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Comparative uncertainty: theory and automation

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In the last decades, qualitative approaches to probabilistic uncertainty are receiving wider and wider attention. We propose a characterization of partial preference orders through an uniform axiomatic treatment of a variety of qualitative uncertainty notions. To this aim we prove a representation result that connects qualitative notions of partial uncertainty to their numerical counterparts. We describe an executable specification, in the declarative framework of Answer Set Programming, that constitutes the core engine for qualitative management of uncertainty. Some basic reasoning tasks are also identified.
KEY WORDS: Uncertainty order relations, qualitative uncertainty frameworks, partial assessments, answer set programming.

1. Introduction

Where are all the numbers coming from?
—Peter Cheeseman 1988

Numerous formalisms for dealing with uncertainty have been studied during the last century. Most of the proposed probabilistic models of decision under uncertainty rely on numerical measures and representations. All of them originate from amendments of the well-known Probability measure (Savage 1972), usually aimed at generalizing it to better fit different peculiarities of specific application fields. As a matter of fact, several authors criticized such use of the classical theory of probability, by emphasizing that common people, in expressing their intuitive judgments on the likelihood of events, often (deliberately) violate the postulates of such theory. It seems reasonable that the human mental process yielding a judgment is guided by a number of heuristics which unconsciously "act behind the scenes". Subjective, psychological, and environmental factors may influence this process (Fox 1999; Fox and See 2003; Kahneman *et al.* 1982; Suppes 1974). For instance, different descriptions of the same event often give rise to substantially different judgments. Paradigmatic example of this phenomenon is the well-known

Ellsberg Paradox (Ellsberg 1961; Fishburn 1986). A way of circumventing these problems consists in dropping some of the postulates characterizing probability measures. Most of the proposals in this direction weaken the additivity property of probability functions. This could be done in several manners, obtaining alternative notions of uncertainty measures: Possibility and Necessity measures, Belief and Plausibility functions, 0-monotone and 0-alternating functions, etc. For a survey the reader can refer to (Klir and Folger 1988; Nguyen and Walker 1997; Walley 1996), among others. (App. A briefly summarizes some basic notions about uncertainty measures from the quantitative point of view.)

Actually, the numerical approach itself has been criticized *in toto*, because all of the proposed numerical frameworks inevitably suffer from important drawbacks:

— The hardness, for human beings, to elicit precise numerical values;
— The difficulty of expressing a complete evaluation.

The former problem may originate from lack of adequate knowledge or expertise, or from incapability of people to correctly estimate numerical values and numerically grade their preferences. It seems, in fact, more plausible that assessments of uncertainty are based on comparative qualitative judgments (Brafman and Tennenholtz 1997; Cheeseman 1988; Dubois *et al.* 2002; Parsons 1994; Redelmeier *et al.* 1995; Tversky 1974). Moreover, other aspects may make numerical elicitations not appropriate from a practical point of view. Indeed, studies concerning the sensitiveness of automated decision systems with respect to the accuracy of the given numerical assessments (e.g., (Pradhan *et al.* 1996)) suggest that there exist fields of applications where there could be little advantage in forcing the user to express precise quantifications of preferences. Hence, rough assessments or even qualitative treatments could yield similar results with less complex modeling and, often, lower computational efforts. To obviate such weakness of numerical models, *qualitative approaches* have been proposed in the last decades and are receiving wider and wider attention, either as theoretical tools to deal directly with belief management (Bilgiç 2001; de Cooman 1997; Dubois 1986), or inside the more articulated framework of decision-making theory (see, for example, (Dubois *et al.* 1997; Dubois *et al.* 2003; Dubois *et al.* 1997; Giang and Shenoy 2001)). The central idea of such methodologies is to grade uncertainty about the truth of propositions, through comparisons expressing the judgment of “less or more believed to be true”. This operationally translates into the use of order relations in place of numerical grades. The notions of qualitative probabilities, qualitative plausibility, and so on, are then introduced. (Often, the terms *comparative* probabilities, *comparative* plausibility, etc., are used.)

Similar reasons originate the difficulty of expressing a *complete* evaluation. In the context of qualitative models (but the same argument applies to the numerical case) it is doubtful that a set of preferences expressed by a human being constitutes a complete (i.e., total) order of the domain of discernment. There could be several reasons that prevent the subject of the analysis from providing such complete information. (S)he may be incapable or reluctant to describe a total order in very rich contexts, for instance, because of substantial difficulties in comparing every possible event, or because (s)he is interested in reasoning about a restricted portion of the domain (Brafman and Tennenholtz 1997; Halpern 1997; Kaplan and Fine 1977; Lehmann 1996; Tversky and Simonson 1993). This problem can be circumvented by adapting to the qualitative framework the pioneering approach proposed by de Finetti in the context of Probability measures (de Finetti 1931; de Finetti 1974). Namely, by introducing the so called *partial models*, i.e. qualitative assessments defined only on some of the situations at hand and intended to be restrictions of some complete models. (Then, we deal with partial capacities, partial probabilities, and so on.) This approach allows the analyst of the problem to focus his/her evaluation on the situations really judged relevant, w.r.t. the problem at hand. This also leaves open the possibility to enlarge the model to other scenarios that could enter on the scene later.

The article is organized as follows. In the next section we introduce basic notions and provide an axiomatic view of the most studied uncertainty orders. (Notice that we focus on the treatment of partial orders, even if total relations can easily be dealt with by exploiting the very same machinery.) Sec. 3 illustrates the potentialities of the declarative approach, by describing two reasoning tasks that exploit preference orders. In Sec. 4 we briefly describe a tool for declarative management of preference orders. Such a tool is ultimately based on a successful form of declarative programming, namely Answer Set Programming (Lifschitz 1999; Marek and Truszczyński 1999). Finally, we draw some conclusions and outline future developments. For the ease of the reader, Appendix A briefly summarizes some basic notions about uncertainty measures from the quantitative point of view. Appendix B recalls the main features of Answer Set Programming, with particular emphasis on its application to our context.

2. Axiomatization of partial preference orders

Let us recall some notions on uncertainty orders. The domain of discernment is represented by a finite set of events $\mathcal{E} = \{E_1, \ldots, E_n\}$ (among them, ϕ and Ω denote the impossible and the sure event, respectively). The events are seen as the relevant propositions on which the subject of the analysis expresses his/her opinion. As mentioned, $\mathcal{E}$ does not necessarily represent a full model, i.e. it does not comprehend all elementary situations and all of their combinations. For this reason, a crucial component of partial assessments is the knowledge of the logical relationships (incompatibilities, implications, combinations, equivalences, etc.) holding among events. Such relationships are usually expressed by stating a collection $\mathcal{C}$ of constraints on the events (as well as, on conjunctions and disjunctions of events). By taking into account the constraints $\mathcal{C}$, the family $\mathcal{E}$ spans a minimal Boolean algebra $\mathcal{A}_{\mathcal{E}}$ containing $\mathcal{E}$ itself. Note that $\mathcal{A}_{\mathcal{E}}$ is only implicitly defined via $\mathcal{E}$ and $\mathcal{C}$ and it is not a part of the assessment. Anyway, $\mathcal{A}_{\mathcal{E}}$ can be referenced as a supporting structure.

Definition 2.1. Let $\mathcal{E}$ be a set of events and $\mathcal{C}$ a collection of constraints on $\mathcal{E}$. $\mathcal{A}_{\mathcal{E}}$ is the minimal algebra $(\mathcal{A}, \cup, \cap, ^{-}, \phi, \Omega)$ satisfying $\mathcal{C}$ and such that $\mathcal{E} \subseteq \mathcal{A}$.

Such an algebra induces a lattice structure on $\mathcal{A}$. The *atoms* of $\mathcal{A}_{\mathcal{E}}$ are the minimal elements of the (sub-)lattice $\mathcal{A}_{\mathcal{E}} \setminus \{\phi\}$. Then, each event corresponds to a set of atoms and $\mathcal{A}_{\mathcal{E}}$ is (partially) ordered by set inclusion.

An useful piece of notation: in what follows, given any binary relation R, the writing $\neg(A\mathsf{R}B)$ means that the pair $\langle A, B\rangle$ does not belong to R.

Definition 2.2. Let $\mathcal{A}_\mathcal{E}$ be an algebra of events. A binary relation $\preccurlyeq^*$ over $\mathcal{A}$ is a *(total) preference order* if it satisfies the following conditions:

(A1) $\preccurlyeq^*$ is a pre-order, i.e. it is reflexive, transitive, and total;†
(A2) $\phi \preceq^* \Omega$ and $\neg(\Omega \preceq^* \phi)$ (*non-triviality*);
(A3) for all events A, B, $A \subseteq B \to (A \preccurlyeq^* B)$ (*monotonicity*).

