We do not observe any significant difference with respect to the type of graph
- In these cases the approximation ratio is smaller than the previously reported ones

Computational time

Graph Type	n	k	Time per iteration (sec)
RGG	100	50	0.0121
	300	150	0.3274
	1000	500	15.0086
BA	100	50	0.0122
	300	150	0.3291
	1000	500	14.2448

Graph Type	n	k	Time per iteration (sec)
PMED	100	50	0.0116
	300	150	0.3191
	900	450	10.9442
ER	100	50	0.0137
	150	75	0.0392

Table : Average computational time required for each iteration when $k = n/2$

The computational time of the iterations in the extreme cases, i.e. $k = 1$ or $n = k$ is always < 0.0001

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Conclusions

- We studied the minimum k-storage problem from the theoretical and experimental viewpoints
- Directed graphs:
  - There exists a polynomial-time exact algorithm for trees
  - The problem is not in APX
- Undirected graph:
  - There exists a polynomial-time exact algorithm for bounded-treewidth graphs
  - The problem is not approximable within a factor of $1 + \frac{1}{e}$, unless $P = NP$
  - There exists a constant-factor polynomial-time approximation algorithm based on local search
  - This algorithm performs very well in practical scenarios

Thank you for your attention