Three-variable statements of set-pairing*

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Abstract

The approach to algebraic specifications of set theories proposed by Tarski and Givant inspires current research aimed at taking advantage of the purely equational nature of the resulting formulations for enhanced automation of reasoning on aggregates of various kinds: sets, bags, hypersets, etc. The viability of the said approach rests upon the possibility to form ordered pairs and to decompose them by means of conjugated projections. Ordered pairs can be conceived of in many ways: along with the most classic one, several other pairing functions are examined, which can be preferred to it when either the axiomatic assumptions are too weak to enable pairing formation *à la* Kuratowski, or they are strong enough to make the specification of conjugated projections particularly simple, and their formal properties easy to check within the calculus of binary relations.

Keywords: Set theory, Calculus of binary relations, Pairing, Automated reasoning, Aggregates.

Dedicated to Denis Richard

1 Introduction and background

> *In the first place, there were three kinds of human beings, not merely the two sexes, male and female, as at present: there was a third kind as well, which had equal shares of the other two, $\cdots$. Secondly, the form of each person was round all over, with back and sides encompassing it every way, $\cdots$. Terrible was their might and strength, and the thoughts of their hearts were great, that they even conspired against the gods.*
> —Plato, Symposium, The Speech of Aristophanes (189a-193e)

In his epochal paper [44], Zermelo calls *axiom of elementary sets* a postulate asserting that:

- there is a set, $\emptyset$, which is devoid of elements;
- a singleton set $\{x\}$ can be formed out of any object x of the domain of discourse; and, more generally,
- an unordered pair $\{x, y\}$ can be formed out of objects x, y whatsoever.

In the original list of postulates for set theory proposed by Zermelo, this postulate occupies the second position, after the *extensionality* axiom asserting that distinct sets cannot have precisely the same elements.

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Between 1914 and 1921, Norbert Wiener, Felix Hausdorff, and Kazimierz Kuratowski devised encodings of ordered pairs based on unordered pairs (cf. [43, 31, 33]), such as

$$\llcorner\, x,\, y\, \lrcorner \;\; =_{\text{Def}} \;\; \big\{\{\{x\},\emptyset\},\{\{y\}\}\big\};$$
$$\big(\,x,\, y\,\big) \;\; =_{\text{Def}} \;\; \big\{\{\,x,y\,\},\{\,x\,\}\big\}.$$

Let us place ourselves in the framework of a set theory which does not cater for individuals or proper classes: then extensionality can be stated as simply as

$$\textbf{(E)} \qquad \forall\, x\, \forall\, y\, \big(\, x \neq y \;\rightarrow\; \exists\, v\, (\, v \in x \;\leftrightarrow\; v \notin y\,)\big),$$

and Zermelo's postulate of elementary sets can be decomposed as the conjunction of the following *null-set axiom* and *axiom of unordered pairs*:

$$\textbf{(N)} \qquad \exists\, z\, \forall\, v\; v \notin z, \qquad\qquad \textbf{(P)} \qquad \forall\, x\, \forall\, y\, \exists\, p\, \forall\, v\, \big(\, v \in p \;\leftrightarrow\; (\, v = x \;\vee\; v = y\,)\big).$$

Several studies (cf., among many others, [13, 29, 27]) indicate the number of distinct variables to be a significant measure of complexity for sentences. From this angle, one may be led to think that **(P)** is somewhat deeper than **(E)**, because it involves 4 variables instead of 3. Alfred Tarski, however, discovered a sentence **(OP)** which is logically equivalent to **(P)**, involves three variables altogether, and explicitly asserts the existence of *ordered pairs* (cf. [11, pp. 341–343], [41], and [42, p. 129]). We will recall how Tarski succeeded in formulating **(OP)** in three variables in Sec.3.

As Tarski already pointed out in the early 1950s and then discussed in depth in [42], an important by-product of having the elementary set postulate recast in three variables is that any first-order theory of sets to which **(N)** and **(P)** belong (either as axioms or as theorems) can, through this rendering, be translated into the algebraic formalism which developed in the 1940s (cf. [40, 28, 11]) from the far-reaching studies on logic carried out by Peirce and Schröder in the late 19^{th} and early 20^{th} century. Recently, this approach to the algebraic formalization of set theory inspired some research aimed at automating equational set-reasoning (cf. [21, 16, 22]).

In this paper we will consider a version of the elementary set postulate which is a bit stronger than the one, $\textbf{(N)} \wedge \textbf{(P)}$, discussed above. In conjunction with **(N)**, our postulate has clauses catering for the single-element insertion and removal operations $x, y \overset{\text{with}}{\longmapsto} x \cup \{y\}$ and $x, y \overset{\text{less}}{\longmapsto} x \setminus \{y\}$:

$$\textbf{(W)} \qquad \forall\, x\, \forall\, y\, \exists\, w\, \forall\, v\, \big(\, v \in w \;\leftrightarrow\; (\, v \in x \;\vee\; v = y\,)\big),$$
$$\textbf{(L)} \qquad \forall\, x\, \forall\, y\, \exists\, \ell\, \forall\, v\, \big(\, v \in \ell \;\leftrightarrow\; (\, v \in x \;\wedge\; v \neq y\,)\big).$$

Taking advantage of the presence of **(E)**, exploiting a notion of ordered pair which slightly differs from the one due to Kuratowski, and proceeding in a way similar (but much simpler) to the way **(P)** was restated as **(OP)**, we will succeed in recasting $\textbf{(N)} \wedge \textbf{(W)} \wedge \textbf{(L)}$ as a 3-variable sentence.

More generally, we will consider various weak theories of sets which result from adopting as axioms some of the sentences in Figure 1. These sentences are provable within important classic theories of sets: e.g. within full Zermelo-Fraenkel (where **(P)** is sometimes deduced from the *replacement axiom scheme* as shown in [30, pp. 9–10]); or within Tarski's theory [39] of finite sets (equipollent to Peano arithmetic, cf. [42]).

Our theories hence retain, in the small, valuable traits. On the other hand, by leaving some of the sentences in Figure 1 out of our selection of axioms, we can frame our investigation inside less classic but nevertheless useful variants of set theory: recall that *bags* (also called *multi-sets*, cf. e.g. [14]) do not meet extensionality, **(E)**, and *hypersets* (cf. [1, 2]) meet neither regularity, **(R)**, nor the weaker acyclicity assumption $(\mathbf{A}^n)$. The theory consisting solely of $(\mathfrak{I})$, $(\mathfrak{D})$, and $(\mathfrak{P})$, is known to be an ideal target first-order theory into which to translate modal systems of propositional logic (cf. [12, 3, 4]): in the translation, the power-set operator $\mathscr{P}(\cdot)$ corresponds to the necessity operator $\Box$. From the standpoint of this '$\Box$-*as*-$\mathscr{P}$' translation, the weakness of the axiomatic system is a virtue rather than a defect: if, e.g., **(E)** were postulated too, this would set an undesirable limitation to the usability of this theory in the study of non-classic logics.

(E)	$\forall x \forall y \big(\forall v (v \in x \leftrightarrow v \in y) \rightarrow x = y\big)$
(N)	$\exists z \forall v \neg v \in z$
(P)	$\forall x \forall y \exists p \forall v \big(v \in p \leftrightarrow (v = x \vee v = y)\big)$
(W)	$\forall x \forall y \exists w \forall v \big(v \in w \leftrightarrow (v \in x \vee v = y)\big)$
(L)	$\forall x \forall y \exists \ell \forall v \big(v \in \ell \leftrightarrow (v \in x \wedge \neg v = y)\big)$
($\mathfrak{I}$)	$\forall x \forall y \exists w \forall v \big(v \in w \leftrightarrow (v \in x \wedge v \in y)\big)$
($\mathfrak{D}$)	$\forall x \forall y \exists d \forall v \big(v \in d \leftrightarrow \neg(v \in x \leftrightarrow v \in y)\big)$
($\mathfrak{P}$)	$\forall x \exists p \forall v (v \in p \leftrightarrow v \subseteq x)$
(R)	$\forall x \exists r \big((r \in x \vee r = x) \wedge \neg\exists v (v \in r \wedge v \in x)\big)$
($\mathbf{A}^n$)	$\forall x_0 \cdots \forall x_n (x_0 \in x_1 \wedge \cdots \wedge x_{n-1} \in x_n \rightarrow \neg x_n \in x_0)$

$n = 0, 1, 2, \ldots$

Figure 1: Toolkit for assembling weak theories of aggregates

We will seek *pairing notions* which are easily amenable to a 3-variable formulation under different (and inequivalent) selections of the axioms. As mentioned above, one reason for undertaking this quest, is that any such pairing notion can be used as the keystone of an *equational* variable-free rendering of the theory under focus, or of any axiomatic extension of it. Indeed, a fully equipollent axiomatic system of the theory can be obtained via a classic translation (cf. Figure 5 in Sec.4) of first-order predicate logic into the Schröder-Tarski *calculus of binary relations*.

The pairing notion revolves around *conjugated (quasi-)projections*: one names so two functions ℓ, r which are so defined on the universe $\mathcal{V}$ of sets (not necessarily on the whole of it) as to ensure that for any given sets a_0, a_1 there is at least one set b such that $\ell(b) = a_0$ and $r(b) = a_1$. Before proceeding to the definition of ℓ and r, one usually has in mind a specific *pairing operation* $p(\cdot,\cdot)$ by which the desired b can be found out of given a_0, a_1 simply by determining $b = p(a_0, a_1)$; notice, however, that b is not required to be unique in general. Formally, in the calculus of relations (cf. Sec.4) the single-valuedness and pairing properties which ℓ, r must fulfill can be stated as follows:

$$(\ell^{\smile} \circ \ell) - \boldsymbol{\iota} = \emptyset, \qquad (r^{\smile} \circ r) - \boldsymbol{\iota} = \emptyset, \qquad \ell^{\smile} \circ r = \mathbb{1}.$$

Whenever one proposes a concrete specification of projections ℓ and r, one must either postulate or *prove within* the calculus of relations that ℓ, r meet these conditions.

Through Skolemization of the first-order sentences **(N)**, **(W)**, **(L)**, **($\mathfrak{I}$)**, **($\mathfrak{D}$)**, and **($\mathfrak{P}$)**, one brings into play: a constant, namely $\emptyset$, which designates a void set; dyadic operation symbols, namely $\cdot$ with $\cdot$ and $\cdot$ less $\cdot$, which designate single-element insertion and removal; additional dyadic operation symbols, namely $\cdot \cap \cdot$ and $\cdot\triangle\cdot$, which designate intersection and symmetric difference; and a monadic symbol, $\mathscr{P}(\cdot)$, which designates the power-set operation. Further dyadic operations can then be introduced as follows:

$$\{x, y\} =_{\text{Def}} (\emptyset \text{ with } x) \text{ with } y \quad (\text{and } \{x\} =_{\text{Def}} \{x, x\}), \tag{1}$$

$$(x, y) =_{\text{Def}} \big\{\{x, y\}, \{x\}\big\}, \tag{2}$$

$$x \circledast y =_{\text{Def}} \{x \text{ less } y,\ x \text{ with } y\}, \tag{3}$$

$$\langle x, y\rangle =_{\text{Def}} \{y\} \circledast x, \tag{4}$$

$$[x, y] =_{\text{Def}} (x \circledast y) \circledast x, \tag{5}$$

$$[\![x, y]\!] =_{\text{Def}} x \text{ with } \big(y \text{ with } (y \text{ with } x)\big), \tag{6}$$

$$\ulcorner x, y\urcorner =_{\text{Def}} \Big\{\{x\}, \big\{\{x\}, \{y, \{y\}\}\big\}\Big\}, \tag{7}$$

$$\wr x, y\int =_{\text{Def}} \text{ if } x = y \text{ then } \mathscr{P}(x) \text{ else } \mathscr{P}(x)\triangle\mathscr{P}(y) \text{ fi}, \tag{8}$$

$$(\!|x, y|\!) =_{\text{Def}} \wr\wr x, y\int, \wr x, x\int\int. \tag{9}$$

Of these, (2), (4), (5), (6), and (7) can be regarded as acceptable pairing operations (cf. Sec.2) in a full-fledged set theory, whereas one cannot retrieve unambiguously x or y from either $\{x, y\}$ or $\wr x, y\int$ (e.g., $\{x, y\}$ equals $\{y, x\}$ under extensionality), and only y can be retrieved with certainty from $x \circledast y$ (x could be either one of the elements of $x \circledast y$). As regards (9), its acceptability depends on how cleverly one defines the $\subseteq$ relator, as we will discuss in Sec.6.

