## Contents

1. Introduction  
   - Chapter 1. QUASIVARIATIONAL SYSTEMS: LOCAL EXISTENCE  
     1. Preliminaries and notations  
     2. Some results on convex analysis  
     3. Hamiltonian systems  
     4. Local existence  
   - Chapter 2. QUASIVARIATIONAL SYSTEMS: GLOBAL NONEXISTENCE  
     1. Main hypotheses  
     2. Variational identities  
     3. A crucial property  
     4. Preliminary Lemmas  
     5. Main result  
   - Chapter 3. QUASIVARIATIONAL SYSTEMS: GLOBAL EXISTENCE  
     1. Main hypotheses  
     2. Global existence  
   - Chapter 4. APPLICATIONS TO ELLIPTIC SYSTEMS  
     1. Global non existence of radial solutions  
     2. Global existence of radial solutions  
   - Chapter 5. APPLICATIONS TO ELLIPTIC EQUATIONS  
     1. Non existence of entire solutions
Bibliography 91
1. Introduction

In a recent paper [21] Naito and Usami proved the non existence of non negative entire solutions of the differential inequality

\[ \text{div}(A(|\nabla u|)\nabla u) \geq f(u), \quad x \in \mathbb{R}^n, \quad n \geq 2, \quad (1.1) \]

where \( A : \mathbb{R}^+ \to \mathbb{R}^+, \) \( sA(s) \in C(\mathbb{R}_0^+; \mathbb{R}) \cap C^1(\mathbb{R}^+; \mathbb{R}), \) with

\[ A(s) > 0 \quad \text{and} \quad [sA(s)]' > 0 \quad \text{for} \quad s > 0, \]

and \( f \in C(\mathbb{R}_0^+; \mathbb{R}), \) \( f \) is non decreasing,

\[ f(0) = 0 \quad \text{and} \quad f(u) > 0 \quad \text{for} \quad u > 0. \]

Important examples of operators \( A \) are the \( m \)-Laplacian operator

\[ A(s) = s^{m-2}, \quad s > 0, \quad m > 1, \quad (1.2) \]

and the mean curvature operator

\[ A(s) = \frac{1}{\sqrt{1 + s^2}}, \quad s \geq 0. \quad (1.3) \]

Naito and Usami proved that if

\[ \lim_{s \to \infty} sA(s) < \infty, \]

then the only non negative entire solution is \( u \equiv 0, \) while if

\[ \lim_{s \to \infty} sA(s) = \infty, \]

then (1.1) has infinitely many positive solutions if

\[ \int_{H^{-1}(F(s))}^\infty \frac{ds}{H^{-1}(F(s))} = \infty \quad (1.4) \]

and no non negative solutions, except \( u \equiv 0, \) if

\[ \int_{H^{-1}(F(s)/n)}^\infty \frac{ds}{H^{-1}(F(s)/n)} < \infty, \quad (1.5) \]
where \( F(u) = \int_0^u f(s)ds \), and \( H(s) \) is the strictly increasing function

\[
H(s) = s^2 A(s) - \int_0^s \sigma A(\sigma) d\sigma, \quad s > 0.
\]

To prove non existence, the idea in [21] is to show, using a comparison principle, that if (1.1) has a non trivial entire solution then necessarily it also has a non trivial entire radial solution \( v = v(|x|) \). The next step is to show that (1.1) has no nontrivial entire radial solutions.

It is rather striking that the analogue of condition (1.4) at 0, that is

\[
\int_0^1 \frac{ds}{H^{-1}(F(s))} = \infty,
\]

is a necessary and sufficient condition for the validity of the strong maximum principle for the inequality

\[
\text{div}(A(|\nabla u|) \nabla u) \leq f(u).
\]

This has been recently shown in two papers by Pucci, Serrin and Zou [34] and Pucci and Serrin [33].