If $\preccurlyeq^*$ is a total preference order, $\sim^*$ is its *symmetric factor*, i.e. $\forall E_1, E_2\, (E_1 \sim^* E_2 \leftrightarrow E_1 \preccurlyeq^* E_2 \wedge E_2 \preccurlyeq^* E_1)$. Moreover, $\prec^*$ is the *asymmetric factor* of $\preccurlyeq^*$, i.e. $\forall E_1, E_2\, (E_1 \prec^* E_2 \leftrightarrow E_1 \preccurlyeq^* E_2 \wedge \neg(E_2 \sim^* E_1))$.

In order to deal with partial assessments, it is convenient to consider two distinct but correlated preference relations. Roughly speaking, the two relations model weak and strict user's preferences, respectively. The following definition formalizes this concept.

Definition 2.3. Let $\preccurlyeq$ and $\prec$ be binary relations over a set of events $\mathcal{E}$, such that $E_1 \prec E_2 \;\to\; E_1 \preccurlyeq E_2$. The pair $\langle \preccurlyeq, \prec \rangle$ is a *weak preference structure* for $\mathcal{E}$ (*w.p.s.*, for short) if there exists a total preference order $\preccurlyeq^*$ over $\mathcal{A}_\mathcal{E}$ such that: $\forall E_1, E_2 \in \mathcal{E}\, \big((E_1 \preccurlyeq E_2 \;\to\; E_1 \preccurlyeq^* E_2) \wedge (E_1 \prec E_2 \;\to\; E_1 \prec^* E_2)\big)$.

Notice that Definition 2.3 does not require either $\preccurlyeq$ or $\prec$ to be total orders, or $\prec$ to be the asymmetric factor of $\preccurlyeq$. On the other hand, it is required that $\preccurlyeq^*$ extends $\preccurlyeq$, and that $\prec^*$ (the asymmetric factor of $\preccurlyeq^*$) extends $\prec$.

Example 2.1. Consider the following situation, concerning a decision making problem in gastroenterology.‡ Let us consider three possible diseases that a patient might suffer: peptic ulcer, gastric cancer, and biliar disease. The symptoms that might be associated to these diseases are jaundice, weight loss, and dark stools. Presence of jaundice indicates biliar disease, weight loss can be associated to gastric cancer, dark stools might indicate peptic ulcer or gastric cancer. From data provided by the hospitals we know that the incidence of peptic ulcer is greater than the incidence of gastric cancer, while biliar disease affect the majority of the patients. Clearly, information useful to make a diagnosis are age and sex of the patient: peptic ulcer and gastric cancer are more frequent in men; biliar disease are more often complained by women. Moreover, in male population, incidence of ulcer is greater then the incidence of biliar disease. As regards age, we can reasonably affirm that older people are more subject to peptic ulcer or gastric cancer than young people. This scenario can be so represented:

$\mathsf{GC} \equiv$ *The real state of suffering from gastric cancer*	$\mathsf{JA} \equiv$ *Jaundice symptoms*
$\mathsf{PU} \equiv$ *The real state of suffering from peptic ulcer*	$\mathsf{WL} \equiv$ *Weight loss symptoms*
$\mathsf{BD} \equiv$ *The real state of suffering from biliar disease*	$\mathsf{DS} \equiv$ *Dark stools*

Let M (resp., W) denote the event *The patient is male* (resp., *female*), and OA (resp., YA) denote the event *The patient is old* (resp., *young*).

† As usual, $\preccurlyeq^*$ is total over $\mathcal{A}$ if for all events $E_1, E_2 \in \mathcal{A}$ it holds that $E_1 \preccurlyeq^* E_2$ or it holds that $E_2 \preccurlyeq^* E_1$. Notice that it could be the case that for $E_1 \neq E_2$, both $E_1 \preccurlyeq^* E_2$ and $E_2 \preccurlyeq^* E_1$ hold.

‡ This example is hypothetical and for illustrative purpose only. It is not intended to express any clinical competence.

Finally, let us focus on the simplified situation in which any patient suffers from at most one disease. The knowledge about diseases and symptoms can be so described in terms of logical constraint: $\mathsf{JA} \cap \mathsf{GC} = \mathsf{JA} \cap \mathsf{PU} = \mathsf{WL} \cap \mathsf{PU} = \phi$, $\mathsf{WL} \cap \mathsf{BD} = \mathsf{DS} \cap \mathsf{BD} = \phi$, $\mathsf{GC} \cap \mathsf{PU} = \mathsf{GC} \cap \mathsf{BD} = \mathsf{PU} \cap \mathsf{BD} = \phi$, $\mathsf{GC} \cup \mathsf{PU} \cup \mathsf{BD} = \Omega$, $\mathsf{OA} \cap \mathsf{YA} = \mathsf{M} \cap \mathsf{W} = \phi$, $\mathsf{OA} \cup \mathsf{YA} = \mathsf{M} \cup \mathsf{W} = \Omega$. Moreover, due to events' meaning, it seems reasonable to describe a w.p.s. as follows: $\phi \prec \mathsf{GC} \prec \mathsf{PU} \prec \mathsf{BD} \prec \Omega$, $\mathsf{GC} \cap \mathsf{W} \prec \mathsf{GC} \cap \mathsf{M}$, $\mathsf{PU} \cap \mathsf{W} \prec \mathsf{PU} \cap \mathsf{M}$, $\mathsf{BD} \cap \mathsf{M} \prec \mathsf{BD} \cap \mathsf{W}$, $\mathsf{BD} \cap \mathsf{M} \prec \mathsf{PU} \cap \mathsf{M}$, $\mathsf{YA} \cap (\mathsf{GC} \cup \mathsf{PU}) \preccurlyeq \mathsf{OA} \cap (\mathsf{GC} \cup \mathsf{PU})$.

The following property can be easily verified.

Proposition 2.1. For any w.p.s. $\langle \preccurlyeq, \prec \rangle$ for $\mathcal{E}$, the following properties hold:

(A1') if there exist $E_1, \ldots, E_n \in \mathcal{E}$ such that $E_1 \preccurlyeq E_2 \preccurlyeq \ldots \preccurlyeq E_n \preccurlyeq E_1$, then $\neg(E_i \prec E_j)$ for any $i, j \in \{1, \ldots, n\}$;
(A2') $\neg(\Omega \preccurlyeq \phi)$;
(A3') for all $E_1, E_2 \in \mathcal{E}$, $E_1 \prec E_2 \rightarrow E_2 \not\subseteq E_1$.

Conditions (A1')–(A3') ensure the existence of a total preference order $\preccurlyeq^*$ which enlarges $\langle \preccurlyeq, \prec \rangle$. Considering numerical approaches to uncertainty, Capacities measures (Choquet 1954) constitute the most general framework, as they express "common sense" behaviors. Any reasonable relation $\preccurlyeq$ must be representable by a partial Capacity (i.e., a restriction of a Capacity measure to the set of events at hand). This corresponds to the satisfaction of the conditions (A1')–(A3') (de Cooman 1997).

Further differentiations among uncertainty notions are done by considering the specific way of combining distinct pieces of information (e.g., as mentioned, for Probabilities additivity is adopted). Within the numerical context, this yields a taxonomy of numerical measures. By following (Dubois 1986; Walley and Fine 1979; Wong *et al.* 1991), a classification of various preference notions has been proposed in (Capotorti and Vantaggi 2000) according to their compliance to the numerical models.

In what follows we revise the treatment presented in (Capotorti *et al.* 1998; Capotorti and Vantaggi 2000) by adapting it to the notion of w.p.s. The correspondence between a qualitative uncertainty notion and a numerical measure is given in terms of a representability result. More formally, we have the following definition.

Definition 2.4. Let $\mathcal{E}$ be a set of events. A total preference order $\preccurlyeq^*$ over $\mathcal{A}_\mathcal{E}$ is said to be *representable* by a numerical measure $f : \mathcal{A}_\mathcal{E} \rightarrow [0, 1]$ if for all $E_1, E_2 \in \mathcal{A}_\mathcal{E}$ it holds that $E_1 \preccurlyeq^* E_2 \leftrightarrow f(E_1) \leqslant f(E_2)$.

A w.p.s. $\langle \preccurlyeq, \prec \rangle$ for $\mathcal{E}$ is said to be *representable* by a partial uncertainty measure $g : \mathcal{E} \rightarrow [0, 1]$ if it admits an enlargement $\preccurlyeq^*$ over $\mathcal{A}_\mathcal{E}$ which is representable by an uncertainty measure $g^* : \mathcal{A}_\mathcal{E} \rightarrow [0, 1]$ extension of g to $\mathcal{A}_\mathcal{E}$.

We refer to any specific class of preference orders by the name of the corresponding numerical notion. The representability results we are going to present legitimate this choice. Notice that in what follows we assume the orders being closed under monotonicity (i.e. for all $E_1, E_2 \in \mathcal{E}$, $E_1 \subseteq E_2 \rightarrow E_1 \preccurlyeq E_2$).

Proposition 2.2 states a necessary and sufficient condition for a w.p.s. to be representable by a belief function. Representability by other uncertainty measures are stated similarly (see below).