By and large, formulating our set axioms in the calculus of relations amounts to finding ways of enforcing, via equalities, that specific pairs ℓ, r of relations constitute conjugated projections—jointly "inverse", in a sense, to an acceptable pairing operation. Historically, what Tarski did in order to provide a 3-variable statement of **(P)** was simply stating that specific relations $\boldsymbol{\pi}_0$, $\boldsymbol{\pi}_1$ (associated with Kuratowski's operation (2)) meet the above mentioned specification of conjugated projections, in particular the condition $\boldsymbol{\pi}_0{}^{\smile} \circ \boldsymbol{\pi}_1 = \mathbb{1}$.

What if we want to state **(W)**, **(L)** in the calculus of relations taking advantage of the assumed availability of **(E)** but without resorting to **(P)**? We have shown elsewhere [18, 19] (thus deepening the result by Michael K. Kwatinetz [32, pp. 55–57]) that **(W)** $\wedge$ **(L)** $\wedge$ **(E)**, taken in isolation from any other axiomatic assumption, cannot be stated in the calculus of relations, insofar as this conjunction is not expressible in three variables. On the other hand, as soon as we add **(N)** to the conjunction, we are under assumptions stronger than **(P)**, and therefore there is hope that we can find a better relational rendering of **(W)** $\wedge$ **(L)** than the one made possible by the formulation of **(P)** mentioned at the end of the preceding paragraph. The comparative ease with which we can achieve such a specification by referring to the operation (5), rather than referring to (4) or (2), indicates that $[x, y]$ is in a sense the best pairing operation, in a weak set theory which one wants to encompass as provable statements **(E)**, **(N)**, **(W)**, and **(L)**. We will discuss this issue in Sec.8.

If we try to withdraw **(N)** from the provable statements, it turns out that $(\mathbf{A}^5)$, in conjunction with **(E)**, can surrogate it in paving the way to the relational rendering of **(W)** $\wedge$ **(L)** (in this case we can rely on the pairing operation $[\![x, y]\!]$, cf. Sec.10). Less importantly (we treat it mainly as a curiosity, cf. Sec.11), **(R)** conjoined with **(N)** $\wedge$ **(W)** $\wedge$ **(L)**, even in the absence of **(E)**, makes the pairing function $\ulcorner x, y \urcorner$ a viable alternative to the classic Kuratowski's operation (x, y), as well as to the operations $\langle x, y \rangle$ and $[x, y]$ proposed by us in this paper, which enter into competition with (x, y) only when **(E)** takes part in the game.

Another special case in which we cannot rely on **(E)**, is the conjunction $(\mathfrak{I}) \wedge (\mathfrak{D}) \wedge (\mathfrak{P})$: as already mentioned, this theory bridges non-classic propositional logics with classic first-order predicate calculus. By singling out projections associated with the pairing operation (9)—cf. Sec.6—, we will hence pave the way to equational rendering of modal propositional calculi.

We acknowledge the assistance of Otter [35], a theorem-prover from the Argonne National Laboratory, in the somewhat slippery algebraic manipulations needed to perform our theoretical exploration reported in this paper. The experimental side of our investigation on set-pairing, which will occasionally emerge (see also, [22, 23, 17]), will be the main focus of Sec.7.

2 Correctness verifications for the proposed pairing functions

Let us briefly state here a basilar result due to Kuratowski in somewhat general terms:

Theorem 1 *Assume that the dyadic constructs $d(\cdot,\cdot)$ and $\cdot\, E \,\cdot$ represent a function and a relation satisfying the condition*

$$v\, E\, d(x, y) \leftrightarrow (v = x \,\vee\, v = y).$$

Then, the function $d(d(x, y), d(x, x))$ satisfies the pairing condition

$$d(d(x, y), d(x, x)) = d(d(u, v), d(u, u)) \rightarrow (x = u \,\wedge\, y = v).$$

Proof. Given a set p of the form $p = d((x, y), d(x, x))$, we can determine its first component x as the unique set v such that $(\forall\, z\, E\, p)(v\, E\, z)$. If there is no set $w \neq x$ satisfying the condition $(\exists\, z\, E\, p)(w\, E\, z)$, then we can conclude that the second component y of p coincides with the already determined x. Otherwise, after observing that there are at most two sets w such that $(\exists\, z\, E\, p)(w\, E\, z)$, i.e., formally, that $\exists\, w_0\, \exists\, w_1\, \big(\forall\, w\, ((\exists\, z\, E\, p)(w\, E\, z) \leftrightarrow (w = w_0 \,\vee\, w = w_1))\big)$, we can determine y as the only set $u \neq x$ such that $(\exists\, z\, E\, p)(u\, E\, z)$. $\square$

Careful examination of the above proof that it only relies on first-order principles: not even the extensionality axiom intervenes in it. On the other hand, save for the case of the pair $(\cdot,\cdot)$, the theorems which follow rely on set-theoretic postulates which we will point out in full only much later, namely in Sections 8, 10, and 11. This enables us to keep the exposition intuitive for the time being.

The following theorem contains a corollary of the preceding one, and two variants of it which follow from set-theoretic axioms which we are leaving as understood to keep our presentation simple and intuitive.

Theorem 2 *The functions $x, y \mapsto (x, y)$, $x, y \mapsto \langle x, y\rangle$, and $x, y \mapsto [x, y]$ satisfy the pairing conditions*

$$(x, y) = (u, v) \rightarrow (x = u \wedge y = v).$$
$$\langle x, y\rangle = \langle u, v\rangle \rightarrow (x = u \wedge y = v),$$
$$[x, y] = [u, v] \rightarrow (x = u \wedge y = v).$$

Proof. In the case of $(\cdot,\cdot)$ the desired result ensues immediately from Theorem 1 in view of how this operation is defined from the function $\{\cdot,\cdot\}$ resulting from the Skolemization of **(P)**. The other two cases are settled as follows. We readily have that any set of the form $z \circledast w$ owns exactly two elements, one of which owns w as a member whereas the other does not. Moreover, one of these two elements is z; and w plainly is the only entity belonging to one and only one of them. Both sets $\langle x, y\rangle$ and $[x, y]$ have the form $\cdot \circledast x$, and hence we can uniquely retrieve x from either of them. As concerns y, we must argue differently in the two cases, referring to the respective definitions. In the former case we observe that $\{y\}$, which determines y uniquely, must be a member and actually the sole singleton member of $\langle x, y\rangle$. In the latter case, we observe that $[x, y]$ consists of a singleton (to which x does not belong) and the doubleton $x \circledast y$; it is easy, hence, to determine $x \circledast y$ and, subordinately, y also in this case. □

In preparation for another similar theorem regarding two other pairing notions, we prove the following:

Lemma 3 *Let n be any fixed natural number. Assuming the acyclicity of membership, no set p can have more than one member s such that for any $v \in p \;\mathsf{less}\; s$ there exist $x_1, \ldots, x_n$ satisfying $v \in x_1 \in \cdots \in x_n \in s$.*

Proof. Assuming by contradiction that p, s, t and $x_1, \ldots, x_n, y_1, \ldots, y_n$ are such that $s \in p$, $t \in p$, $s \neq t$ hold along with $t \in x_1 \in \cdots \in x_n \in s$ and $s \in y_1 \in \cdots \in y_n \in t$, we would come to the conclusion that s occurs in a membership cycle, conflicting with the acyclicity assumption $(\mathbf{A}^{2\cdot n+1})$. □

Corollary 4 *The functions $x, y \mapsto [\![x, y]\!]$ and $x, y \mapsto \ulcorner x, y\urcorner$ satisfy the respective pairing conditions*

$$[\![x, y]\!] = [\![u, v]\!] \rightarrow (x = u \wedge y = v),$$
$$\ulcorner x, y\urcorner = \ulcorner u, v\urcorner \rightarrow (x = u \wedge y = v).$$

Proof. By inspection of the definition of $[\![x, y]\!]$, and recalling Lemma 3, we see that $y \;\mathsf{with}\; (y \;\mathsf{with}\; x)$ is the only member s of $[\![x, y]\!]$ such that $v \in x_1 \in x_2 \in x_3 \in s$ holds (for suitable x_1, x_2, x_3) for any other member v of $[\![x, y]\!]$. To retrieve x from $[\![x, y]\!]$ it clearly suffices to determine $[\![x, y]\!] \;\mathsf{less}\; s$. Then, after similarly observing that there is only one member t of s such that w belongs to some element of t for any other member w of s, namely $t = y \;\mathsf{with}\; x$, we can retrieve the second component y of $[\![x, y]\!]$ by determining $s \;\mathsf{less}\; t$.

Likewise, by inspection of the definition of $\ulcorner x, y\urcorner$, which is a special doubleton, we see that $\{x\}$ can be determined as the only member that belongs to the other member of $\ulcorner x, y\urcorner$. Then, after exploiting our knowledge of $\{x\}$ to determine both x and $\{y, \{y\}\}$, we can again determine y as the only member of $\{y, \{y\}\}$ which belongs to the other member. □

Let us postpone the proof of the pairing condition

$$(\!|x, y|\!) = (\!|u, v|\!) \rightarrow (x = u \wedge y = v)$$

to Sec.6, because this is somewhat subtler than the proofs supplied above. For the time being, we just say that this proof will have a close analogy with the verification regarding the “standard” pair $(\cdot,\cdot)$ carried out in Theorem 2. However, in exploiting Theorem 1 (with $\wr\cdot,\cdot\wr$ in place of $d(\cdot,\cdot)$) for that proof, one clearly cannot take E to be $\in$. The set $\wr x, y\wr$, in fact, does not necessarily contain both x and y as elements; even worse, it can have an arbitrarily large cardinality. We will need to introduce a suitable definition of set inclusion and to devise for E a special relation which conveniently mimics the membership relation.

3 Historical notes concerning the formal notion of ordered pair

> *On peut considérer la notion de couple comme un* signe fondamental $\cdots$ *Mais on peut aussi exprimer la notion de couple à l'aide des autres signes fondamentaux (oú d'abréviations qui s'y ramènent): il suffit de prendre comme définition de (x, y) l'ensemble $\{\{x\},\{x,y\}\}$ $\cdots$ —il est en effet visible qu'en définissant ainsi un couple on satisfait à l'axiome fondamental donnant la condition d'égalité de deux couples. Toutefoit cette seconde méthode met l'accent sur un aspect de la notion de couple qui est parfaitement dénué d'interêt $\cdots$, la seule et unique question ayant une importance mathématique étant en effet de connaître les conditions pour que deux couples soient égaux.*
>
> —Roger Godement, Cours d'Algèbre, 1966

To understand Tarski's idea on how to specify **(P)** in three variables, one should bear in mind the encoding $(\,x,\,y\,)$ of ordered pairs devised by Kuratowski and accept also the set $\{\,\{\,x,y\,\},\{\,x\,\},\emptyset\,\}$ as a legitimate—though redundant—encoding for the same ordered pair.