The purpose of this thesis is to extend the results of Naito and Usami [21] in two ways:

(i) by considering systems of elliptic differential equations,

(ii) by including in the divergence operator a diffusion term of the form \( g(u) \).

Thus we are interested in the existence and non existence of solutions \( u : \mathbb{R}^n \to \mathbb{R}^N \) of elliptic systems of the form

\[
\text{div}(g(u)A(|\nabla u|) \nabla u) - \nabla_u g(u)A(|\nabla u|) = f(r, u), \quad r = |x|, \quad x \in \mathbb{R}^n,
\]

(1.6)

where \( \nabla u \) denotes the Jacobian matrix and where

\[
A(s) = \int_0^s \sigma A(\sigma) d\sigma, \quad s > 0.
\]

The functions \( A, f, g \) satisfy
(I1) \( A : \mathbb{R}^+ \to \mathbb{R}^+ \) is continuous, \( s \mapsto sA(s) \) is strictly increasing in \( \mathbb{R}^+ \) and
\[
\lim_{s \to 0^+} sA(s) = 0;
\]

(I2) \( g : D \to \mathbb{R}_0^+ \) is of class \( C^1 \), where \( D = \mathbb{R}^N \) or \( D = \mathbb{R}^N \setminus \{0\} \).

(I3) there exists a non negative function \( F \in C^1(\mathbb{R}_0^+ \times \mathbb{R}^N; \mathbb{R}) \), with \( F(r, 0) = 0 \) for all \( r \geq 0 \), such that
\[
\nabla_u F(r, u) = f(r, u) \quad \text{and} \quad F_r(r, u) \geq 0
\]
for all \((r, u) \in \mathbb{R}_0^+ \times \mathbb{R}^N\).

Note that the assumptions on \( A \) required in (I1) are significantly weaker than the ones of Naito and Usami.

The partial differential system (1.6) is the Euler-Lagrange equation of the functional
\[
J(u) = \int_{\mathbb{R}^n} \{g(u)A(|\nabla u|) + F(|x|, u)\} dx. \tag{1.7}
\]

The prototype for the diffusion term that we have in mind is
\[
g(u) = |u|\gamma, \tag{1.8}
\]
where \( \gamma \in \mathbb{R} \) (possibly negative), consequently
\[
D = \begin{cases} \mathbb{R}^N & \text{if } \gamma \geq 0, \\ \mathbb{R}^N \setminus \{0\} & \text{if } \gamma < 0. \end{cases} \tag{1.9}
\]

As noted above, the proof in \([21]\) of non existence of entire solutions, not necessarily radial, of (1.6) with \( N = 1 \) is based on maximum principle. When dealing with elliptic systems the situation becomes significantly more complicated than in the scalar case, since in general \textit{comparison or maximum principles do not hold}.

Moreover even a variational approach seems far from trivial if one wants to include terms of the type (1.8). Indeed, in the simpler case, when \( A(s) = s^{m-2} \), the
functional (1.7) reduces to
\[ J(u) = \int_{\mathbb{R}^n} \{ g(u) |\nabla u|^m + F(|x|, u) \} dx, \]
which does not fall within usual settings, namely when
\[ 0 < \frac{1}{c} \leq g(u) \leq C \quad \text{for all} \quad u \in \mathbb{R}^N; \]
see, e.g., the papers of Arcoya and Boccardo [1] for \( N = 1 \), and of Arioli and Gazzola [2] for \( N > 1 \), and the references contained therein. For these reasons in this thesis when \( N > 1 \) we restrict our attention to radial solutions \( u = u(|x|) \) of (1.6) and thus we are interested in the existence and non existence of radial solutions of the ordinary differential system
\[
[g(u)A(|u'|)|u'|' + \frac{n-1}{r} g(u)A(|u'|)|u'|' - \nabla u g(u) A(|u'|) = f(r, u), \quad r > 0,
\]
\[ u(0) = u_0 \neq 0, \quad u'(0) = 0, \quad (1.10) \]
where \( u' = du/dr \) and \( r = |x| \).