Proposition 2.2 (Comparative belief). A w.p.s. $\langle \preccurlyeq, \prec \rangle$ for $\mathcal{E}$ can be extended to a total preference order $\preccurlyeq^*$ over $\mathcal{A}_\mathcal{E}$ representable by a belief function if and only if for all $X, Y, Z, W \in \mathcal{E}$ s.t. $X \subset Y$, $Z \subset W \subset Y$, $W \setminus Z \subseteq Y \setminus X$ the following implication holds:

$$\text{(B')} \qquad X \sim Y \rightarrow \neg(Z \prec W).$$

Proof. ($\Rightarrow$). By hypothesis and by the characterization theorem for total comparative beliefs (Wong *et al.* 1991), there exists a total order $\preccurlyeq^*$ extending $\langle \preccurlyeq, \prec \rangle$ and satisfying the following condition (which characterizes complete beliefs):

(B) $\forall A, B, C \in \mathcal{A}_\mathcal{E}$ s.t. $A \subset B, B \cap C = \phi, A \prec^* B \rightarrow A \cup C \prec^* B \cup C$.

Let us assume that (B') does not hold. Then there exist $X, Y, Z, W \in \mathcal{E} \subseteq \mathcal{A}_\mathcal{E}$ s.t. $X \subset Y$, $Z \subset W \subset Y$, and $W \setminus Z \subseteq Y \setminus X$ with $X \sim^* Y$ and $Z \prec^* W$. Hence, by monotonicity and transitivity of $\preccurlyeq^*$ we have

$$X \sim^* X \cup Z \sim^* X \cup W \sim^* Y. \tag{1}$$

From $W \setminus Z \subseteq Y \setminus X$ and $Z \subset W$, we easily obtain that $(X \setminus Z) \cap \overline{X \setminus W} = X \cap (W \setminus Z) = \phi$ and that $\overline{X \setminus Z} \cap (X \setminus W) = X \cap (Z \setminus W) = \phi$. Then, since $(X \cup Z) \setminus Z = X \setminus Z$ and $(X \cup W) \setminus W = X \setminus W$, it follows that

$$(X \cup Z) \setminus Z = (X \cup W) \setminus W \tag{2}$$

Since $(X \setminus W) \cap W = \phi$ and $Z \prec^* W$, by (2) and (B) we obtain $Z \cup (X \setminus W) = Z \cup ((X \cup W) \setminus W) = Z \cup ((X \cup Z) \setminus Z) = X \cup Z \prec^* W \cup (X \setminus W) = X \cup W$. This contradicts (1). Thus, (B') must hold.

($\Leftarrow$). Let us observe that, since $\langle \preccurlyeq, \prec \rangle$ is a w.p.s., by Proposition 2.1, (A2') and (A3') hold for $\langle \preccurlyeq, \prec \rangle$. This allows us to add to $\langle \preccurlyeq, \prec \rangle$ the relationships $\phi \prec \Omega$ and $E_1 \preccurlyeq E_2$ for all $E_1, E_2 \in \mathcal{A}_\mathcal{E}$ s.t. $E_1 \subseteq E_2$, without introducing inconsistency (i.e., the presence of both $E_i \prec E_j$ and $E_j \preccurlyeq E_i$, for some E_i, E_j). Hence, without loss of generality, we can assume from the beginning $\langle \preccurlyeq, \prec \rangle$ such that $\phi \prec \Omega$ holds and $\preccurlyeq$ is closed by monotonicity.

Let us describe a construction of the enlargement $\preccurlyeq^*$. First, we construct an enlargement $\langle \preccurlyeq', \prec' \rangle$ over $\mathcal{A}_\mathcal{E}$ closed under axioms (B) and (B'), i.e. such that

$$\forall A, B, C \in \mathcal{A}_\mathcal{E}, A \subset B, B \cap C = \phi \quad A \prec' B \rightarrow A \cup C \prec' B \cup C \tag{3}$$

$$\forall X, Y, Z, W \in \mathcal{A}_\mathcal{E}, X \subset Y, Z \subset W \subset Y, W \setminus Z \subseteq Y \setminus X \quad X \sim' Y \rightarrow Z \sim' W \tag{4}$$

Such an enlargement can be obtained by applying the following procedure.

Set $\langle \preccurlyeq', \prec' \rangle \equiv \langle \preccurlyeq, \prec \rangle$ and $\mathcal{E}' \equiv \mathcal{E}$
Repeat as long as it is possible
($\uparrow$) **If** $\exists A, B \in \mathcal{E}'$ s.t. $A \subset B, A \prec' B$ **and** $\exists C \in \mathcal{A}_\mathcal{E} \setminus \{\phi\}$ s.t. $B \cap C = \phi$ **and** $\neg(A \cup C \prec' B \cup C)$ **then**
  Add $A \cup C \prec' B \cup C$ to $\prec'$
  Set $\mathcal{E}' = \mathcal{E}' \cup \{A \cup C, B \cup C\}$

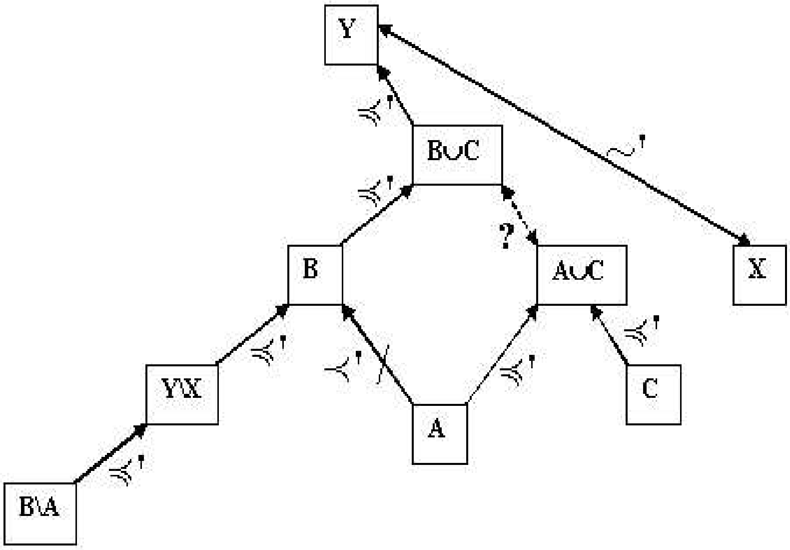

Fig. 1. A potential conflict configuration in the construction of the enlargement: considering the relationship holding between $(A \cup C)$ and $(B \cup C)$ (the edge marked by "?"), the step (↑) would declare that $(A \cup C) \prec' (B \cup C)$, while (↓) would impose $(A \cup C) \sim' (B \cup C)$.

(↓) **If** $\exists X, Y \in \mathcal{E}'$ s.t. $X \subset Y, X \sim' Y$ **and** $\exists C \in \mathcal{A}_{\mathcal{E}}$ s.t. $Y \cap C \neq \phi$ **and** $\neg(X \cap C \sim' Y \cap C)$ **then**
Add $X \cap C \preccurlyeq' Y \cap C$ and $Y \cap C \preccurlyeq' X \cap C$ to $\preccurlyeq'$
Set $\mathcal{E}' = \mathcal{E}' \cup \{X \cap C, Y \cap C\}$

At each iteration of the **Repeat**-loop, the step (↑) enlarges the relation $\prec'$ without affecting $\preccurlyeq'$. Similarly, each step (↓) enlarges the relation $\preccurlyeq'$ without influencing $\prec'$. Each execution of steps (↑) and (↓) preserves the validity of condition (B') for the w.p.s. $\langle \preccurlyeq', \prec' \rangle$ over $\mathcal{E}'$, while (B) is not falsified. By construction, (B) is satisfied at the end of procedure. Moreover, we have that steps (↑) and (↓) do not conflict, i.e. it never happens that there exist a $C \in \mathcal{A}_{\mathcal{E}} \setminus \{\phi\}$ and $A, B, X, Y \in \mathcal{E}'$, with $A \subset B$, $B \cap C = \phi$, $X \subset Y$, $(B \cup C) \subset Y$, and $B \setminus A \subseteq Y \setminus X$, but s.t. $A \prec' B$ and $X \sim' Y$ (cf. Figure 1, where we depict events as nodes of a graph whose edges represent order relationships).