By way of first approximation, **(OP)** can be formulated as follows:

$$\textbf{(OP)} \qquad \forall\, x\,\forall\, y\,\exists\, q\,(\,q\,\pi_0\,x \;\wedge\; q\,\pi_1\,y\,),$$

where the abbreviating relators π_0 and π_1 designate conjugated projections associated with ordered pairs of the above kind and are defined as follows:

$$q\,\sigma\,x \quad \leftrightarrow_{\text{Def}} \quad \exists\, s\,\big(\,x\in s \;\wedge\; s\in q \;\wedge \neg\,\exists\, u\,(\,u\in s \;\wedge\; u\neq x\,)\big),$$

viz., there is a singleton s in q to which x belongs;

$$q\,\pi_0\,x \quad \leftrightarrow_{\text{Def}} \quad q\,\sigma\,x \;\wedge\, \neg\,\exists\, v\,(\,q\,\sigma\,v \;\wedge\; v\neq x\,),$$

viz., there is a unique singleton s in q, and x belongs to s;

$$q\,\pi_1\,y \quad \leftrightarrow_{\text{Def}} \quad \exists\, w\,\big(\,y\in w \;\wedge\; w\in q\,\big) \;\wedge\, \neg\,\exists\, z\,\big(\,\exists\, t\,(\,z\in t \;\wedge\; t\in q\,) \;\wedge\; \neg q\,\pi_0\,z \;\wedge\; z\neq y\,\big),$$

viz., q has either the form$\{\{x,y\},\{x\}\}$ or the form $\{\{x,y\},\{x\},\emptyset\}$, for some x.

Then, in unfolding π_0 and π_1 within **(OP)** according to their definitions, one should judiciously rename bound variables so as to bring no variables other than x, y, and q into play. In particular the conjunct $q\,\pi_0\,x$, once fully unfolded, will be

$$\exists\, y\,\big(\,x\in y \;\wedge\; y\in q \;\wedge \neg\,\exists\, q\,(\,q\in y \;\wedge\; q\neq x\,)\big) \;\wedge\; \neg\,\exists\, y\Big(\exists\, x\,\big(\,y\in x \;\wedge\; x\in q \;\wedge \neg\,\exists\, q\,(\,q\in x \;\wedge\; q\neq y\,)\big) \;\wedge\; y\neq x\Big).$$

Likewise, $q\,\pi_1\,y$ unfolds within **(OP)** into

$$\exists\, x\,(\,y\in x \;\wedge\; x\in q\,) \;\wedge\; \neg\exists\, x\,\big(\,\exists\, y\,(\,x\in y \;\wedge\; y\in q\,) \;\wedge\; \neg\,q\,\pi_0\,x \;\wedge\; x\neq y\,\big),$$

where $q\,\pi_0\,x$ should be unfolded, in its turn, as before.

Even though **(OP)** and **(P)** can be shown to be logically equivalent to each other, the intuitive meaning of **(OP)** differs from the one of **(P)**. Notice, however, that if **(OP)** (which is readily seen to logically follow from **(P)**) is assumed, then, in view of the single-valuedness of π_b for $b = 0, 1$ (to wit, $\forall q \forall u \forall v \big((q\pi_b u \wedge q\pi_b v) \rightarrow u = v \big)$), the following becomes an intuitively acceptable 3-variable rendering of **(P)**:

$$\forall q \Big(\big((\exists v \, q \, \pi_0 \, v) \wedge (\exists v \, q \, \pi_1 \, v) \big) \rightarrow \exists p \forall v \, \big(v \in p \;\leftrightarrow\; (q \, \pi_0 \, v \;\vee\; q \, \pi_1 \, v) \big) \Big).$$

Under the assumption **(OP)** one could, with equal ease, get 3-variable formulations of **(W)** and **(L)**; e.g., **(W)** could be stated as follows:

$$\forall q \Big((\exists v \, q \, \pi_0 \, v) \rightarrow \exists p \forall v \, \big(v \in p \;\leftrightarrow\; (q \, \pi_1 \, v \;\vee\; \exists p (q \, \pi_0 \, p \wedge v \in p)) \big) \Big).$$

On the other hand, notice that

$$\forall q \, \exists p \forall v \, \big(v \in p \;\leftrightarrow\; (q \, \pi_0 \, v \;\vee\; q \, \pi_1 \, v) \big)$$

would *not* be an acceptable rendering of **(P)**; in fact, should there be a q devoid of both π_0-image and π_1-image, then the set p corresponding to such a q as here specified would be null.

4 The calculus of binary relations

We will now outline the ground, fully equational, formalism to be exploited in subsequent treatment of set-pairing. In recalling the basic concepts of the calculus of relations, we will slightly adapt the notions developed in [42] (cf. also [38] and [5]) as an evolution of the algebraic approach to logic first proposed by Augustus De Morgan, Charles Sanders Peirce, and Ernst Schröder.

In the *calculus of relations* one can both specify properties of binary (i.e., dyadic) relations, and infer properties ensuing from such specifications. We consider only *homogeneous* relations (see [38, Chapter 2]), to wit, relations over an unspecified yet fixed domain $\mathcal{U}$ of discourse. The signature of the language $\mathcal{L}^{\times}$ underlying this calculus consists of the following symbols:

- Constants $\emptyset$, $1\!\!1$, and ι.
- Another symbol $\in$, of arity 0 like constants but freely interpretable.
- Primitive Boolean operators, $\cdot$ and $+$ (intersection and symmetric difference of relations, both dyadic), and the Peircean operators $\circ$ (composition, dyadic) and $^{\smile}$ (conversion, monadic). In terms of these one can express other constructs such as $\sqcup$ and $-$ (dyadic union and difference), and $\overline{}$ (complementation, monadic). We will assume that the priorities of these operators are decreasing relative to the ordering $\overline{}, {}^{\smile}, \circ, \cdot, +, \sqcup, -$.

Semantics can be assigned to the terms of this signature by simply fixing a nonempty domain $\mathcal{U}$, choosing a subset $\in^{\Im}$ of the Cartesian square $\mathcal{U} \times \mathcal{U}$ as interpretation of $\in$, and then interpreting in the usual manner the basic constants and constructs:

$$\begin{array}{l}
\emptyset^{\Im} =_{\mathrm{Def}} \emptyset, \qquad 1\!\!1^{\Im} =_{\mathrm{Def}} \mathcal{U} \times \mathcal{U}, \qquad \iota^{\Im} =_{\mathrm{Def}} \{(a,a) \mid a \text{ in } \mathcal{U}\}; \\
(Q \cdot R)^{\Im} =_{\mathrm{Def}} \{ (a,b) \text{ in } Q^{\Im} \mid (a,b) \text{ in } R^{\Im} \}; \\
(Q + R)^{\Im} =_{\mathrm{Def}} \{ (a,b) \text{ in } Q^{\Im} \mid (a,b) \text{ not in } R^{\Im} \} \cup \{ (a,b) \text{ in } R^{\Im} \mid (a,b) \text{ not in } Q^{\Im} \}; \\
(Q \circ R)^{\Im} =_{\mathrm{Def}} \{ (a,b) \text{ in } 1\!\!1^{\Im} \mid \text{there are pairs } (a,c) \text{ in } Q^{\Im} \text{ such that } (c,b) \text{ in } R^{\Im} \}; \\
(Q^{\smile})^{\Im} =_{\mathrm{Def}} \{ (b,a) \text{ in } 1\!\!1^{\Im} \mid (a,b) \text{ in } Q^{\Im} \}.
\end{array}$$

Throughout this paper, the privileged domain $\mathcal{U}$ of discourse is meant to be the universe of all sets, namely the von Neumann's *cumulative hierarchy* $\mathcal{V}$, cf. [34, pp. 100–102]; however, each theory which gets focused expresses only one facet of full-fledged set theory; therefore, it is perfectly legal and consistent with its axioms to interpret it over some domain of "aggregates" much more loosely constrained than $\mathcal{V}$. For example, we could take our domain to be the collection of all *hereditarily finite* sets drawn from $\mathcal{V}$.

Properties of relations can be stated through *equalities* $Q = R$ whose sides Q, R are expressions built from the above constants and operators.

The language $\mathcal{L}^{\times}$ can be extended profitably with many derived operators (e.g. $P \sqcup Q =_{\mathrm{Def}} P + Q + P \cdot Q$, $P - Q =_{\mathrm{Def}} P + P \cdot Q$, $\overline{P} =_{\mathrm{Def}} \not{P} =_{\mathrm{Def}} P + \mathbb{1}$) and with a number of shorthand pieces of notation for equalities, as illustrated in Figure 2.

$P \sqsubseteq Q$	$\leftrightarrow_{\mathrm{Def}}$	$P - Q = \emptyset$
$\mathsf{RUniq}(P)$	$\leftrightarrow_{\mathrm{Def}}$	$P^{\smile} \circ P \sqsubseteq \boldsymbol{\iota}$
$\mathsf{LUniq}(P)$	$\leftrightarrow_{\mathrm{Def}}$	$\mathsf{RUniq}(P^{\smile})$
$\mathsf{funcPart}(P)$	$=_{\mathrm{Def}}$	$P - P \circ \overline{\boldsymbol{\iota}}$
$\mathsf{valve}(P, Q)$	$=_{\mathrm{Def}}$	$P - \overline{\boldsymbol{\iota}} \circ (P - Q)$
$\mathsf{syq}(P, Q)$	$=_{\mathrm{Def}}$	$\overline{P^{\smile} \circ \overline{Q}} \cdot \overline{\overline{P}^{\smile} \circ Q}$
$\mathsf{noy}(P)$	$=_{\mathrm{Def}}$	$\mathsf{syq}(P, P)$

Figure 2: Definitional extensions of the basic relational language

In order to characterize the behavior of the relational constructs, a number of axioms are adopted. Figure 3 shows an axiomatization involving the primitive constructs. The choice of such *logical* axioms is a preparatory step for the development of an inference machinery for relational reasoning—and, subordinately, for set-reasoning.

$P \cdot Q$	$=$	$Q \cdot P$
$P \cdot (Q + R) + P \cdot Q$	$=$	$P \cdot R$
$(P \star_1 Q) \star_1 R$	$=$	$P \star_1 (Q \star_1 R)$
$\boldsymbol{\iota} \circ P$	$=$	P
$P^{\smile\smile}$	$=$	P
$(P \star_2 Q)^{\smile}$	$=$	$Q^{\smile} \star_2 P^{\smile}$
$((P + Q) + P \cdot Q) \circ R$	$=$	$(Q \circ R + P \circ R) + Q \circ R \cdot P \circ R$
$Q \cdot (Q \circ P + \mathbb{1}) \circ P^{\smile}$	$=$	$\emptyset$
$\mathbb{1} \cdot P$	$=$	P

$\star_1 \in \{+, \cdot, \circ\}$ and $\star_2 \in \{\cdot, \circ\}$

Figure 3: Logical axioms of the calculus of relations

The issue of translating first-order theories into the calculus of relations has been treated, among others, in [7, 22]. In [16, 23], in particular, it is shown how the Zermelo-Fraenkel set theory can be recast, in its entirety, within the calculus of relations. This task amounts to enhancing the logical axioms with a number of *proper* axioms aimed at restraining the possible interpretations of the primitive symbol $\in$. Figure 4 gives an example of this, by showing translated versions of **(E)**, **(N)**, **(OP)**, **(R)**, and **($\mathbf{A}^n$)**, where the steps in the formalization of **(OP)** reflect the ideas discussed in Sec.3. (The notation in Figures 4 and 8 complies with the one we have used in [16]. The noy operator was introduced by Jacques Riguet in 1948.)