The canonical models for (1.10) are given by the initial value problem
\[
[|u'|^{m-2}u']' + \frac{n-1}{r} |u'|^{m-2}u' = |u|^{p-2}u, \quad r > 0,
\]
\[ u(0) = u_0 \neq 0, \quad u'(0) = 0, \quad (1.11) \]
where \( m, p > 1, \quad n \geq 1 \), or more generally
\[
[|u'|^{m-2}u']' - \frac{n}{m} |u'|^{m-2}u |u'|^{m} + \frac{n-1}{r} |u|^{\gamma} |u'|^{m-2}u' = |u|^{p-2}u, \quad r > 0,
\]
\[ u(0) = u_0 \neq 0, \quad u'(0) = 0, \quad (1.11) \]
where \( \gamma \in \mathbb{R}, \quad m, p > 1, \quad n \geq 1 \). We shall see that the problem of existence and non existence of solutions of (1.11) is strictly related to the competition between the terms \( |u'|^{m-2}u' \) and \( |u|^{p-2}u \). To better illustrate this fact we consider the simple case where \( N = n = 1 \) in (1.11), namely the initial value problem
\[
[|u'|^{m-2}u']' = |u|^{p-2}u, \quad r > 0,
\]
\[ u(0) = u_0 > 0, \quad u'(0) = 0. \quad (1.12) \]
This example in the case $m = 2$ and $p > 2$ was given in [30]. Let $J_R = [0, R),
0 < R \leq \infty$, denote the maximal interval of existence a solution $u$ of (1.12). We
claim that
\[ u(r) > u_0, \quad u'(r) > 0, \quad r \in (0, R). \tag{1.13} \]
Indeed, since $u(0) = u_0 > 0$ there exists $r_0 \in (0, R)$ such that
\[ u(r) > 0 \quad \text{for all} \quad r \in [0, r_0). \]
In turn
\[ [|u'|^{m-2}u']' = u^{p-1} > 0 \quad \text{for all} \quad r \in (0, r_0). \]
Consequently the function $|u'|^{m-2}u'$ is strictly increasing and since $u'(0) = 0$ then
\[ |u'|^{m-2}u' > 0 \quad \text{for all} \quad r \in (0, r_0). \]
Hence
\[ u'(r) > 0 \quad \text{for all} \quad r \in (0, r_0). \]
Now let $r_1 \in (0, R)$ be the first $r > r_0$ such $u'(r) > 0$ in $(0, r_1)$ and $u'(r_1) = 0$.
Integrating (1.12) from 0 to $r_1$ we get a contradiction, namely
\[ 0 = [u'(r_1)]^{m-1} - [u'(0)]^{m-1} = \int_0^{r_1} u^{p-1} ds > 0. \]
Hence (1.13) is proved.

Consequently, by direct integration of (1.12) from 0 to $r > 0$, one has
\[ \frac{m-1}{m} (u')^m = \frac{u^p}{p} - \frac{u_0^p}{p}, \tag{1.14} \]
which can be written as
\[ u^{-p/m}u' = \left( \frac{m}{p(m-1)} \left[ 1 - \left( \frac{u_0}{u} \right)^p \right] \right)^{1/m}, \quad r \geq 0. \tag{1.15} \]
Since $u$ is strictly increasing for $r \geq r_2 > 0$ we have
\[ \frac{u_0}{u(r)} \leq 1 - \varepsilon, \quad \varepsilon \in (0, 1), \]
and so

\[ u^{-p/m}u' \geq \left( \frac{m}{p(m-1)} \varepsilon \right)^{1/m} \text{ for all } r \geq r_2. \]