Things are so because, otherwise, (B') could be falsified in $\mathcal{E}'$ by putting $Z \equiv (A \cup C)$, $W \equiv (B \cup C)$ and X, Y as they are. Finally, by construction of $\mathcal{E}'$, after each iterations it holds that $\mathcal{A}_{\mathcal{E}'} \equiv \mathcal{A}_{\mathcal{E}}$. This fact, since $\mathcal{A}_{\mathcal{E}}$ is finite, guarantees the termination of the procedure. At this point, $\preccurlyeq'$ can be closed under monotonicity over $\mathcal{A}_{\mathcal{E}}$ preserving (B'). The w.p.s. $\langle \preccurlyeq', \prec' \rangle$ so obtained can be enlarged to obtain $\langle \preccurlyeq'', \prec'' \rangle$ as follows: first add all "inverted" pairs (E_j, E_i) such that $E_i \preccurlyeq' E_j$ and such addition does not create intransitive cycles; then, impose a strict order for all the remaining pairs:

$$\preccurlyeq'' \quad =_{\text{Def}} \quad \preccurlyeq' \cup \{E_j \preccurlyeq'' E_i : E_i \preccurlyeq' E_j \text{ and axiom (A1') is preserved}\}$$
$$\prec'' \quad =_{\text{Def}} \quad \prec' \cup \{E_i \prec'' E_j : E_i \preccurlyeq'' E_j \text{ and } \neg(E_j \preccurlyeq'' E_i)\}$$

Note that, by (3) and (4), the w.p.s. $\langle \preccurlyeq', \prec' \rangle$ already contains all the equivalences $X \sim' Y$ and all the strict preferences $A \prec' B$ over $\mathcal{A}_{\mathcal{E}}$ that axioms (B) and (B') could involve, hence (B) is satisfied by the w.p.s. $\langle \preccurlyeq'', \prec'' \rangle$.

If we close $\langle \preccurlyeq'', \prec'' \rangle$ by transitivity, we obtain that (the closure of) $\preccurlyeq''$ induces a partial order relation over the quotient set $\mathcal{A}_{\mathcal{E}}/\sim''$. In what follows, for the sake of simplicity, let us use $\preccurlyeq''$ also to denote such induced order over $\mathcal{A}_{\mathcal{E}}/\sim''$. Moreover, let $\preccurlyeq''|_S$ denote the restriction of $\preccurlyeq''$ to the set S. The complete order $\preccurlyeq^*$ can be obtained by this procedure:

Set $\mathcal{A}$ to $\mathcal{A}_{\mathcal{E}}/\sim''$ and $\mathcal{F}$ to the lattice $(\mathcal{A}, \preccurlyeq'')$
Repeat until $\mathcal{A} \neq \phi$
  Set $\mathcal{M}$ to be the set of minimal elements of the lattice $\mathcal{F}$
  Set $E_i \sim^* E_j$ for all $E_i, E_j \in \mathcal{M}$
  Set $E_h \prec^* E_k$ for all $E_h \in \mathcal{M}$ and $E_k \in \mathcal{A} \setminus \mathcal{M}$
  Set $\mathcal{A} \equiv \mathcal{A} \setminus \mathcal{M}$ and $\mathcal{F} \equiv (\mathcal{A} \setminus \mathcal{M}, \preccurlyeq''|_{\mathcal{A}\setminus\mathcal{M}})$

This procedure preserves (B). In fact, since $\preccurlyeq'$ was closed by monotonicity, the w.p.s. $\langle \preccurlyeq'', \prec'' \rangle$ already predicates on all pairs of elements among $A, B, A \cup C$, and $B \cup C$ (with $A \subset B$ and $B \cap C = \phi$), while the construction of $\preccurlyeq^*$ does not impose further constraints among them. Hence, by the characterization theorem for beliefs (Wong *et al.* 1991), $\preccurlyeq^*$ is representable by a belief function. □

An analogous property can be stated for comparative 0-monotonicity. Propositions 2.3 relates the qualitative notions to the corresponding quantitative measures. The proof proceeds along the lines of Proposition 2.2 but employing, in place of condition (B), the appropriate characterization of total preference.

Proposition 2.3 (Comparative 0-monotonicity). Let $\langle \preccurlyeq, \prec \rangle$ be a w.p.s. for $\mathcal{E}$. Then $\langle \preccurlyeq, \prec \rangle$ can be extended to a total preference order $\preccurlyeq^*$ over $\mathcal{A}_{\mathcal{E}}$ representable by a 0-monotone function[§] if and only if for all $X, Y, Z \in \mathcal{E}$ s.t. $X \subset Y, Z \subseteq Y \setminus X$ it holds that

$$\text{(0M')} \qquad X \sim Y \rightarrow \neg(\phi \prec Z).$$

Proof. (Sketch) By recalling that complete comparative 0-monotone orders are characterized by the axiom

$$\text{(0M)} \quad \forall E, H \in \mathcal{A}_{\mathcal{E}} \text{ s.t. } E \cap H = \phi, \quad \phi \prec^* E \rightarrow H \prec^* E \cup H,$$

the algorithm outlined in the proof of Proposition 2.2 must be adapted to 0-monotone orders by modifying the steps (↑) and (↓) as follows:

(↑) **If** $\exists E, H \in \mathcal{E}'$ s.t. $E \cap H = \phi, \phi \prec' E$ **and**
  $\neg(H \prec' E \cup H)$ **then**
    Add $H \prec' E \cup H$ to $\prec'$
    Set $\mathcal{E}' = \mathcal{E}' \cup \{E \cup H\}$

[§] 0-monotonicity is often referred to as *super-additivity*, cf. Appendix A.

($\downarrow$) **If** $\exists X, Y \in \mathcal{E}'$ s.t. $X \subset Y, X \sim' Y$ **and** $\exists Z \in \mathcal{A}_{\mathcal{E}}$ s.t. $Z \subseteq Y \setminus X$ **and** $\neg(\phi \sim' Z)$ **then**
Add $\phi \sim' Z$ to $\preccurlyeq'$
Set $\mathcal{E}' = \mathcal{E}' \cup \{Z\}$

Figure 2 shows how a potential violation of (0M) (obtainable by putting $Z \equiv E$) is avoided in virtue of (0M'). $\square$

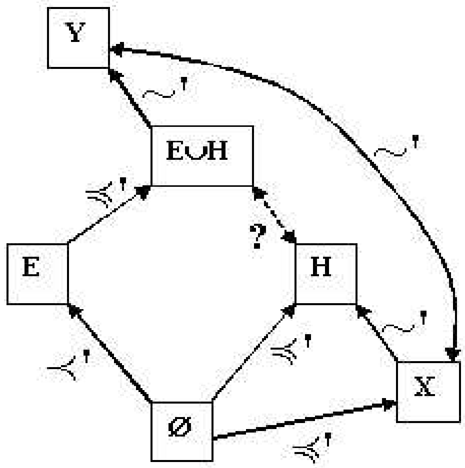

Fig. 2. A critical configuration in the enlargement procedure for comparative 0-monotonicity orders (cf. Figure 1 and Prop. 2.3).

Other notions of preference orders (Proposition 2.4) are treated uniformly and the proofs of the corresponding representability results proceeds along the same lines sketched in Propositions 2.2 and 2.3.

Proposition 2.4. Let $\langle \preccurlyeq, \prec \rangle$ be a w.p.s. for $\mathcal{E}$. The following properties hold:
Comparative plausibility. $\langle \preccurlyeq, \prec \rangle$ can be extended to a total preference order $\preccurlyeq^*$ over $\mathcal{A}_{\mathcal{E}}$ *representable by a plausibility function* (equivalently, by a n-alternating function, with $n \geq 2$) if and only if for all $X, Y, Z, W \in \mathcal{E}$ s.t. $X \subset Y$, $Z \subset W \subset Y$, $W \setminus Z = Y \setminus X$ it holds that

$$\text{(PL')} \qquad X \prec Y \rightarrow \neg(Z \sim W).$$

Compare (PL') with the corresponding axiom (PL) characterizing complete comparative plausibility orders:

$$\text{(PL)} \quad \forall A, B, C \in \mathcal{A}_{\mathcal{E}} \text{ s.t. } A \subseteq B, B \cap C = \phi, A \sim^* B \rightarrow A \cup C \sim^* B \cup C.$$

Comparative 0-alternation. $\langle \preccurlyeq, \prec \rangle$ can be extended to a total preference order $\preccurlyeq^*$ over $\mathcal{A}_{\mathcal{E}}$ representable by a 0-alternating function[¶] if and only if or all $X, Y \in \mathcal{E}$ s.t. $X \subset Y$,

[¶] 0-alternation is often referred to as *sub-additivity*, cf. Appendix A.

it holds that

$$\text{(0A')} \qquad X \sim Y \rightarrow \neg(X \cup (\Omega \setminus Y) \prec \Omega).$$

(Recall that complete comparative 0-alternating orders can be characterized by

$$\text{(0A)} \quad \forall K, F \in \mathcal{A}_{\mathcal{E}} \text{ s.t. } K \cup F = \Omega, \quad F \prec^* \Omega \rightarrow K \cap F \prec^* K. \ )$$

To the best of our knowledge, there exists no purely qualitative characterization of comparative probability. This notion seems to have an intrinsically numerical character. Among the possible characterizations proposed in literature, the following one is drawn from (Coletti 1990):

Proposition 2.5 (Comparative probability). A w.p.s. $\langle \preccurlyeq, \prec \rangle$ for $\mathcal{E}$ can be extended to a total order $\preccurlyeq^*$ over $\mathcal{A}_{\mathcal{E}}$ representable by a probability function if and only if
(CP) for any $X_1, \ldots, X_n, Y_1, \ldots, Y_n \in \mathcal{E}$, with $Y_i \preccurlyeq X_i, \forall i = 1, \ldots, n$, such that for some $r_1, \ldots, r_n > 0$ $\sup\left(\sum_{i=1}^{n} r_i(a_i - b_i)\right) \leq 0$, implies that $X_i \sim Y_i$, for all $i = 1, \ldots, n$ (a_i, b_i are the indicator functions of X_i, Y_i, resp.).