While from the side of quantified predicate calculus we can easily focus on various pairing operations, from the side of the calculus of relations it turns out to be more convenient to focus on the *conjugated projections* ℓ, r associated with pairs of each kind. Formalized within $\mathcal{L}^{\times}$, the conditions ℓ, r must meet are:

$$\ell^{\smile} \circ r = \mathbb{1}, \qquad \mathsf{RUniq}(\ell), \qquad \mathsf{RUniq}(r).$$

Notice that the first of these directly reflects into the equational formulation of **(OP)**. When it comes to formulating weak set theories in the said terms (in the way just illustrated), the single-valuedness of ℓ and r often comes for free, thanks to the fact (to be shown in Sec.7) that $Q \circ Q^{\smile} \sqsubseteq \boldsymbol{\iota}$ yields that $\mathsf{valve}(P, Q) \circ \mathsf{valve}^{\smile}(P, Q) \sqsubseteq \boldsymbol{\iota}$. The condition $\ell^{\smile} \circ r = \mathbb{1}$, on the other hand, must be imposed more or less explicitly.

	$\ni$	$=_{\text{Def}}$	$\in^{\smile}$		$\ni\ni$	$=_{\text{Def}}$	$\ni \circ \ni$
	$\boldsymbol{\sigma}$	$=_{\text{Def}}$	$\ni \circ (\ni - \ni \circ \overline{\boldsymbol{\iota}})$				
	$\boldsymbol{\pi}_0$	$=_{\text{Def}}$	$\boldsymbol{\sigma} - \boldsymbol{\sigma} \circ \overline{\boldsymbol{\iota}}$		$\boldsymbol{\pi}_1$	$=_{\text{Def}}$	$\ni\ni - (\ni\ni - \boldsymbol{\pi}_0) \circ \overline{\boldsymbol{\iota}}$
(E)	$\boldsymbol{\iota}$	$=$	$\mathsf{noy}(\in)$	**(N)**	$\mathbb{1}$	$=$	$\overline{\mathbb{1} \circ \in} \circ \mathbb{1}$
(OP)	$\mathbb{1}$	$=$	$\boldsymbol{\pi}_0{}^{\smile} \circ \boldsymbol{\pi}_1$	**(R)**	$\mathbb{1} \circ \in$	$=$	$\mathbb{1} \circ (\in - \ni \circ \in)$
(A^n)	$\emptyset$	$=$	$\underbrace{\in \circ \cdots \circ \in}_{n+1 \text{ factors}} \cdot\, \boldsymbol{\iota}$				

Figure 4: Peircean specification of a very weak set theory

5 Translating first-order theories into the calculus of relations

Over half century ago (cf. [42, pp. 95–145]), Tarski discovered an effective procedure for reducing each sentence of the language underlying any first-order theory of membership which includes the pair axiom to an equivalent sentence involving three variables only. This procedure enables global translation of such a theory into a purely equational extension of the calculus of relations. A variant of Tarski's original procedure (today presumably lost) was later found, independently, by J. Donald Monk and by Roger Maddux. Our own interest in such a translation is our expectation that, thanks to it, we can gain better service from today's theorem provers run in autonomous mode. In our own experience, in fact, a prover generally demonstrates higher performances when confronted with purely equational theories than with theories which more fully exploit the symbolic first-order apparatus.

As recapitulated in Figure 5, Maddux' general method associates a relational expression $E_\varphi = \mathsf{mdx}(\varphi)$ with any first-order *formula* φ of the set-theoretic language devoid of constants and function symbols whose only primitive predicate symbols are $=$ and $\in$. This translation presupposes that conjugated projections ℓ, r are available; in terms of these one can easily specify the parameters L, R on which the translation depends, for example as follows: $L =_{\text{Def}} \ell \sqcup (\boldsymbol{\iota} - \ell \circ \mathbb{1})$ and $R =_{\text{Def}} r \sqcup (\boldsymbol{\iota} - r \circ \mathbb{1})$. (Any equation of the calculus of relations can easily be translated, in its turn, into a 3-variable first-order sentence (cf., e.g., [21]). Consequently one can, via Maddux' translation and thanks to **(OP)** and to the assumed single-valuedness of ℓ and r, restate in three variables any first-order sentence.)

To understand Figure 5, refer to an enumeration $\mathsf{v}_0, \mathsf{v}_1, \mathsf{v}_2, \ldots$ of all individual variables, and to an interpretation $\Im$; for all a in the universe $\mathcal{U}$, and for all natural numbers i, let a_i be the value for which $(a, a_i) \in \mathsf{th}(i)^{\Im}$ holds. The definitions are so given as to ensure that

$$E_\varphi^{\Im} \;=\; \{\, (a,b) \in \mathbb{1}^{\Im} \mid \Im \models \varphi(a_0, \ldots, a_i) \,\}$$

holds provided that no variable v_j with $i < j$ belongs to $\mathsf{freeVars}(\varphi)$, i.e., occurs free in φ. It should hence be clear that the equation $E_\varphi = \mathbb{1}$, viz. $\mathsf{Maddux}(\varphi)$, has the same truth-value as φ when φ is a sentence.

In spite of its very appealing conceptual simplicity, Maddux' translation tends to produce utterly long equations; we are confident that more efficient translation algorithms can be designed, and have undertaken a research in this direction (cf. [7] and [20]), without yet exploiting the full generality of conjugated projections.

Assume $L^{\smile}\circ L \sqcup R^{\smile}\circ R \sqsubseteq \iota$, $L^{\smile}\circ R = L\circ 1\!\!1 = R\circ 1\!\!1 = 1\!\!1$. Let $i, j = 0, 1, 2, \ldots$; $\overrightarrow{V}$ stand for a list of variables, and φ, ψ, χ stand for formulae.		
$\mathsf{th}(0)$	$=_{\text{Def}}$	L,
$\mathsf{th}(i+1)$	$=_{\text{Def}}$	$R \circ \mathsf{th}(i)$
$\mathsf{sibs}([\,])$	$=_{\text{Def}}$	$1\!\!1$,
$\mathsf{sibs}([\mathsf{v}_i \vert \overrightarrow{V}])$	$=_{\text{Def}}$	$\mathsf{sibs}(\overrightarrow{V}) \cdot \big(\mathsf{th}(i) \circ \mathsf{th}^{\smile}(i)\big)$
$\mathsf{mdx}\big(\mathsf{v}_i = \mathsf{v}_j\big)$	$=_{\text{Def}}$	$\big(\mathsf{th}(i) \cdot \mathsf{th}(j)\big) \circ 1\!\!1$
$\mathsf{mdx}\big(\mathsf{v}_i \in \mathsf{v}_j\big)$	$=_{\text{Def}}$	$\big((\mathsf{th}(i) \circ \in) \cdot \mathsf{th}(j)\big) \circ 1\!\!1$
$\mathsf{mdx}(\neg\,\varphi)$	$=_{\text{Def}}$	$\overline{\mathsf{mdx}(\varphi)}$
$\mathsf{mdx}(\varphi \wedge \psi)$	$=_{\text{Def}}$	$\mathsf{mdx}(\varphi) \cdot \mathsf{mdx}(\psi)$
$\mathsf{mdx}(\exists\,\overrightarrow{V}\,\varphi)$	$=_{\text{Def}}$	$\mathsf{sibs}\big(\mathsf{freeVars}(\exists\,\overrightarrow{V}\,\varphi)\big) \circ \mathsf{mdx}(\varphi)$
$\mathsf{Maddux}(\chi)$	$\leftrightarrow_{\text{Def}}$	$\mathsf{mdx}(\chi) = 1\!\!1$

Figure 5: Translation of first-order formulae/sentences into relational expressions/equations

6 A pairing device for non-classic logics

To improve readability, let us recast here in Skolemized form the theory Ω whose axioms are $(\mathfrak{I})$, $(\mathfrak{D})$, and $(\mathfrak{P})$ of Figure 1:

$$\begin{array}{l} \forall\,x\,\forall\,y\,\forall\,v\,\big(v \in x \cap y \;\leftrightarrow\; (v \in x \,\wedge\, v \in y)\big)\,, \\ \forall\,x\,\forall\,y\,\forall\,v\,\big(v \in x \triangle y \;\leftrightarrow\; (v \in x \;\leftrightarrow\; \neg v \in y)\big)\,, \\ \forall\,x\,\forall\,v\,(v \in \mathscr{P}(x) \;\leftrightarrow\; v \subseteq x)\,. \end{array}$$

As said in Sec.1, Ω was originally conceived as a target first-order framework into which mono-modal propositional logics can be translated uniformly (cf. [9, Chapter 12]): in the translation, the converse $\ni$ of membership acts as a relation which includes immediate accessibility between possible worlds; accordingly, $\cap$ and $\triangle$ play the role of the classic connectives of conjunction and exclusive disjunction; and $\mathscr{P}$ corresponds to the necessity operator $\Box$. One can view Ω as being an extremely weak theory of "aggregates" which becomes a genuine set theory only after appropriate postulates, such as the extensionality axiom **(E)** and the pair axiom **(P)** are added to it. On the other hand, if the extensionality axiom were included in Ω, this would set an undesirable limitation to its usability in the study of non-classic logics; and a similar objection can be raised against postulates, such as **(R)** or $(\mathbf{A}^n)$, entailing the acyclicity of membership. Certain enrichments of Ω with new postulates, e.g. the addition of the pair axiom, do not jeopardize applicability of the $\Box$-as-$\mathscr{P}$ translation method; nevertheless such enrichments appear to be unjustified unless they are shown to yield some technical—perhaps computational—advantages.

The Tarski-Maddux' result summarized in Sec.5 seems to favor the addition of the pair axiom to Ω; however, we will propose below an even less committing way of translating (a variant Ω' of) Ω into the calculus of relations, taking advantage of the fact that the historical result just recalled also holds for theories where an analogue of the pair axiom, of the form

$$\forall\,x\,\forall\,y\,\exists\,q\,\forall\,v\big(\quad v \;\mathsf{in}\; q \quad\leftrightarrow\quad (v = x \,\vee\, v = y) \quad\big)\,,$$

can be derived from the axioms. The only requirement, in regard to this, is that "v in q" be a formula which involves three variables altogether and has v and q as its sole free variables. To achieve our translation purpose, we just have to retouch the one axiom which characterizes the power-set operator $\mathscr{P}(\cdot)$ so that it behaves more naturally when the extensionality axiom is missing. Our proposed replacement for the axiom $(\mathfrak{P})$ of Ω simply consists in adopting, *instead of* the usual definition $v \subseteq x \leftrightarrow (\forall\,u \in v)(u \in x)$, the somewhat less appealing syntactic definition of $\subseteq$:

$$v \subseteq x \quad \leftrightarrow_{\text{Def}} \quad \big((\forall\,u \in v)(u \in x) \rightarrow (\forall\,u \in x)(u \in v)\big) \rightarrow v = x$$

(that is, $v \subseteq x$ holds if and only if either $v = x$ or every element of v belongs to x whereas x has some element not belonging to v). The rationale of this revision is that $\mathscr{P}(x)$ *would otherwise lack* *the ability to discriminate* between x and any other set x' satisfying $\forall u(u \in x \leftrightarrow u \in x')$, and would consequently be unusable for any pair-encoding device. Under the so revised axiom $(\mathfrak{P})$, even without extensionality axiom, it is clear that exactly one p, let us call it $\mathscr{P}(x)$, corresponds to each x so that the elements of p are precisely x and all of its strict subsets $v \subset x$, where

$$v \subset x \leftrightarrow_{\mathrm{Def}} v \subseteq x \wedge v \neq x.$$

Likewise, to any q there corresponds at most one a such that $q \,\mathsf{max}\, a$ holds, where

$$q \,\mathsf{max}\, a \quad \leftrightarrow_{\mathrm{Def}} \quad a \in q \wedge \big(\forall u \in q\big)\big(u \in \mathscr{P}(a)\big)\,;$$

but, unlike $\mathscr{P}$ which is total, max is a *partial* function of its first operand.