By integrating the inequality above and using (1.13) we get

\[ \frac{m}{m-p}[u(r)]^{(m-p)/m} \geq (r-r_2) \left( \frac{m}{p(m-1)} \varepsilon \right)^{1/m}, \quad r \geq r_2, \]

and if \( p > m \) we deduce

\[ \frac{m}{p-m}[u(r_2)]^{(m-p)/m} \geq (r-r_2) \left( \frac{m}{p(m-1)} \varepsilon \right)^{1/m}, \quad r \geq r_2. \]

Thus \( r \) cannot be \( \infty \) if \( p > m \). More precisely we obtain that any solution of (1.12) cannot be continued to \( \mathbb{R}^+ \), indeed both \( u \) and \( u' \) will approach \( \infty \) at some finite value \( r^* > 0 \), see (1.14).

If \( 1 < p \leq m \) from (1.15) and from the fact that \( u \) is strictly increasing we deduce that

\[ u^{-p/m}u' \leq \left( \frac{m}{p(m-1)} \right)^{1/m}, \]

consequently by integration we get if \( p < m \)

\[ \frac{m}{m-p}[u(r)]^{(m-p)/m} \leq \frac{m}{m-p}u_0^{(m-p)/m} + \left( \frac{m}{p(m-1)} \right)^{1/m} r^* > 0, \]

while if \( p = m \) we get

\[ u(r) \leq u_0 e^{r(m-1)^{-1/m}}, \quad r > 0. \]

Hence in both cases we obtain that \( u \) and \( u' \) are bounded in every bounded set, by using also (1.14). This implies that \( u \) can be continued to the entire \( \mathbb{R}^+ \) if \( 1 < p \leq m \).

Thus we have proved that, global solutions of (1.12) exist if and only if \( p \leq m \). This fact holds also for more general systems. Indeed as corollaries of the main results of the thesis we obtain the following

**Corollary 1.** The elliptic system

\[
\text{div}(|u|^{\gamma} |\nabla u|^{m-2} \nabla u) - \frac{\gamma}{m} |u|^{\gamma-2} u |\nabla u|^m = |u|^{p-2} u, \tag{1.16}
\]

with \( m > 1 \) and \( \gamma \in \mathbb{R} \),
(i) admits a local radial solution;
(ii) does not admit nontrivial entire radial solutions if
\[ 1 < m < p, \quad -m - p(m - 1) < \gamma < p - m; \]

(iii) admits entire radial solutions, namely every local radial solution of (1.16) can be continued to all of \( \mathbb{R}^n \), if
\[ \gamma \geq p - m. \]

Note that the condition \( m < p \) in (ii) is equivalent to (1.5), indeed in the case of the \( m \)-Laplacian, it results that \( H(s) = \frac{m - 1}{m} s^m \) and \( F(u) = |u|^p/p \) then \( H^{-1}(F(\sigma)) = c \sigma^{p/m} \), for some constant \( c \). Hence the integral in (1.5) converges if and only if \( m < p \).

Next we consider the generalized mean curvature operator and we deduce the following

**Corollary 2.** The elliptic system

\[
\text{div}\left(|u|^{\gamma}(1 + |\nabla u|^2)^{m/2-1}\nabla u\right) - \frac{\gamma}{m}|u|^{\gamma - 2}u\left[(1 + |\nabla u|^2)^{m/2} - 1\right] = |u|^{p-2}u, \quad (1.17)
\]

with \( 1 < m \leq 2, \ p > 1 \),

(i) admits a local radial solution;
(ii) does not admit nontrivial entire radial solutions if
\[ 1 < m < p, \quad -m - p(m - 1) < \gamma < p - m; \]

(iii) admits entire radial solutions, namely every local solution of (1.17) can be continued to all of \( \mathbb{R}^n \), if
\[ \gamma \geq p - m. \]

Finally we consider the mean curvature operator, which must be treated separately since it does not satisfy the assumptions of the main Theorem 15 of Chapter 2.
Corollary 3. Assume that $p > 1$. Then the system
\[ \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = |u|^{p-2}u \quad \text{in } \mathbb{R}^n \]
does not admit non trivial entire radial solution.