Axiom (CP) involves quantitative notions (e.g., indicator functions and summations) and its verification requires numerical elaborations. Nevertheless, it is possible to (qualitatively) state a *necessary*, but not sufficient, condition for representability of an order through a probability function.

Proposition 2.6 (*Weak* comparative probability). If $\langle \preccurlyeq, \prec \rangle$ can be extended to a total preference order $\preccurlyeq^*$ over $\mathcal{A}_{\mathcal{E}}$ representable by a probability function then for all $X, Y, Z \in \mathcal{E}$ s.t. $X \cap Z = Y \cap Z = \phi$, it holds that

$$\text{(WC)} \qquad X \preccurlyeq Y \rightarrow \neg(Y \cup Z \prec X \cup Z).$$

We mentioned that the introduction of different classes of orders is motivated by the presence of practical situations where a strictly probabilistic approach is not viable. Here is a simple example.

Example 2.2. Let A, B, and C be three companies, each of them a potential buyer of a firm that some other company wants to sell. In spite of being distinct, A and C belong to the same holding. Hence, the following uncertainty order about which company will be the buyer, could reflect specific information about the companies' strategies (by abuse of notation, let A denote the event "the company A buys the firm", and similarly for B and C): $\phi \prec A \prec B \prec B \cup C \prec A \cup C \prec \Omega$. Since A, B and C are incompatible, it is immediate to see that the order relation is not representable by a probability because it violates axiom (WC), while it can be managed in line with belief functions behaviors because it agrees with axiom (B').

3. Reasoning tasks for preference orders

In this section we describe two reasoning tasks that exploit preference orders. Such tasks can be seen as the basic constituents of expert systems and decision-support tools that handle qualitative knowledge in form of comparative assessments.

Qualitative framework detection. This is a classification task: Given a (partial) assessment (which means a description of domain of discernment, constraints, and preferences), the goal consists in detecting which is the most stringent among all compatible uncertainty frameworks. Notice that, by proceeding in this way, we actually invert the usual attitude towards qualitative management of uncertainty. In fact, specific axioms are usually set in advance, so that only relations satisfying them are admitted. Here, on the contrary, given a fixed preference relation, the goal consists in ascertaining which are the reasonable rules to work with.

Considering an assessment as the outcome of a reasoning process performed by an agent (human or not), detecting the correct uncertainty framework provides useful information about the cognitive schema of the agent. Selecting the most restrictive framework (among those appropriate) clearly corresponds to adopting a sort of "cautious approach" in interpreting agent's thought. This guides one in determining agent's conceptualization of uncertainty (i.e., its way of expressing lack of information and variability of phenomena) and its (implicit) model of the problem at hand.

Such a detection process can be, for instance in a multi-agent system, of great help in constructing more informed representation of (other) agents' models of reality. This translates in better strategies in agent modeling, decision making, and plan recognition, i.e., the attempt of inferring the plans of other agents by communicating with them or by observing their behaviors.

The following is an example of framework detection.

Example 3.1. Suppose a physician wants to perform a preliminary assessment about the reliability of a test for SARS (Severe Acute Respiratory Syndrome). Up to his/her knowledge, the SARS diagnosis is based on moderate or severe respiratory symptoms and on the positivity or indeterminacy of an adopted clinical test about the presence of the SARS-associated antibody coronavirus SARS-CoV. The elements appearing in his/her analysis can be summarized as follows:
$\mathsf{A} \equiv$ *Normal respiratory symptoms*, $\mathsf{B} \equiv$ *Moderate respiratory symptoms*,
$\mathsf{C} \equiv$ *Severe respiratory symptoms*, $\mathsf{D} \equiv$ *Moderate or severe respiratory symptoms*,
$\mathsf{E} \equiv$ *Death from pulmonary diseases*, $\mathsf{F} \equiv$ *Positive or indeterminate clinical test*,
subject to these (logical) restrictions: $\mathsf{A}\cap\mathsf{B}=\emptyset$, $\mathsf{B}\cap\mathsf{C}=\emptyset$, $\mathsf{A}\cap\mathsf{C}=\emptyset$, $\mathsf{A}\cup\mathsf{B}\cup\mathsf{C}=\Omega$, $\mathsf{D}=\mathsf{B}\cup\mathsf{C}$, $\mathsf{E}\subset\mathsf{C}$, $\mathsf{F}\cap\mathsf{A}=\emptyset$. Consider the w.p.s. $\langle\preccurlyeq,\prec\rangle$ so described: $\emptyset\prec\mathsf{C}$, $\mathsf{C}\prec\mathsf{B}$, $\mathsf{B}\preccurlyeq\mathsf{A}$, $\mathsf{C}\prec\mathsf{D}$, $\mathsf{E}\prec\mathsf{C}$, $\mathsf{E}\prec\mathsf{D}$, $\mathsf{F}\prec\mathsf{A}$, $\mathsf{A}\cup\mathsf{E}\sim\mathsf{A}\cup\mathsf{C}$. Due to events' meaning, such order seems reasonable. Considering the axioms described in Sec. 2, it is easy to verify that such w.p.s. agrees with the basic axioms, however it cannot be managed by using either a Probability or a Belief function, since it does not satisfy the corresponding axioms. Nevertheless, one can use comparative plausibility, 0-monotone or 0-alternating functions. (See Example 4.1 below.)

Qualitative inference. An interesting task, strictly related to the previous one, consists in inferring new knowledge on the basis of a partial model. This ultimately amounts to finding an extension of a preference relation so as to take into account one or more further events extraneous to the initial assessment. Clearly, this should be achieved in a way that the extension retains the same character of the initial order (e.g., both should satisfy the same axioms). More precisely, let an initial (partial) assessment be given, expressed as a w.p.s. $\langle \preccurlyeq, \prec \rangle$ over set of known events $\mathcal{E}$. Assume that $\langle \preccurlyeq, \prec \rangle$ satisfies the axioms characterizing a specific class, say $\mathfrak{C}$, of orders. Consider a new event S (not in $\mathcal{E}$), implicitly described by a collection $\mathcal{C}'$ of set-theoretical constraints involving the events of $\mathcal{E}$. In the spirit of (Coletti 1990, Thm. 3), the problem can be formulated as: *Determine which is the "minimal" extension* $\langle \preccurlyeq^+, \prec^+ \rangle$ *(over* $\mathcal{E} \cup \{\mathsf{S}\}$*) of* $\langle \preccurlyeq, \prec \rangle$*, induced by the new event, which still belongs to the class* $\mathfrak{C}$. In other words, we are interested in ascertaining how the new event S must relate to the members of $\mathcal{E}$ in order that $\langle \preccurlyeq^+, \prec^+ \rangle$ still is in $\mathfrak{C}$. To this aim we want to determine the sub-collections $\mathcal{L}_{\mathsf{S}}$, $\mathcal{WL}_{\mathsf{S}}$, $\mathcal{U}_{\mathsf{S}}$, and $\mathcal{WU}_{\mathsf{S}}$, of $\mathcal{E}$ so defined:

$$\begin{array}{rll} E \in \mathcal{L}_{\mathsf{S}} & \text{iff} & \text{no extension } \preccurlyeq^* \text{ of } \preccurlyeq \text{ can infer that } \mathsf{S} \preccurlyeq^* E \\ E \in \mathcal{WL}_{\mathsf{S}} & \text{iff} & \text{no extension } \preccurlyeq^* \text{ of } \preccurlyeq \text{ can infer that } \mathsf{S} \prec^* E \\ E \in \mathcal{U}_{\mathsf{S}} & \text{iff} & \text{no extension } \preccurlyeq^* \text{ of } \preccurlyeq \text{ can infer that } E \preccurlyeq^* \mathsf{S} \\ E \in \mathcal{WU}_{\mathsf{S}} & \text{iff} & \text{no extension } \preccurlyeq^* \text{ of } \preccurlyeq \text{ can infer that } E \prec^* \mathsf{S} \end{array}$$

Consequently, in order to satisfy the axioms characterizing $\mathfrak{C}$, any weak preference structure $\langle \preccurlyeq^+, \prec^+ \rangle$ extending $\langle \preccurlyeq, \prec \rangle$ must, at least, impose that:

$$\begin{array}{llll} E \prec^+ \mathsf{S} & \forall E \in \mathcal{L}_{\mathsf{S}}, \qquad & E \preccurlyeq^+ \mathsf{S} & \forall E \in \mathcal{WL}_{\mathsf{S}}, \\ \mathsf{S} \prec^+ E & \forall E \in \mathcal{U}_{\mathsf{S}}, \qquad & \mathsf{S} \preccurlyeq^+ E & \forall E \in \mathcal{WU}_{\mathsf{S}}. \end{array}$$