In our revised version Ω' of Ω, one can conceive an analogue of the unordered pair $\{a, b\}$ to be $\mathscr{P}(a)$ when $a = b$ and to have the same elements as $\mathscr{P}(a)\triangle\mathscr{P}(b)$ when $a \neq b$. Actually, we have already introduced in Sec.1 the notation $\wr a, b \int$ for such an "unordered pair" and have also pointed out how to construct from it the exotic 'ordered pair' $(\!|a, b|\!)$ entirely analogous to the traditional Kuratowski's pair (a, b). Theorem 1 entails that this pair behaves as desired, as we are going to see in Theorem 5 below. With this rationale in mind, we can characterize as follows a "pseudo-membership" which meets the formal analogue seen above of the pair axiom:

$$\begin{aligned} b \,\mathsf{in}\, q \ \leftrightarrow_{\mathrm{Def}} & \Big(b \in q \wedge \big(\neg\exists\, d \in q\big)\big(b \subset d\big)\Big) \ \vee \\ & \exists\, a\Big(q \,\mathsf{max}\, a \wedge b \subset a \wedge \forall\, d\Big(d \in q \leftrightarrow \big(d \in \mathscr{P}(a) \wedge d \notin \mathscr{P}(b)\big)\Big)\Big)\,. \end{aligned}$$

To see that in can be specified in three variables, it suffices to observe that since max is single-valued, the *definiens* of the predicate in can be rewritten—with a harmless variable renaming—as follows:

$$\begin{aligned} \Big(b \in q \wedge \big(\neg\exists\, d \in q\big)\big(b \subset d\big)\Big) \quad \vee \quad & \Big(\exists\, d\, (q \,\mathsf{max}\, d \wedge b \subset d) \\ & \wedge \forall\, d\Big(d \in q \leftrightarrow \Big(\big(\neg d \in \mathscr{P}(b)\big) \wedge \exists\, b\big(q \,\mathsf{max}\, b \wedge d \in \mathscr{P}(b)\big)\Big)\Big)\Big)\,. \end{aligned}$$

Theorem 5 *The relation* in *and the functions* $x, y \mapsto \wr x, y \int$ *satisfy the conditions*

$$\begin{aligned} & v \,\mathsf{in}\, \wr x, y \int \leftrightarrow (v = x \vee v = y), \\ & (\!|x, y|\!) = (\!|u, v|\!) \rightarrow (x = u \wedge y = v). \end{aligned}$$

Proof. The second conclusion of the present theorem readily follows from the first, thanks to Theorem 1 (where we take in and $\wr \cdot, \cdot \int$ as E and $d(\cdot, \cdot)$, respectively) and by the definition of $(\!|\cdot, \cdot|\!)$. Hence we need only concentrate on the proof of the first claim in what follows.

Of the four possible cases, which are (1) $x = y$, (2) $x \neq y$ but neither $x \subset y$ nor $y \subset x$, (3) $y \subset x$, and (4) $x \subset y$, we need only consider the first three. In case (1), $\wr x, y \int$ equals $\mathscr{P}(x)$ by definition, and hence we must show that $b \,\mathsf{in}\, \mathscr{P}(x)$ holds if and only if $b = x$. On the one hand, in fact, $x \,\mathsf{in}\, \mathscr{P}(x)$ follows from the definition of in because $x \subseteq x$ and $d \subseteq x \rightarrow \neg(x \subset d)$. On the other hand, no b other than x can satisfy $b \,\mathsf{in}\, \mathscr{P}(x)$. Indeed, assuming by contradiction that $x \neq b$ and $b \,\mathsf{in}\, \mathscr{P}(x)$, b should meet one of the two disjunct of the definition of in; but it cannot meet the first, else we would get into the contradiction $b \in \mathscr{P}(x) \wedge x \neq b \wedge \neg(b \subset x)$ (since $x \in \mathscr{P}(x)$). It cannot meet the second either, because this would lead to the contradiction $b \subset x \wedge (b \in \mathscr{P}(x) \leftrightarrow (b \in \mathscr{P}(x) \wedge \neg b \in \mathscr{P}(b)))$.

In case (2), $\wr x, y \int$ equals $\mathscr{P}(x)\triangle\mathscr{P}(y)$ and x, y both belong as $\subset$-maximal elements to it; hence, $x, y \,\mathsf{in}\, \wr x, y \int$. On the other hand, $\wr x, y \int$ has no maximum and no element v of $\wr x, y \int$ distinct from x and y can be maximal, because v is included in either x or y; therefore, $v \,\mathsf{in}\, \wr x, y \int$ cannot hold.

In case (3), $y \subset x$ trivially yields $\mathscr{P}(y) \subset \mathscr{P}(x)$, $\wr x, y \int = \mathscr{P}(x) \setminus \mathscr{P}(y)$, and $x \in \wr x, y \int$. Hence, $\mathscr{P}(x)\ \mathsf{max}\ x$, and $\wr x, y \int\ \mathsf{max}\ x$, and hence $x\ \mathsf{in}\ \wr x, y \int$; on the other hand, $y\ \mathsf{in}\ \wr x, y \int$ holds by virtue of the second clause of the definition of in. There can be no $z\ \mathsf{in}\ \wr x, y \int$ other than x, y, because x is the only maximum and y is the only maximal subset of x not belonging to $\wr x, y \int$. □

This leads to the equational specification of Ω' shown in Figure 6.

$$
\begin{array}{lll|lll}
\supset & =_{\mathrm{Def}} & \ni \circ \notin - \not\ni \circ \in & \subset & =_{\mathrm{Def}} & \supset^{\smile} \\
\mathscr{P} & =_{\mathrm{Def}} & (\in - \supset \circ \notin) - \overline{\iota + \supset} \circ \in & \boldsymbol{\mu} & =_{\mathrm{Def}} & (\mathscr{P} \circ \overline{\not\ni \circ \in}) \cdot \in \\
\mathsf{in} & =_{\mathrm{Def}} & \multicolumn{4}{l}{(\in - \subset \circ \in) \sqcup \Big(\subset \circ \boldsymbol{\mu} - \mathscr{P} \circ \ni \circ \in - \mathbb{1} \circ (\in - \in \circ \mathscr{P}^{\smile} \circ \boldsymbol{\mu}) - \mathscr{P} \circ \not\ni \circ (\in \circ \mathscr{P}^{\smile} \circ \boldsymbol{\mu} - \in) \Big)} \\
\boldsymbol{\vartheta} & =_{\mathrm{Def}} & (\mathsf{in} - \overline{\iota} \circ \mathsf{in}) \circ \mathsf{in} & & & \\
\boldsymbol{\varphi} & =_{\mathrm{Def}} & \boldsymbol{\vartheta} - \overline{\iota} \circ \boldsymbol{\vartheta} & \boldsymbol{\psi} & =_{\mathrm{Def}} & \mathsf{in} \circ \mathsf{in} - \overline{\iota} \circ (\mathsf{in} \circ \mathsf{in} - \boldsymbol{\varphi}) \\
\hline
\multicolumn{3}{c}{\mathscr{P} \circ \mathbb{1} \;=\; \mathbb{1}} & \multicolumn{3}{c}{\boldsymbol{\varphi} \circ \boldsymbol{\psi}^{\smile} \;=\; \mathbb{1}} \\
\multicolumn{3}{c}{\mathbb{1} \circ \mathsf{syq}(\in, \in \circ \boldsymbol{\varphi} \cdot \in \circ \boldsymbol{\psi}) \;=\; \mathbb{1}} & \multicolumn{3}{c}{\mathbb{1} \circ \mathsf{syq}(\in, \in \circ \boldsymbol{\varphi} + \in \circ \boldsymbol{\psi}) \;=\; \mathbb{1}}
\end{array}
$$

Figure 6: Peircean specification of Ω'

7 Automated equational set-reasoning

Previous research (cf. [22], for instance) revealed the possibility of exploiting a first-order theorem-prover to experiment with axioms like the ones in Figure 3 and the ones on sets we have examined so far (cf. Figure 4). As a continuation of this line of research, in what follows we report on a number of experiments developed with the theorem-prover Otter on the set-theoretical notion of ordered pair. The remaining of this paper is principally focused on such experiments.

Otter was not conceived specifically for automating the calculus of relations: actually, to rely on Otter, we must *emulate* that calculus via a corresponding first-order theory whose intended models are relation algebras (cf. [26]). This theory, sometimes called the *arithmetic of relation algebras* [11], has in its language individual variables ranging over relations, which we can use conveniently in place of the schematic variables (such as P, Q, R of Figure 3) often used at the meta-level to represent relational expressions whatsoever of the calculus. The language of the arithmetic of relations also provides propositional connectives, which occasionally play a role in our experimentation: implication $\rightarrow$, in particular, surrogates the entailment relation $\vdash^{\times}$. Anyway, the only sentences which Otter must handle while emulating the calculus of relations are, in essence, universal closures of equations or of "quasi-equations" of the form

$$
\Big(\bigwedge_{i=1}^{n} L_i = R_i \Big) \;\rightarrow\; L_0 = R_0 \,.
$$

This is what ensures a certain overall computational efficiency with our approach.

One must be aware that the arithmetic of relations lacks completeness. This limitation originates from the existence of models for this theory which comply with its axioms without being isomorphic to relation algebras. Although this drawback disappears when the existence of conjugated projections is either postulated or *provable* (cf. [42, Chapter 8]), incompleteness implies that proofs such as those of Theorems 1 and 2, Lemma 3, and Corollary 4, do not necessarily have, *a priori*, counterparts in the arithmetic or in the calculus of relations. This is what constitutes the challenge in our task of verifying *within* the calculus of relations that particular pairs of relations are, under specific set-theoretic axioms, conjugated projections.

The first achievement of our experimental activity consisted in proving a collection of general algebraic laws mainly related to single-valuedness (cf. Figure 7). These laws—which can be thought of as having deeper semantic content than those in Figure 3—constitute a solid ground for the development of further experimentation.

Let us start by considering the following postulate:

$$
\mathsf{RUniq}(\mathsf{syq}(P, \in)),
$$

i.	$\mathsf{LUniq}(P), \mathsf{LUniq}(Q)$	$\vdash^{\times}$	$\mathsf{LUniq}(P \circ Q)$
ii.	$\mathsf{LUniq}(Q)$	$\vdash^{\times}$	$\mathsf{LUniq}\big(\mathsf{valve}(P, Q)\big)$
iii.	$(\mathbf{E})$	$\vdash^{\times}$	$\mathsf{RUniq}\big(\mathsf{syq}(\mathsf{P}, \in)\big)$
iv.	$Q \circ Q \cdot \boldsymbol{\iota} = \emptyset$	$\vdash^{\times}$	$\mathsf{RUniq}\big(P - P \circ (\overline{\boldsymbol{\iota}} - Q)\big)$
v.		$\vdash^{\times}$	$\mathsf{RUniq}\big(\mathsf{funcPart}(P)\big)$
vi.	$(\mathbf{A}^{2 \cdot n+1})$	$\vdash^{\times}$	$\mathsf{RUniq}(\gamma_{n+1})$

where $\gamma_n =_{\mathrm{Def}} \ni - \ni \circ (\overline{\boldsymbol{\iota}} - \underbrace{\in \circ \cdots \circ \in}_{n \text{ factors}})$

Figure 7: Basic lemmas concerning single-valued relations, proved with Otter

where P ranges over all relational expressions.