Note that in the vectorial case case $N > 1$ there is no obvious change of variable which eliminates the term $|u|^\gamma$ from the divergence. Moreover it is worth noting that Corollary 1 implies in particular that any local radial solution of the corresponding system may not be extended to all of $\mathbb{R}^n$.

In the scalar case $N = 1$ it is easy to see that if $u$ is any local non negative radial solution of (1.10) with $u_0 > 0$, then $u = u(|x|)$ is increasing and thus, if the ball $B(0, R)$ is its maximal domain of existence, then either
\[ \lim_{r \to R^-} u(r) = \infty \quad \text{or} \quad \lim_{r \to R^-} u'(r) = \infty. \]

The situation is significantly more complicated in the vectorial case $N > 1$ as no monotonicity is available. Nevertheless we may still prove that

Corollary 4. The elliptic system
\[ \text{div}(|u|^\gamma |\nabla u|^{m-2} \nabla u) - \frac{\gamma}{m} |u|^{\gamma-2} u |\nabla u|^m = |u|^{p-2}u, \]
with
\[ 1 < m < p \quad \text{and} \quad -m - p(m - 1) < \gamma < p - m, \]
adopts a one parameter family of solutions $u : B(0, R) \to \mathbb{R}^N \setminus \{0\}$ such that
\[ \lim_{|x| \to R^-} |u(|x|)| = \infty. \]

In the other situation, namely when we have global solutions, we obtain the following results

Corollary 5. Let
\[ p, m > 1 \quad \text{and} \quad \gamma \geq p - m, \]
then the elliptic system
\[
\text{div}(\gamma^2|u|^{m-2}\nabla u) - \frac{\gamma}{m}|u|^{\gamma-2}u|\nabla u|^m = |u|^{p-2}u,
\] (1.18)

admits a one parameter family of non trivial entire radial solutions, namely \( u = u(|x|), u : \mathbb{R}^n \to \mathbb{R}^N \setminus \{0\} \) such that
\[
\lim_{|x| \to \infty} |u(|x|)| = \infty,
\]
that is (1.18) admits entire radial solutions.

**Corollary 6.** Let
\[
1 < m \leq 2, \quad p > 1 \quad \text{and} \quad \gamma \geq p - m,
\]
then the elliptic system
\[
\text{div}(\gamma^2(1 + |\nabla u|^2)^{m/2-1}\nabla u) - \frac{\gamma}{m}|u|^{\gamma-2}u[(1 + |\nabla u|^2)^{m/2} - 1] = |u|^{p-2}u, \quad (1.19)
\]
admits a one parameter family of non trivial entire radial solutions \( u = u(|x|), \)
\( u : \mathbb{R}^n \to \mathbb{R}^N \setminus \{0\} \) such that
\[
\lim_{|x| \to \infty} |u(|x|)| = \infty,
\]
that is (1.19) admits entire radial solutions.

When \( N = 1 \), Corollary 1 can be improved by giving up the lower bound for \( \gamma \) in (ii). Indeed this lower bound is peculiar of the technique used to treat the vectorial case (see Remark 1 of Chapter 2). Moreover by using a weak comparison principle due to Pucci, Serrin and Zou in [34] and utilizing the same argument of Naito and Usami in [21], we can obtain non existence of all solutions, radial or not, of the equation
\[
\text{div}(g(u)A(|\nabla u|)\nabla u) - g'(u)A(|\nabla u|) = f(|x|, u), \quad x \in \mathbb{R}^n. \quad (1.20)
\]
More precisely, we obtain the following global result.
Proposition 1. Assume that the functions $A, f, g$ satisfy conditions (I1)-(I3) with $N = 1$. Suppose that

$$g \text{ is non increasing}$$

and that there exists a non decreasing function $\varphi \in C(\mathbb{R}^+_0; \mathbb{R})$, with $\varphi(0) = 0$, such that

$$\frac{f(r, u)}{g(u)} \geq \varphi(u) \quad \text{for all } r \geq 0 \text{ and all } u > 0.$$ 