Example 3.2. Consider the weak preference structure of Example 3.1 and the new event:

$$\mathsf{S} \equiv \textit{The real state of having SARS}$$

subject to these restrictions: $\mathsf{S} \subset \mathsf{F}$ and $\mathsf{F} \cap \mathsf{E} \subset \mathsf{S}$. Since in Example 3.1 we discovered that the initial preference relation satisfies axiom (PL'), we want to impose such axiom and compute the extension of the initial order. We will see in Example 4.2, that in order to satisfy (PL'), the following relationships among events have to be verified by any extension involving the new event S: $\mathsf{S} \prec \mathsf{A} \cup \mathsf{C}$, $\mathsf{S} \prec \mathsf{A} \cup \mathsf{E}$, $\mathsf{S} \prec \mathsf{D}$, $\mathsf{S} \prec \mathsf{A}$, $\mathsf{S} \prec \Omega$, $\emptyset \preccurlyeq \mathsf{S}$, $\mathsf{S} \preccurlyeq \mathsf{F}$. In other words, we have $\emptyset \in \mathcal{WL}_{\mathsf{S}}$, $\{\mathsf{A} \cup \mathsf{C}, \mathsf{A} \cup \mathsf{E}, \mathsf{D}, \mathsf{A}, \Omega\} \subseteq \mathcal{U}_{\mathsf{S}}$, and $\mathsf{F} \in \mathcal{WU}_{\mathsf{S}}$. This shows that, apart from obvious relations induced by monotonicity, no significant constraint involving S can be inferred. Since S and E can be freely compared, this result suggests that either further investigation about relevance of the clinical test or a revision of the initial preference relation, should be performed.

The availability of automated tools able to extend preference orders, whenever new knowledge is acquired, directly suggests applications in expert systems and decision-support tools. In automated diagnosis, planning, or problem solving, to mention some examples, one could easily imagine scenarios where knowledge is not entirely available from the beginning. We could outline how a rudimentary inference process could develop, by identifying the basic steps an automated agent should perform:

0) Acquisition of an initial collection of observations about the object of the analysis, together with a (qualitative) partial preference assessment;

1) Detection of which is the most adequate (i.e., the most discriminant) uncertainty framework;

2) Whenever new knowledge becomes available, refine agent's description of the real world by performing order extension.

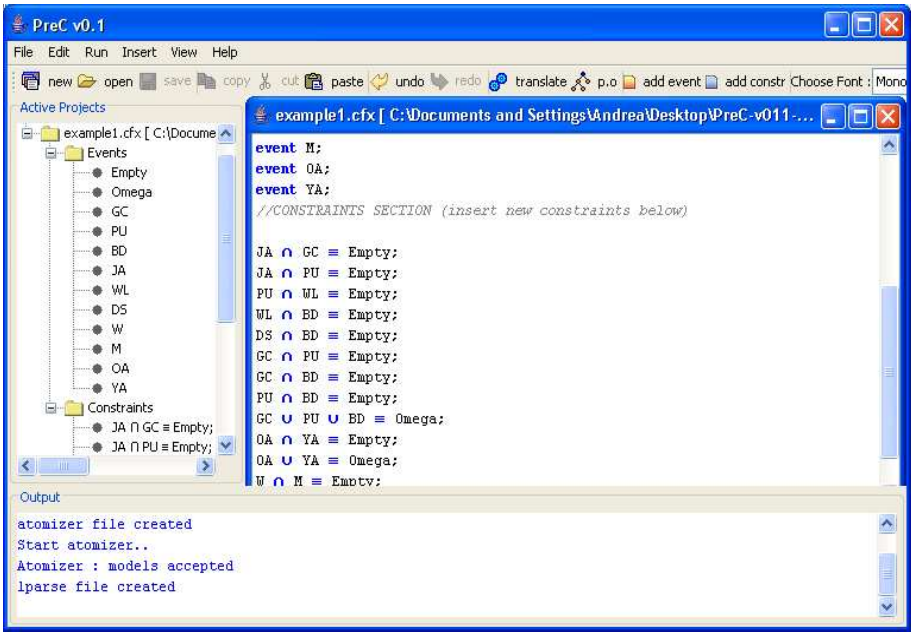

Fig. 3. Main window of the *Pre*ference *C*runcher

The results of step 2) could be then exploited to guide further investigations on the real world, in order to obtain new information. Then, the process will be repeated until further pieces of knowledge are obtainable or an enough accurate degree of belief is achieved.

4. *PreC*: Working with preference orders

Apart from qualitative probabilities, all axioms in Sec. 2 are of direct declarative reading, as they involve only logical and preference relations. It is then rather immediate to provide an executable declarative specification of them. In fact, their declarative character supports a straightforward translation within the logical framework of Answer Set Programming (ASP, for short, App. B briefly summarizes some basic notions about ASP). Thus, we immediately obtain an executable ASP-specification suitable to perform the tasks described in the previous section. More specifically, this is done by exploiting an ASP-solver (in our case SModels (ASP)).

The ASP-specification is obtained as follows. We start by defining in ASP the predicates prec(·,·), precneq(·,·), and equiv(·,·), to render the relators $\preccurlyeq$, $\prec$, and $\sim$, resp. The characterization of potential legal answer sets (cf., App. B) is done by asserting properties of prec(·,·), precneq(·,·), and equiv(·,·). (Auxiliary predicates/functions set-theoretical constructs, such as event(·), subset(·,·), diffset(·,·), etc. are of immediate reading.) For instance (A3') is rendered by weeding out all answer sets where $A \subseteq B \wedge B \prec A$ holds for some events A and B:

```
:- event(A;B), subset(A,B), precneq(B,A).
```

As regards preference classification, let us consider one of the axioms of Sec. 2, say (B'). The following rule is also of immediate reading:

```
failsB :- event(X;Y;Z;W), subset(X,Y), X!=Y, subset(Z,W), Z!=W, subset(W,Y),
          W!=Y, subset(diffset(W,Z),diffset(Y,X)), equiv(X,Y), precneq(Z,W).
```

Namely, the fact failsB is true (i.e., belongs to the answer set) whenever there exist events falsifying (B'). All other axioms can be treated similarly.

When an ASP-solver is fed with such program and a description of a preference relation (i.e., a set of facts of the forms prec(·,·), precneq(·,·), equiv(·,·)), different outcomes may be obtained:

a) If no answer set is produced, then the input w.p.s. violates some basic requirement, such as axioms (A1')–(A3').

b) Otherwise, if an answer set is generated, there exists a numerical (partial) model representing the input w.p.s. The presence in the answer set of a fact of the form fails$\mathcal{C}$ (say failsB), witnesses that the corresponding axiom ((B') in the case) is violated. Consequently, the given order (as well as any of its extensions) is not compatible with the uncertainty framework ruled by $\mathcal{C}$.

Example 4.1. Consider the situation described in the Example 3.1. The w.p.s. $\langle \preceq, \prec \rangle$ can be so described in the syntax of ASP:

```
precneq(N,C) :- empty(N).             precneq(C,B).    precneq(C,D).
precneq(E,C).                         precneq(E,D).    precneq(F,A).
equiv(unionset(A,E),unionset(A,C)).   prec(B,A).
```

As expected, if such ASP specification is given as input to smodels, the answer set found includes the facts failsB1 and failsWC, testifying that the given assessment does not satisfy either of the axioms (B') and (WC).

A similar treatment can be done to implement the order extension task. In this case, the input knowledge consists in a set of events together with a collection of logical constraints and preferences, a description of a the new event, and one or more axioms to be imposed. The handling of the imposed axioms is done by ASP-rules of the form:

```
:- holdsB, event(X;Y;Z;W), subset(X,Y), X!=Y, subset(Z,W), Z!=W, subset(W,Y),
   W!=Y, subset(diffset(W,Z),diffset(Y,X)), equiv(X,Y), precneq(Z,W).
```

Compare this rule with the similar one introduced above for implementing the classification task. Rules of this kind, i.e., with empty head, are called *constraints* (in the sense described in App. B, see also (Baral 2003)). A constraint is satisfied when at least one of the atoms in its body is not satisfied. By such rules we declare "undesirable" any extension for which an axiom is violated. Intuitively speaking, whenever the fact holdsB is true, in order to satisfy the above rule, at least one of the other facts must be false. (Notice that these facts are all true exactly when (B') is violated.) In order to activate this constraint (i.e. to impose axiom (B')) it suffices to add the fact holdsB to the input of the solver. Since in general more than one extension is possible, the collections $\mathcal{L}_\mathsf{S}$, $\mathcal{WL}_\mathsf{S}$, $\mathcal{U}_\mathsf{S}$, and $\mathcal{WU}_\mathsf{S}$ can be obtained by computing the intersection Cn of all the answer