The first task in Otter-based set-reasoning consisted in (automatically) proving the following property:

$$\mathsf{noy}(\in) = \boldsymbol{\iota} \quad \vdash^{\times} \quad \mathsf{RUniq}(\mathsf{syq}(P, \in)) \ . \tag{10}$$

A proof of this fact was obtained via this sequence of intermediate steps, where the last law is equivalent to $\mathsf{RUniq}(\mathsf{syq}(P^{\smile}, \in))$:

- $\mathsf{syq}(P^{\smile}, \in)^{\smile} \circ \mathsf{syq}(P^{\smile}, \in) \sqsubseteq \overline{\not\ni \circ P^{\smile}} \circ \overline{\overline{P} \circ \in}$, from laws on $\circ$ and $\overline{}$;
- $\mathsf{syq}(P^{\smile}, \in)^{\smile} \circ \mathsf{syq}(P^{\smile}, \in) \sqsubseteq \overline{\not\ni \circ \overline{P^{\smile}}} \circ \overline{P \circ \not\in}$, from laws on $\circ$ and $\overline{}$;
- $\mathsf{syq}(P^{\smile}, \in)^{\smile} \circ \mathsf{syq}(P^{\smile}, \in) \sqsubseteq \mathsf{noy}(\in)$, from laws on inclusion;
- $\mathsf{syq}(P^{\smile}, \in)^{\smile} \circ \mathsf{syq}(P^{\smile}, \in) \sqsubseteq \boldsymbol{\iota}$, from **(E)** and the laws on inclusion.

As reported in [10], also a proof of the converse of the above law (10) was obtained by using Otter. Consequently, we have (automatically) certified that the equality $\mathsf{RUniq}(\mathsf{syq}(P, \in))$ is an alternative formulation of the extensionality axiom **(E)**:

$$\mathsf{noy}(\in) = \boldsymbol{\iota} \quad \dashv\vdash^{\times} \quad \mathsf{RUniq}(\mathsf{syq}(P, \in)).$$

A crucial law among those in Figure 7 is *iv*. Let us now briefly sketch the proof of this law as generated with Otter. A preliminary step was introducing the following definition:

$$\mathsf{protoFuncPart}(P, Q) \quad =_{\mathrm{Def}} \quad P - (P \circ Q).$$

The leading derivation steps yielding the desired proof of *iv* are:

- $\mathsf{funcPart}(P) \ = \ \mathsf{protoFuncPart}(P, \overline{\boldsymbol{\iota}})$,
- $\mathsf{protoFuncPart}(P, Q) \cdot \mathsf{protoFuncPart}(P, Q) \circ Q \ = \ \emptyset$,
- $\mathsf{protoFuncPart}(P, Q)^{\smile} \circ \mathsf{protoFuncPart}(P, Q) \sqsubseteq \overline{Q}$,
- $\mathsf{protoFuncPart}(P, Q)^{\smile} \circ \mathsf{protoFuncPart}(P, Q) \sqsubseteq \overline{Q} \cdot \overline{Q}^{\smile}$,
- $Q \circ Q \sqsubseteq \overline{\boldsymbol{\iota}} \vdash^{\times} \mathsf{RUniq}(\mathsf{protoFuncPart}(P, \overline{\boldsymbol{\iota}} \cdot \overline{Q}))$,
- $Q \circ Q \cdot \boldsymbol{\iota} = \emptyset \vdash^{\times} \mathsf{RUniq}(\mathsf{protoFuncPart}(P, \overline{\boldsymbol{\iota}} - Q))$.

Each of the above proof steps was derived by using the axioms (cf. Figure 3) and a collection of lemmas on relational constructs (cf. [22]). The complete proof, as well as some details such as timings and settings of Otter's parameters, can be found in [17].

As corollaries of *iv*, Otter easily obtained the proofs of *v* (timing: 0.01 sec., length: 3) and of several instances of the scheme *vi* of Figure 7 (for the case $n = 3$, the length of the generated proof is 4 and it was obtained in 0.75 sec; while for $n = 5$ a proof of length 5 was obtained in 0.02 sec.).

A number of laws regarding functionality (i.e., right uniqueness) was obtained with Otter; here are some of them:

law	length of proof	time (sec.)	generated clauses	kept clauses
$\mathsf{RUniq}(\emptyset)$	1	0.06	917	109
$\mathsf{RUniq}(\iota)$	1	0.06	917	109
$\mathsf{RUniq}(P) \vdash^{\times} \mathsf{RUniq}(P \cdot Q)$	7	1.86	26575	4371
$\mathsf{RUniq}(P)\,,\ \mathsf{RUniq}(Q) \vdash^{\times} \mathsf{RUniq}(P \circ Q)$	6	0.05	926	217

Analogous laws on left uniqueness, such as *i* in Figure 7, were then easily obtained by exploiting the definition of LUniq (cf. Figure 2) together with basic lemmas on relational constructs.

In order to obtain an automated proof of the law *ii* of Figure 7, it was convenient to prove a few lemmas regarding the valve operator, among which:

law	length of proof	time (sec.)	generated clauses	kept clauses
$\mathsf{valve}(P,Q) \sqsubseteq P$	2	1.11	11435	6041
$\mathsf{valve}(P,Q) \sqsubseteq \overline{\iota} \circ (P \cdot \overline{Q})$	5	1.10	12334	5893
$R \circ \mathsf{valve}(P,Q) \sqsubseteq R \circ \overline{\iota} \circ (P \cdot \overline{Q})$	4	0.94	19114	2324
$\mathsf{valve}(P,Q) \cdot R \sqsubseteq P \cdot R$	5	0.20	3791	670
$P \sqsubseteq Q \vdash^{\times} \mathsf{valve}(P,Q) = P$	5	0.90	11600	4079
$\mathsf{LUniq}(Q) \vdash^{\times} \mathsf{LUniq}(\mathsf{valve}(P,Q))$	12	66.27	253318	15441

8 Expressibility of (E) ∧ (N) ∧ (W) ∧ (L) in 3 variables

In our own formalization of the axiom of elementary sets, very much like in Tarski's one, the notion of ordered pair will be the hinge of the formulation in three variables. The pairs we have in mind are as follows:

$$\langle\, x,\, y \,\rangle \quad =_{\mathrm{Def}} \quad \{\ \{\, y\,\}\ \mathsf{less}\ x\,,\ \{\, y\,\}\ \mathsf{with}\ x\ \}$$

where the binary functions less and with, and the constant $\emptyset$, result from the Skolemization of **(L)**, **(W)**, and **(N)**, respectively, and

$$\{\, v,\, w\,\} \quad =_{\mathrm{Def}} \quad (\emptyset\ \mathsf{with}\ v)\ \mathsf{with}\ w\,, \qquad\qquad \{\, v\,\} \quad =_{\mathrm{Def}} \quad \{\, v,\, v\,\}\,.$$

Although the structure of such pairs only marginally departs from the above-recalled Kuratowski's pair notion, we need to assume the extensionality axiom, **(E)**, which is not necessary with the traditional approach.

Proceeding in a way similar (but much simpler) to the way **(P)** got restated as **(OP)**, we achieve the following restatement of **(N)** ∧ **(W)** ∧ **(L)**:

$$\textbf{(D)} \quad \forall x\, \forall y\, \exists d\, \Big(\, y \in d\ \wedge\ \forall v \big(\, \exists w (\, v \in w\ \ \wedge\ \ w \in d\,) \\ \wedge\ \exists \ell (\, v \notin \ell\ \ \wedge\ \ \ell \in d\,)\ \ \leftrightarrow\ \ v = x\,\big)\Big)\,,$$

which under the renaming $v \mapsto y\,,\ w \mapsto x\,, \ell \mapsto x$ of bound variables becomes a 3-variable sentence. This **(D)** says that one can build the set $\big\{\, y\ \mathsf{less}\ x,\ y\ \mathsf{with}\ x\,\big\}$ out of sets x, y whatsoever. Only indirectly, it enables one to form singletons, the null set $\emptyset$, and ordered pairs of the form $\langle x, y\rangle$.[1]

[1] Remark that the prenex normal form of **(E)**[∧ **(R)**] ∧ **(N)** ∧ **(W)** ∧ **(L)** has quantificational prefix ∃∀∀∃∃∃∀, whereas the quantificational prefix of **(E)**[∧ **(R)**] ∧ **(D)** is ∀∀∃∃∃∀∀∀. Hence, if we take the number of quantifier alternations as a complexity measure (cf. [15]), **(D)** is simpler then **(N)** ∧ **(W)** ∧ **(L)**.

As a matter of fact, by bringing **(D)** into Skolemized form we get

$$\textbf{(D}'\textbf{)}\quad Y \in (Y \circledast X) \wedge \forall v\Big(\exists w\big(v \in w \;\wedge\; w \in Y \circledast X\big) \wedge \exists \ell\big(v \notin \ell \;\wedge\; \ell \in Y \circledast X\big) \;\leftrightarrow\; v = X\Big),$$

where uppercase variables are meant to be universally bound. This is equivalent to the conjunction of **(N)**, **(W)**, and **(L)**, in the following sense:

- under **(N)**, **(W)**, and **(L)**, one can define

  $$X \circledast Y \;=_{\text{Def}} \{\, X \,\mathsf{less}\, Y \,,\; X \,\mathsf{with}\, Y \,\}$$

  and then derive **(D′)**;
- under **(E)** and **(D′)**, one can prove that

  $$\begin{array}{ll} \textbf{(W}'\textbf{)} & \exists w \in Y \circledast X \,\forall v(\, v \in w \;\leftrightarrow\; v \in Y \,\vee\, v = X\,)\,, \\ \textbf{(L}'\textbf{)} & \exists \ell \in Y \circledast X \,\forall v(\, v \in \ell \;\leftrightarrow\; v \in Y \,\wedge\, v \neq X\,)\,, \\ \textbf{(N}'\textbf{)} & \exists s \in (Y \circledast X) \circledast Y \,\exists e \in s \,\exists z \in s \circledast e \,\forall v\, v \notin z\,, \end{array}$$

  whence **(W)**, **(L)**, and **(N)** readily follow.

$\in\in$	$=_{\text{Def}}$	$\in \circ \in$		$\notin\in$	$=_{\text{Def}}$	$\overline{\in} \circ \in$
				mix	$=_{\text{Def}}$	$\in\in \cdot \notin\in$
$\boldsymbol{\lambda}$	$=_{\text{Def}}$	$\mathsf{valve}(\mathsf{mix}, \emptyset)$		$\boldsymbol{\varrho}$	$=_{\text{Def}}$	$\mathsf{valve}(\in\in, \boldsymbol{\lambda})$
			(D)	$\mathbb{1}$	$=$	$\boldsymbol{\lambda} \circ \ni$

Figure 8: Peircean specification of a strengthened axiom **(D)** of elementary sets, and of projections $\boldsymbol{\lambda}^{\smile}, \boldsymbol{\varrho}^{\smile}$ pertaining to it

Figure 8 shows a translation of **(D)** into the calculus of relations, along with a Peircean specification of conjugated projections $\boldsymbol{\lambda}^{\smile}, \boldsymbol{\varrho}^{\smile}$ which correspond to our notion $\langle x, y\rangle$ of ordered pair very much like the expressions $\boldsymbol{\pi}_0, \boldsymbol{\pi}_1$ in Figure 4 designate projections associated with Kuratowski's pair notion. In the calculus of relations, it can easily be proved that

$$\mathsf{LUniq}(\boldsymbol{\lambda}) \;\text{ and }\; \mathsf{LUniq}(\boldsymbol{\varrho}), \;\text{ viz., } \boldsymbol{\lambda}^{\smile}, \boldsymbol{\varrho}^{\smile} \text{ designate partial functions.}$$

These laws can easily be proved from law *ii* of Figure 7 and the simple lemma $\mathsf{LUniq}(\emptyset)$. Otter was able to prove $\mathsf{LUniq}(\boldsymbol{\lambda})$ in 0.01 seconds by producing a proof of length 3. Then, a proof of length 3 of $\mathsf{LUniq}(\boldsymbol{\varrho})$ was obtained as an immediate corollary.