If

$$\lim_{s \to \infty} sA(s) = \infty$$

and $\varphi$ satisfies the condition

$$\int_{-\infty}^{\infty} \left( H^{-1}\left( \frac{1}{n} \int_{t}^{\infty} \varphi(\tau) d\tau \right) \right)^{-1} dt < \infty,$$

then equation (1.20) does not admit any entire positive solution.

Moreover, we obtain the following corollary

Corollary 7. The elliptic equation

$$\text{div} (u^\gamma |\nabla u|^{m-2} \nabla u) - \frac{\gamma}{m} u^{\gamma-1} |\nabla u|^m = u^{p-1} \quad \text{in } \mathbb{R}^n, \quad p, m > 1,$$

do not admit any positive entire solutions if

$$1 < m < p \quad \text{and} \quad \gamma < p - m,$$

while it admits entire solutions if

$$\gamma \geq p - m.$$

Note that the above corollary is a consequence of the theorem above only for the case $\gamma < 0$.

Our methods work also for general elliptic systems of the type

$$\text{div} \nabla \tilde{\mathcal{L}}(x, u, \nabla u) = \tilde{\mathcal{L}}(x, u, \nabla u), \quad x \in \mathbb{R}^n, \quad (1.21)$$
where as noted in [31] and [14], $\tilde{\mathcal{L}}$ has the form $\mathcal{L}(r, u, |\nabla u_1|, \ldots, |\nabla u_N|)$ and

$$\mathcal{L}(r, u, v) = G(u, v) + F(r, u), \tag{1.22}$$

where the function $G(u, \cdot)$ is strictly convex in $\mathbb{R}^N$ for all $u \in \mathbb{R}^N \setminus \{0\}$ and $F$ is, as above, the potential with respect to the variable $u$ of the force $f$. In particular, radial solutions of (1.21) satisfy the following ordinary system

$$[r^{n-1}\nabla \mathcal{L}(r, u, u')]' = r^{n-1}\nabla_u \mathcal{L}(r, u, u'), \quad r = |x|, \quad x \in \mathbb{R}^n,$$

where $\nabla = \nabla_u$ denotes the gradient operator with respect to the third variable. The system above becomes by (1.22)

$$[r^{n-1}\nabla G(u, u')]' = r^{n-1}\{\nabla_u G(u, u') + f(r, u)\}. \tag{1.23}$$

More generally, we study the following quasi variational system

$$[\nabla G(u, u')]' - \nabla_u G(u, u') + Q(r, u, u') = f(r, u), \tag{1.24}$$

where $Q$ is a continuous damping term and $f$ is a continuously differential nonlinear driving force.

Moreover the system (1.24) may be considered as the motion equation for a holonomic dynamical system with $N$ degrees of freedom, whose Lagrangian is defined in (1.22) and whose dynamics are governed by a general nonlinear damping term $Q = Q(r, u, v)$. The variables $u_i$ represent appropriate Lagrangian or generalized coordinates, while the $v_i$’s are stand–in variables for the derivatives $u'_i$.

We recall that system (1.24) becomes variational if

$$Q(r, u, v) = \frac{h'(r)}{h(r)} \nabla G(u, v),$$

where

$$h \in C^1(\mathbb{R}^+; \mathbb{R}^+_0) \cap C(\mathbb{R}^+_0; \mathbb{R}^+_0),$$

such that

$$h(0) = 0, \quad h(r) > 0 \quad \text{for} \quad r > 0.$$
In this case system (1.24) can be written as
\[
[h(r) \nabla G(u, u')]' - h(r) \nabla_u G(u, u') = h(r)f(r, u). \tag{1.25}
\]

Thus we immediately see that (1.23) is the special case of (1.25) when \( h(r) = r^{n-1} \).