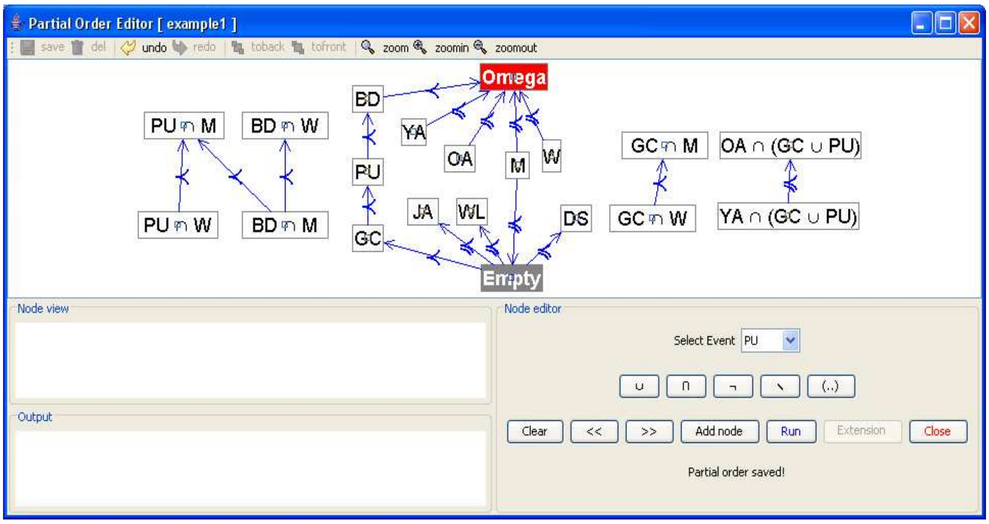

Fig. 4. Visual management of preference orders (cf., Example 2.1)

sets. (Or, equivalently, by computing the set of logical consequences of the ASP-program. Notice that, in general, *Cn* needs not to be an answer set by itself. Some ASP-solvers offer the computation of *Cn* as a built-in feature. This is not the case for smodels so a post-processing of smodels'output is needed.) This allows one to detect the minimal extension of the preference relation which is mandatory for each total order.

Example 4.2. Consider the Examples 4.1 and 3.2. As expected, the following facts belong to each answer set and are obtained by filtering smodels' output: precneq(S,AC), precneq(S,AE), precneq(S,D), precneq(S,A), precneq(S,Omega), prec(Empty,S), prec(S,F), where AC, AE, Omega, and Empty are instantiated to the events A∪C, A∪E, Ω, and $\emptyset$, respectively.

The executable specifications we briefly outlined in this section (together with the ASP-solver and a C-library of functions designed to efficiently handling sets and operations on sets) constitute the core inference-engine of the prototypical tool *PreC* (*Pre*ference *C*runcher). This tool is aimed at assisting the user in interactively dealing with (partial) preference orders and qualitative uncertainty. *PreC* offers an user-friendly and mouse-oriented interface to input, modify, and manage assessments; to activate the reasoning tasks; and to handle order extensions as described in the previous sections. In this manner the user does not interact directly with the ASP-solver and does not handle any ASP-specification. Actually, both the solver and the ASP code could be changed or improved in a transparent manner. Moreover, new uncertainty notions and new tasks can be included to extend the tool. Figure 3 depicts the main window of *PreC*, through which the user can describe, manage, and modify his/her assessments. The visualization of preference orders in form of graphs (cf., Figure 4) allows the user to modify or extend the set of events, for instance in preparation of the execution of one of the inference tasks of Sec. 3.

Conclusions

In this article we proposed an uniform characterization of partial preference orders that provides a strong connection between the axiomatizations of partial preference orders and the corresponding axiomatizations for complete orders. Moreover, the uniform axiomatic treatment presented in this paper straightforwardly translates into an executable specification in a declarative programming framework. This allowed us to explore the potentialities offered by Answer Set Programming for building decision support systems based on qualitative judgments. Hence, an implementation of what could be thought of as a kernel of an inference engine sprouted almost naturally. Moreover, the highly declarative character of the encoding of the axioms into executable ASP-specifications makes it possible to easily modify and extend the treatment to deal with new notions of uncertainty. Similarly, the discovery of alternative axiomatizations of an uncertainty notion can be immediately "plugged-in" the framework just modifying or adding suitable ASP rules (consider, for instance, the case of comparative Probabilities, for which no qualitative characterization has been found to date). Alternatively, a challenging goal for future research consists in completing our approach so as to handle comparative Probabilities through integration with efficient numerical approaches such as linear optimization tools (e.g., column generation techniques, cf. (Jaumard *et al.* 1991), among others). More in general, we envisage the design of a full-blown automated system which integrates different (in some way complementary) techniques and methods for uncertainty management; comprehending mixed numerical/qualitative assessments and extending the range of applicability to conditional probability frameworks (Coletti and Vantaggi 2006).

Appendix A. Gentle introduction to uncertainty measures

In this appendix we briefly describe the various generalizations of Probability measures referred to in this article. The following material is far from being an exhaustive and complete treatment. We give just a informal introduction to the subject. The interested reader can refer to the widely available literature. Introductory treatment of the relationships between Probability measures, Belief functions and Possibility measures, can be found in (Klir and Folger 1988; Nguyen and Walker 1997; Walley 1996), to mention some among many.

We consider, as domain of interest, a set Ω of possibilities (Ω is often referred to as *sample space*). For our purposes it is sufficient to consider the case of a finite domain. An *event* is then defined as a subset of Ω. In order to introduce uncertainty measures, we consider any algebra $\mathcal{A}$ (on Ω), consisting of a set of subsets of Ω, such that $\Omega \in \mathcal{A}$, and closed under union and complementation.

All of the measures we are going to introduce are (normalized) monotone real-valued functions over an algebra. Such functions are usually called (*Choquet*) *Capacities* (Choquet 1954), even if they are referred also as *fuzzy measures* or *Sugeno measures*.

Definition A.1. A real-valued function F on 2^Ω is a *Capacity* if it holds that

$$F(\phi) = 0, \quad F(\Omega) = 1, \quad \text{and for all } A, B \subseteq \Omega \;\; A \subseteq B \to F(A) \leqslant F(B).$$

Let $CAP(\Omega)$ denotes the class of Capacities over Ω. The notion of Capacity is often considered to be too general to be of interest by itself. In fact, by adopting it, apart from monotonicity, there is no other relationship imposed between the uncertainty of a composed event, e.g. $F(A \cup B)$, and the uncertainty of its components $F(A)$ and $F(B)$. In order to reflect different rationales in managing the information, further constraints are imposed on the manner in which uncertainties of composed events are determined. In what follows we describe some of the more interesting measures so obtained by refining Capacities.

The most widely adopted measure of uncertainty is characterized by the additivity property of combination of events: A *Probability* P over Ω is a Capacity which satisfies the following *additivity* requirement: For all $A, B \subseteq \Omega$ s.t. $A \cap B = \phi$ it holds that $P(A \cup B) = P(A) + P(B)$. The class of all Probabilities over Ω is denoted by $PROB(\Omega)$ and, clearly, $PROB(\Omega) \subseteq CAP(\Omega)$.

Definition A.2. Let F_1 and F_2 be two functions on 2^{Ω}. Then, F_1 is the *dual* of F_2 if for each $A \subseteq \Omega$ it holds that $F_1(A) = 1 - F_2(\Omega \setminus A)$.

Note that the dual of a Capacity is a Capacity too. Moreover, the dual of a Probability is the Probability itself.

Additivity, even being widely adopted in "measurement" processes, is usually thought to be too strong requirement. Hence, several generalizations have been proposed. In particular, the following definition characterizes those Capacities satisfying only one of the weak inequalities which, taken together, give additivity.

Definition A.3. Let $g : \mathcal{A}_{\mathcal{E}} \to [0, 1]$ be an uncertainty measure. Then,

— f is said to be a 0-*monotone* (or *super-additive*) function iff $\forall E_1, E_2 \in \mathcal{E}$ with $E_1 \cap E_2 = \phi$ it holds that $f(E_1 \cup E_2) \geqslant f(E_1) + f(E_2)$.

— f is said to be a 0-*alternating* (or *sub-additive*) function iff $\forall E_1, E_2 \in \mathcal{E}$ with $E_1 \cap E_2 = \phi$ it holds that $f(E_1 \cup E_2) \leqslant f(E_1) + f(E_2)$.

Notice that the dual of a 0-monotone function is a 0-alternating function. The class of 0-monotone (resp. 0-alternating) functions is denoted by $0MON(\Omega)$ (resp. $0ALT(\Omega)$).

Let now introduce a pair of dual uncertainty measures that induce the slightly different notions of sub-additivity and super-additivity, respectively.

Definition A.4. Let Π and N be Capacities over Ω.

— Π is a *Possibility* measure (over Ω) if it satisfies the following property: For all $A, B \subseteq \Omega \quad \Pi(A \cup B) = \max\{\Pi(A), \Pi(B)\}$.