We also succeeded in deriving the analogue $\mathbb{1} = \boldsymbol{\lambda} \circ \boldsymbol{\varrho}^{\smile}$ of **(OP)** from **(D)** and **(N)** (cf. [10]); on the other hand, we have been unable to obtain this within the calculus of relations, unless by assuming **(N)**. Nevertheless, we can be sure that **(N)** follows from **(D)** because if we put

$$\boldsymbol{\rho} \;=_{\text{Def}}\; \boldsymbol{\lambda} \circ \left(\in \cdot \ni \circ \overline{\boldsymbol{\iota} \circ \mathsf{mix}}\right)$$

then (much more easily than for $\boldsymbol{\varrho}$) one can prove that $\mathbb{1} = \boldsymbol{\lambda} \circ \boldsymbol{\rho}^{\smile}$, and one can easily derive $\mathsf{LUniq}(\boldsymbol{\rho})$ from **(E)**. In defining this new $\boldsymbol{\rho}$, we have in mind a second variant of Kuratowski's pair, which is

$$[x, y] \;=_{\text{Def}}\; (x \circledast y) \circledast x\,.$$

Otter's proof of $\mathbb{1} = \boldsymbol{\lambda} \circ \boldsymbol{\rho}^{\smile}$ relies on the following lemmas:

law	length of proof	time (sec.)	generated clauses	kept clauses
$P \circ Q = \mathbb{1} \;\vdash^{\times}\; P \circ (Q \cdot Q^{\smile} \circ P^{\smile}) = \mathbb{1}$	3	0.03	169	51
$P \circ R^{\smile} \cdot S \sqsubseteq P \circ (R^{\smile} \cdot P^{\smile} \circ S)$	2	0.25	4499	305
$P \sqsubseteq Q \;\vdash^{\times}\; P \circ (R \cdot P^{\smile} \circ S) \sqsubseteq P \circ (R \cdot Q^{\smile} \circ S)$	4	3.45	16941	5070

From these lemmas, the following corollaries were easily drawn:

- **(D)** $\vdash^{\times}$ $(\in\cdot\boldsymbol{\lambda}\circ\ni)\circ\boldsymbol{\lambda}^{\smile} = \mathbb{1}$ length:3; time:0.03
- $\in\cdot\boldsymbol{\lambda}\circ\ni \sqsubseteq \boldsymbol{\lambda}\circ(\ni\cdot(\overline{\overline{\boldsymbol{\iota}}\circ\mathsf{mix}})^{\smile}\circ\in)$ length:4; time:0.11
- $\mathbb{1} = (\in\cdot\boldsymbol{\lambda}\circ\ni)\circ\boldsymbol{\lambda}^{\smile} \sqsubseteq (\boldsymbol{\lambda}\circ(\ni\cdot(\overline{\overline{\boldsymbol{\iota}}\circ\mathsf{mix}})^{\smile}\circ\in))\circ\boldsymbol{\lambda}^{\smile}$ length:7; time:4.11
- $\mathbb{1} = (\boldsymbol{\lambda}\circ(\ni\cdot(\overline{\overline{\boldsymbol{\iota}}\circ\mathsf{mix}})^{\smile}\circ\in))\circ\boldsymbol{\lambda}^{\smile}$ length:5; time:0.25

from which our thesis readily follows.

Otter was not able to prove $\mathsf{LUniq}(\boldsymbol{\rho})$ from **(E)** in a single shot. Hence the following auxiliary intermediate lemma had to be proved:

$$\textbf{(E)} \quad \vdash^{\times} \quad \mathsf{LUniq}\big(\in\cdot\ni\circ\overline{\overline{\boldsymbol{\iota}}\circ\mathsf{mix}}\big)\ . \tag{11}$$

In order to obtain a proof of (11) in a reasonable amount of time, we proceeded stepwise. The following are the steps performed with Otter. Notice that it was necessary to prove a number of auxiliary lemmas.

- $\overline{\overline{P}\circ Q}\circ Q^{\smile} \sqsubseteq P$,
- $(W\cdot R\circ Q)\circ T \sqsubseteq \boldsymbol{\iota} \vdash^{\times} R\cdot W\circ(Q^{\smile}\cdot T\circ P) = \emptyset$,
- $\overline{P}\cdot R\circ S^{\smile}\circ(Q^{\smile}\cdot T\circ P) = \emptyset \vdash^{\times} (Q\cdot P^{\smile}\circ T^{\smile})\circ S\cdot\overline{P}^{\smile}\circ R = \emptyset$,
- $P\circ R^{\smile}\cdot S = \emptyset \vdash^{\times} (P\circ R\cdot Q)^{\smile}\cdot S = \emptyset$,
- $(R\circ S^{\smile}\cdot\overline{P}\circ Q)\circ T \sqsubseteq \boldsymbol{\iota} \vdash^{\times} (Q\cdot P^{\smile}\circ T^{\smile})\circ S\cdot\overline{P}^{\smile}\circ R = \emptyset$,
- $(P\circ P\cdot\overline{P}\circ P)\circ\overline{\overline{\boldsymbol{\iota}}\circ(P\circ P\cdot\overline{P}\circ P)}^{\smile} \sqsubseteq \boldsymbol{\iota} \vdash^{\times} (P\cdot P^{\smile}\circ\overline{\overline{\boldsymbol{\iota}}\circ(P\circ P\cdot\overline{P}\circ P)})\circ P^{\smile}\cdot\overline{P}^{\smile}\circ P = \emptyset$,
- $(P\cdot P^{\smile}\circ\overline{\overline{\boldsymbol{\iota}}\circ(P\circ P\cdot\overline{P}\circ P)})\circ P^{\smile}\cdot\overline{P}^{\smile}\circ P = \emptyset$,
- $(P\cdot P^{\smile}\circ\overline{\overline{\boldsymbol{\iota}}\circ(P\circ P\cdot\overline{P}\circ P)})\circ(P\cdot P^{\smile}\circ\overline{\overline{\boldsymbol{\iota}}\circ(P\circ P\cdot\overline{P}\circ P)})^{\smile}\cdot\overline{P}^{\smile}\circ P = \emptyset$,
- $(P\cdot P^{\smile}\circ\overline{\overline{\boldsymbol{\iota}}\circ(P\circ P\cdot\overline{P}\circ P)})\circ(P\cdot P^{\smile}\circ\overline{\overline{\boldsymbol{\iota}}\circ(P\circ P\cdot\overline{P}\circ P)})^{\smile}\cdot P^{\smile}\circ\overline{P} = \emptyset$,
- $(\in\cdot\ni\circ\overline{\overline{\boldsymbol{\iota}}\circ(\in\circ\in\cdot\notin\circ\in)})\circ(\in\cdot\ni\circ\overline{\overline{\boldsymbol{\iota}}\circ(\in\circ\in\cdot\notin\circ\in)})^{\smile}\cdot(\not\ni\circ\in\sqcup\ni\circ\notin) = \emptyset$.

The overall time spent in proving these laws was 15.62 seconds. The longest and most heavily time-consuming proof was the one of the last law: length 8 in 6.52 seconds. From the last of the above laws, by the definition of noy and mix, and by assuming **(E)**, we can conclude the proof of (11).

At this point, $\mathsf{LUniq}(\boldsymbol{\rho})$ could be derived readily by means of law i in Figure 7.

9 A digression on the theory (E) ∧ (N) ∧ (W) ∧ (L)

In earlier studies [19], we noticed that investigating **(N)** and **(W)** in isolation from **(L)** is not convenient. To make an example, a set unifiability algorithm which works under **(E)**, **(N)**, **(W)**, and **(R)** can be found even in the absence of **(L)**, and in such a weak axiomatic framework it would also be possible to supply a "disjunctive syllogistic decomposition" (cf. [8]) for systems of the form

$$\begin{cases} \{s_{11},\ldots,s_{1n_1}\} & = & \{d_{11},\ldots,d_{1m_1}\} \\ \{s_{21},\ldots,s_{2n_2}\} & = & \{d_{21},\ldots,d_{2m_2}\} \\ \vdots & \vdots & \vdots \\ \{s_{K1},\ldots,s_{Kn_K}\} & = & \{d_{K1},\ldots,d_{Km_K}\} \end{cases}$$

(in the unknowns s_{ij}, d_{ih}); however, bringing the disjuncts of the decomposition to the pleasant form of normalized systems of set-equations only becomes possible when **(L)** is available (cf. [36]).

Another illustration of how well **(N)**, **(W)**, and **(L)** work together, comes from the Turing-completeness proof for a programming language centered on the associated operations $\emptyset$, with, and less, namely the "*tiny-SETL*" language treated in [9, Chapter 4].

A new argument in favor of treating the triad **(N)**, **(W)**, **(L)** as a single postulate can be drawn from Sec.8: the conjunction of these three sentences can be stated very tersely by an equivalent sentence which involves three variables altogether. Since **(N)** $\wedge$ **(W)** yields **(P)**, something close to Tarski's statement in three variables of **(N)** $\wedge$ **(OP)** would be achievable for **(N)** $\wedge$ **(W)** as well; but the outcome would be much lengthier and more cryptic than for the said triad.

Still other reasons for being interested in the triad **(N)**, **(W)**, **(L)** were highlighted by the study [37] on how these assumptions can be exploited in arithmetizing the syntax and the deductive apparatus of set theory within set theory itself. This study led to the identification of fragments of set theory which are essentially undecidable (as regards satisfiability) and have a very low syntactic complexity.

10 Expressibility of $(\mathbf{E}) \wedge (\mathbf{A}^n) \wedge (\mathbf{W}) \wedge (\mathbf{L})$ in 3 variables

In this section we show that under **(E)**, one can drop the null-set axiom **(N)**, provided that one makes the acyclicity assumption $(\mathbf{A}^n)$ about sets. In this case, in fact, **(W)** and **(L)** suffice to support suitable notions of ordered pair. The ordered pair we deal with in this section is

$$[\![X, Y]\!] \quad =_{\text{Def}} \quad X \mathsf{with}(Y \mathsf{with}(Y \mathsf{with} X)).$$

With this pair we associate the following two relations:

$$\boldsymbol{\alpha} \quad =_{\text{Def}} \quad \mathsf{syq}(\in \cdot \in\in \circ \in\in, \, \in),$$

and

$$\boldsymbol{\beta} \quad =_{\text{Def}} \quad \gamma_3 \circ \mathsf{syq}(\in \cdot \in\in, \, \in),$$

which will act as left and right projections.