— N is a *Necessity* measure (over Ω) if it is the dual of a Possibility measure.

It is immediate to verify that any Possibility measure Π satisfies the *sub-additivity* property, i.e., for all $A, B \subseteq \Omega \;\; \Pi(A \cup B) \leqslant \Pi(A) + \Pi(B)$. Similarly, any Necessity measure N satisfies the *super-additivity* property, i.e., for all $A, B \subseteq \Omega \;\; N(A \cup B) \geqslant N(A) + N(B)$.

A Possibility measure Π is usually induced by a *possibility distribution* (i.e. a fuzzy set) $\pi : \Omega \to [0, 1]$ (Dubois and Prade 1980; Zadeh 1965). The value $\pi(x)$ expresses the possibility of a singleton $x \in \Omega$ to be representative of the concept being considered.

Possibility is then defined by putting $\Pi(A) = \max\{\pi(x) \mid x \in A\}$ for any $A \subseteq \Omega$. The classes of Possibilities and Necessities over Ω are denoted by $POS(\Omega)$ and $NEC(\Omega)$, respectively.

Our last definition regards Belief and Plausibility measures. By adopting the most general formulation, following (Shafer 1976), we have:

Definition A.5. A function $Bel : 2^{\Omega} \to [0,1]$ is a *Belief* measure if it is a Capacity and it satisfies the following condition (known as ∞-*monotonicity*).

$$\text{For each } n \geqslant 1,\ Bel\big(\bigcup_{i=1}^{n} A_i\big) \geqslant \sum_{\phi \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} Bel\big(\bigcap_{i \in I} A_i\big)$$

(where $A_i \subseteq \Omega$ for each i).

Intuitively speaking, a Belief function Bel is usually constructed through a basic assignment of uncertainty, not necessarily being a Capacity, $\mu : 2^{\Omega} \to [0,1]$ so that, for any proposition $A \subseteq \Omega$, $Bel(A) = \sum_{B \subseteq A} \mu(B)$. Belief functions are also called *Capacities monotone of infinite order*. Capacities which satisfy the above condition with the restriction $n \leqslant N$ are then said to be *monotone of order* N (or N-monotone). Dually, if the opposite inequality ($\leqslant$) is considered, the measure is said to be an N-*alternating* Capacity. For $N = 2$ these properties reduce to usual super- and sub-additivity, respectively. The dual of a Belief measure is called *Plausibility* measure. The class of Belief (resp. *Plausibility*) measures is denoted by $BEL(\Omega)$ (resp. $PL(\Omega)$).

The classes of Capacities seen so far are so related:

$$\begin{gathered} CAP(\Omega) \supset 0MON(\Omega) \supset BEL(\Omega) \supset NEC(\Omega) \\ CAP(\Omega) \supset 0ALT(\Omega) \supset PL(\Omega) \supset POS(\Omega) \\ BEL(\Omega) \cap PL(\Omega) \supset PROB(\Omega). \end{gathered}$$

Appendix B. Answer set programming

Let us briefly recall the basics of a successful alternative style of logic programming (Lifschitz 1999; Marek and Truszczyński 1999), known as Answer Set Programming (ASP, for short). In this logical framework, a problem can be encoded—by using a function-free logic language—as a set of properties and constraints which describe the (candidate) solutions. More specifically, an *ASP-program* is a collection of *rules* of the form

$$L_1; \dots; L_k; not\, L_{k+1}; \dots; not\, L_\ell \ \leftarrow\ L_{\ell+1}, \dots, L_m, not\, L_{m+1}, \dots, not\, L_n$$

where $n \geqslant m \geqslant \ell \geqslant k \geqslant 0$ and each L_i is a literal, i.e., an atom A or a negation of an atom $\neg A$. The symbol $\neg$ denotes classical negation, while *not* stands for negation-as-failure (Notice that $','$ and $';'$ stand for logical conjunction and disjunction, respectively.) The left-hand side and the right-hand side of the clause are called *head* and *body*, respectively. A rule with empty head is a *constraint*. (The literals in the body of a constraint cannot be all true, otherwise they would imply falsity.)

Semantics of ASP is expressed in terms of *answer sets* (or equivalently *stable models*, (Gelfond and Lifschitz 1988)). Consider first the case of an ASP-program P which does not involve negation-as-failure (i.e., $\ell = k$ and $n = m$). In this case, a set X of literals is said to be closed under P if for each rule in P, whenever $\{L_{\ell+1}, \ldots, L_m\} \subseteq X$, it holds that $\{L_1, \ldots, L_k\} \cap X \neq \emptyset$. If X is inclusion-minimal among the sets closed under P, then it is said to be an answer set for P. Such a definition is extended to any program P containing negation-as-failure by considering the *reduct* P^X (of P). P^X is defined as the set of rules of the form

$$L_1; \ldots; L_k \leftarrow L_{\ell+1}, \ldots, L_m$$

for all rules of P such that X contains all the literals $L_{k+1}, \ldots, L_\ell$, but does not contain any of the literals $L_{m+1}, \ldots, L_n$. Clearly, P^X does not involve negation-as-failure. The set X is an answer set for P if it is an answer set for P^X.

Once a problem is described as an ASP-program P, its solutions (if any) are represented by the answer sets of P. Notice that an ASP-program may have none, one, or several answer sets.

Let us consider the program P consisting of the two rules $p; q \leftarrow$ and $\neg r \leftarrow p$. Such a program has two answer sets: $\{p, \neg r\}$ and $\{q\}$. If we add the rule (actually, a constraint) $\leftarrow q$ to P, then we rule-out the second of these answer sets, because it violates the new constraint. This simple example reveals the core of the usual approach followed in formalizing/solving a problem with ASP. Intuitively speaking, the programmer adopts a "generate-and-test" strategy: first (s)he provides a set of rules describing the collection of (all) potential solutions. Then, the addition of a group of constraints rules-out all those answer sets that are not desired real solutions. Expressive power of ASP, as well as, its computational complexity have been deeply investigated. The interested reader can refer to (Dantsin *et al.* 2001), among others.

To find the solutions of an ASP-program, an ASP-solver is used. Several solvers have became available (ASP), each of them being characterized by its own prominent valuable features.

Let us give a simple example of ASP-program. (see (Baral 2003; Dovier *et al.* 2005), among others, for a presentation of ASP as a tool for declarative problem-solving). In doing this, we will recall the syntax and the main features of lparse/smodels (one of the available ASP-solvers which we exploited in this article, see (Syrjänen 1999) for a more details). The problem we want to formalize in ASP is the well-known n-*queens* problem: "Given a $n \times n$ chess board, place n queens in such a way that no two of them attack each other". The two clauses below state that a candidate solution is any disposition of the queens, provided that each column of the board contains one and only one queen. The fact that a queen is placed on the n^{th} column and on the m^{th} row is encoded by the atom queen(n,m). (In the syntax of lparse ':-' denotes implication $\leftarrow$. The value of the constant n occurring in the first clause is a parameter of the program supplied to lparse at run-time.)

```
pos(1..n).        1{queen(Col,Row) : pos(Col)}1 :- pos(Row).
```

The second rule is a particular form of constraint available in lparse' language. The general form of such a kind of clauses is

$$k\{\langle \textit{property\_def} \rangle : \langle \textit{range\_def} \rangle\}m \text{ :- } \langle \textit{search\_space} \rangle$$

where: the conditions $\langle \textit{search\_space} \rangle$ in the body define the set of objects of the domain to be checked; the atom $\langle \textit{property\_def} \rangle$ in the head defines the property to be checked; the conjunction $\langle \textit{range\_def} \rangle$ defines the possible values that the property may take on the objects defined in the body, namely by providing a conjunction of unary predicates each of them defining a range for one of the variables that occur in $\langle \textit{property\_def} \rangle$ but not in $\langle \textit{search\_space} \rangle$; k and m are the minimum and maximum number of distinct values that the specified property may take on the specified objects. (Notice that this form of constraint, available in smodels, actually is syntactic sugar, since it can be translated into "proper" ASP-clauses thanks to negation, (Simons 2000; Syrjänen 1999).)

We now introduce two constraints, in order to rule out those placements where two queens control either the same row or the same diagonal of the board (here pos(C;R1;R2) is a shorthand notation for the three facts pos(C), pos(R1), pos(R2)):

```
:- queen(C,R1), queen(C,R2), pos(C;R1;R2), R1<R2.
:- queen(C1,R1), queen(C2,R2), pos(C1;C2;R1;R2), R1<R2, abs(C1-C2)==abs(R1-R2).
```

Here are two of the answer sets produced by smodels (in this case we put n= 8):

```
Answer 1.  queen(4,1)  queen(6,2)  queen(1,3)  queen(5,4)  queen(2,5)  queen(8,6)
           queen(3,7)  queen(7,8)  ...
Answer 2.  queen(4,1)  queen(2,2)  queen(8,3)  queen(5,4)  queen(7,5)  queen(1,6)
           queen(3,7)  queen(6,8)  ...
```

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