Consider that both **(E)** and $(\mathbf{A}^n)$ have already been expressed within the calculus of relations (cf. Figure 4). Moreover, by $(\mathbf{A}^5)$, we immediately obtain $\mathsf{RUniq}(\boldsymbol{\alpha})$ and $\mathsf{RUniq}(\boldsymbol{\beta})$—Otter generated the proofs of these facts in 0.01 (length 2) and 0.06 (length 6) seconds, respectively, by using the laws in Figure 7.

As a consequence of these results, an easy manner to express $(\mathbf{E}) \wedge (\mathbf{A}^n) \wedge (\mathbf{W}) \wedge (\mathbf{L})$ in 3 variables consists in explicitly asserting one further pair axiom:

$$(\mathbf{OP}_1) \qquad \boldsymbol{\alpha}^{\smile} \circ \boldsymbol{\beta} = \mathbb{1}.$$

This ensures that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are a pair of conjugated projections. Notice that this law follows from $(\mathbf{E}) \wedge (\mathbf{A}^5) \wedge (\mathbf{W}) \wedge (\mathbf{L})$ within the predicate calculus. At this point, by means of the pair of conjugated projections $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we can express both **(W)** and **(L)** in three variables, in order to complete the equational rendering of $(\mathbf{E}) \wedge (\mathbf{A}^n) \wedge (\mathbf{W}) \wedge (\mathbf{L})$.

11 Expressibility of $(\mathbf{R}) \wedge (\mathbf{N}) \wedge (\mathbf{W}) \wedge (\mathbf{L})$ in 3 variables

Unlike the pair notions analyzed so far, the ones to be examined in this and in the following section will not benefit from the extensionality axiom **(E)**. Each one of the two pairing functions considered in Sections 8 and 10 has some advantage over Kuratowski's pairing: one leads, in fact, to very simple specifications of the projections and, consequently, to a terse formulation of the conjunction $(\mathbf{N}) \wedge (\mathbf{W}) \wedge (\mathbf{L})$ (provided **(E)** is assumed); the other one, even though more cumbersome, can be exploited in certain contexts where Kuratowski's pairing is not viable, because there is no guarantee that the operation $X \mapsto \{X\}$ can be performed.

Here we are assuming **(N)** and **(W)**—which yield **(P)**; Kuratowski's pairing would hence be viable, but we propose a notion of pair which relies on the axioms **(L)** and **(R)** too. Save for the fact that the associated projections car and cdr are total (which is a rather marginal virtue), we make no claim that these projections are any better than the projections π_0, π_1 discussed in Sec.3 (see also $\boldsymbol{\pi}_0$ and $\boldsymbol{\pi}_1$ in Figure 4). However, since proving that car and cdr meet the formal properties of conjugated projections requires some labor in first-order logic, a labor which we have already afforded with Otter, we can take this as a benchmark from which to start comparing the performances of an automatic theorem-prover confronted with full first-order reasoning on the one hand, and with purely equational reasoning on the other, in carrying out the same task. Currently, we have not provided yet a full equational proof that car and cdr are conjugated projections, and leave this as future work.

The notion of pair we adopt here is:

$$\ulcorner X, Y \urcorner \quad =_{\mathrm{Def}} \quad \Big\{\{X\}, \{\{X\}, \{Y, \{Y\}\}\}\Big\}.$$

Consequently, a pair of conjugated projections car and cdr can be defined as follows:

$$\begin{array}{rcl} \mathsf{parb} & =_{\mathrm{Def}} & \ni - \ni \circ \in, \\ \mathsf{arb} & =_{\mathrm{Def}} & \mathsf{funcPart}(\mathsf{parb}), \\ \mathsf{car} & =_{\mathrm{Def}} & \mathsf{arb} \circ \mathsf{arb}, \\ \mathsf{arb\_lessArb} & =_{\mathrm{Def}} & \mathsf{syq}(\in - \mathsf{arb}^{\smile}, \in) \circ \mathsf{arb}, \\ \mathsf{cdr} & =_{\mathrm{Def}} & \mathsf{syq}(\in \circ\, \mathsf{arb\_lessArb}^{\smile} - \mathsf{arb}^{\smile}, \in) \circ \mathsf{car}. \end{array}$$

Functionality of arb, and then of car, directly follows from laws in Figure 7.

To obtain an automated proof of RUniq(cdr) with Otter, we exploited a few previously proved laws regarding syq:

law	length of proof	time (sec.)	generated clauses	kept clauses
$\overline{Q \circ \overline{Q}} \circ \overline{Q}^{\smile} \circ Q \sqsubseteq \overline{P} \circ Q$	5	0.03	1517	353
$P \circ \overline{Q} \cdot \overline{P} \circ Q \circ \overline{Q}^{\smile} \circ Q \sqsubseteq \overline{P} \circ Q \sqcup P \circ \overline{Q}$	4	0.04	3794	228
$P \circ \overline{Q} \cdot \overline{P} \circ Q \circ \overline{Q}^{\smile} \circ Q \sqcup Q^{\smile} \circ \overline{Q} \sqsubseteq \overline{P} \circ Q \sqcup P \circ \overline{Q}$	12	0.05	2411	298
$\mathsf{syq}(P^{\smile}, Q)^{\smile} \circ \mathsf{syq}(P^{\smile}, Q) \sqsubseteq \mathsf{noy}(Q)$	6	1.73	89272	5686
$\mathsf{syq}(P, Q) \circ Q^{\smile} \sqsubseteq \overline{P}^{\smile}$	14	0.38	24602	1649
$\mathsf{syq}(P, \overline{Q}) \circ \overline{Q}^{\smile} \sqsubseteq P^{\smile}$	12	0.38	24268	1601
$\mathsf{noy}(P) \sqsubseteq \mathsf{noy}(P)^{\smile}$	3	0.04	1260	322
$\boldsymbol{\iota} \sqsubseteq \mathsf{noy}(P)$	8	0.13	3229	1450

By defining equiex $=_{\mathrm{Def}}$ noy($\in$) (i.e., for two sets x and y, $x\,\mathsf{equiex}\,y$ holds if and only if they contain the same elements—recall that in this section we are not postulating the extensionality axiom), Otter was able to draw some consequences of the above laws:

law	length of proof	time (sec.)	generated clauses	kept clauses
$\mathsf{equiex} = \mathsf{equiex}^{\smile}$	5	0.01	41	20
$\mathsf{equiex} \circ \ni \sqsubseteq \ni$	3	0.01	326	182
$\mathsf{equiex} \circ \ni \circ \in \sqsubseteq \ni \circ \in$	2	0.01	432	249
$\mathsf{equiex} \circ \not\ni \sqsubseteq \not\ni$	5	0.34	8858	6866

By using these laws, we obtained the following derivation of the general law RUniq(syq(P, $\in$)$\circ$arb):

- $\overline{\mathsf{parb}} \circ \mathsf{arb}^{\smile} \subseteq \overline{\mathsf{equiex}}$,

- $\mathsf{equiex} \circ \mathsf{parb} \circ \overline{\boldsymbol{\iota}} \subseteq \mathsf{parb} \circ \overline{\boldsymbol{\iota}}$,

- $\mathsf{parb} \circ \overline{\boldsymbol{\iota}} \circ \mathsf{arb}^{\smile} \subseteq \overline{\mathsf{equiex}}$,

- $\overline{\mathsf{parb} \cdot \overline{\mathsf{parb} \circ \overline{\boldsymbol{\iota}}}} \circ \mathsf{arb}^{\smile} \subseteq \overline{\mathsf{equiex}}$,

- $\mathsf{equiex} \circ \mathsf{arb} \subseteq \mathsf{arb}$,

- $\mathsf{syq}(P^{\smile}, \in)^{\smile} \circ \mathsf{syq}(P^{\smile}, \in) \;\subseteq\; \mathsf{equiex}$,

- $(\mathsf{syq}(P^{\smile}, \in) \circ \mathsf{arb})^{\smile}) \circ \mathsf{syq}(P^{\smile}, \in) \circ \mathsf{arb} \;\subseteq\; \mathsf{arb}^{\smile} \circ \mathsf{arb}$.

Then, $\mathsf{RUniq}(\mathsf{syq}(P, \in) \circ \mathsf{arb})$ follows by functionality of arb.

The overall time spent in proving these laws was 0.63 seconds. The longest proof (length 13) was the one of the third law, whereas the most heavily time-consuming proof was the one of the fourth law: length 12 in 0.24 seconds.

As direct consequences, Otter easily obtained $\mathsf{RUniq}(\mathsf{arb\_lessArb})$ and $\mathsf{RUniq}(\mathsf{cdr})$, as desired.

We have been unable till now to obtain an Otter-proof of $\mathsf{car}^{\smile} \circ \mathsf{cdr} = 1\!\!1$ within the calculus of relations. Obtaining such a proof will be our next task in this work. An alternative viable approach could consist in completing our weak set theory by hastily adopting $\mathsf{car}^{\smile} \circ \mathsf{cdr} = 1\!\!1$ as one of our axioms, in analogy with the way we have proceeded in the previous section.

Conclusions

We have proposed many different notions of pair and analyzed the axiomatic assumptions of least commitment under which each of them is acceptable. One may feel that the game of inventing new kinds of pairs is certainly endless (and perhaps pointless) without some definite criterion for choosing the "best" pairing notion. Against an absolute criterion, we argue that quite many different motivations for resorting to pairs exist, and they forcibly lead to different proposals.

One common aim is to enable a syntactically simple "component extraction": for instance—as we have done in this paper—, one may want projections describable in three variables, but then it is convenient to revise even the very simple and minimally committing Kuratowski's pair $(\cdot,\cdot)$, as Tarski did (see beginning of Sec.3). Under scant axiomatic assumption, our own pair $[\cdot,\cdot]$ can be preferred to $(\cdot,\cdot)$, because its projections can be characterized, and their formal properties verified, more simply (see. Sec.8).

The adoption of the pairing function $x, y \stackrel{\mathsf{pp}}{\longmapsto} \{\{x\}, \{\{x\}, y\}\}$ (not discussed in this paper, though akin to our $\ulcorner\cdot,\cdot\urcorner$) was motivated in [37] by the need of keeping the number of quantifier alternations in formulae used for the syntax arithmetization very low, in order to achieve undecidability results as sharp as possible; remarkably, that paper deals with a set theory (without regularity) where $\mathsf{pp}(\cdot,\cdot)$ generally fails to meet the pairing condition $\mathsf{pp}(x, y) = \mathsf{pp}(u, v) \;\rightarrow\; (x = u \wedge y = v)$; notwithstanding, in that context this is acceptable because the operands of pairing always are well-founded encodings of terms and formulae, which suffices to ensure the desired behavior.

A pair such as $\ulcorner\cdot,\cdot\urcorner$ has little appeal from the criterion of three-variable expressibility, however in a first-order set theory it is appealing because its projections can be characterized as simply as

$$\begin{aligned} \mathsf{car}(p) &= \mathsf{arb}(\mathsf{arb}(p)), \\ \mathsf{cdr}(p) &= \mathsf{car}(\mathsf{arb}(p \;\mathsf{less}\; \mathsf{arb}(p)) \;\mathsf{less}\; \mathsf{arb}(p)). \end{aligned}$$

Can you do the same with Kuratowski's pair?

As we recalled in Sec.1, there are extremely weak set theories which cannot be expressed in three variables: for them we cannot find conjugated projections. Where does the borderline lie between one such theory (e.g., $\mathbf{(E)} \wedge \mathbf{(W)} \wedge \mathbf{(L)}$) and one which supports pairing? On the basis of the experimental results in Sections 8, 10, and 11, it becomes clear that proving 3-variable inexpressibility of $\mathbf{(W)} \wedge \mathbf{(L)}$ necessarily calls for the construction of a (non-standard) set-theoretic model which does not fulfill the acyclicity of membership.

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