In the framework of quasi variational systems, it is more customary to denote by \( t \) the scalar variable \( r \) which represents the time variable of the non-autonomous case covered by the quasi variational system (1.24). Throughout the thesis, we consider solutions of the initial value problem
\[
[\nabla G(u, u')]' - \nabla_u G(u, u') + Q(t, u, u') = f(t, u), \quad t > T, \\
u(T) = u_0, \quad u'(T) = 0, \quad T \geq 0
\tag{1.26}
\]
with \( F(T, u_0) > 0 \), namely when the energy associated to any vector solution has negative initial value. Indeed in this situation and under some appropriate growth conditions on \( f \) and on \( G \), which we shall see later, every solution of (1.26) exists locally and is such that
\[
|u(t)| \geq \text{Pos. Const.} > 0, \tag{1.27}
\]
where the constant depends only on \( F(T, u_0) \). This property is crucial because the strict convexity of \( G(u, \cdot) \) is assumed only for all \( u \neq 0 \) so that many prototypes of (1.10), interesting in applications, can be covered. In particular (1.10) is the special case of (1.26) when
\[
G(u, v) = g(u)A(|v|), \quad Q(t, u, v) = \frac{n-1}{t}g(u)A(|v|)v.
\]

The content of the thesis is organized as follows. In Chapter 1 we give some basic notions in convex analysis, we introduce the Legendre transform for differentiable functions and we study the derivations of the Hamiltonian system corresponding to (1.26). Then we prove existence of local solutions of (1.26), which is extremely delicate since the damping term may be singular at time \( t = T \). The proof of this theorem is based on an argument of Leoni in [14].
1. INTRODUCTION

In Chapter 2 we study the non existence of global solutions of (1.26) under appropriate growth conditions on $G$, $F$ and $Q$. One of the main tool used in the proof of Theorem 15, which is the main theorem of Chapter 2, is the variational identity of Pucci and Serrin (see [27], [29] and [31]), together with an argument of Levine and Serrin [18], even if it is adapted to problem (1.26) instead of the abstract evolution problem studied in [18].

Chapter 3 is devoted to the specular problem, namely the existence of global solutions of (1.26). In particular, as noted in the example (1.12), global existence holds in general if either the damping term $Q$ or the action energy $G$ dominates over the driving force $F$. Hence we treat the first case in Theorem 18, while the latter is analyzed in Theorem 19. The proofs of these theorems use ideas of [14], [17] and [30].

Chapter 4 presents the application of the results of the previous chapters to elliptic systems of the general form (1.6). In particular the corollaries presented above are consequences of the main theorems of this chapter. Indeed Theorem 20 deals with non existence of global radial solutions of (1.6), Theorem 21 is devoted to the continuation of every local radial solution of (1.6) to the entire of $[T, \infty)$ while Theorem 22 gives the asymptotic behavior of global radial solutions of (1.6).

Finally in Chapter 5, we study in detail the scalar case, $N=1$, of (1.6). First we put in evidence that that (1.27) is automatic since every solution of (1.10) with $u_0 > 0$ and $N = 1$ is strictly monotone (see Theorem 23 below). Finally, we shall proof non existence of entire solutions, radial or not, of equation (1.20), as stated in the above Theorem 1.
CHAPTER 1

QUASIVARIATIONAL SYSTEMS: LOCAL EXISTENCE

1. Preliminaries and notations

In this chapter we study the existence of local solutions of the initial value problem

\[ \begin{align*}
\frac{d}{dt}G(u, u') - \nabla_u G(u, u') + Q(t, u, u') &= f(t, u), \quad t > T, \\
u(T) &= u_0, \quad u'(T) = v_0,
\end{align*} \]

(1.1)

(1.2)

where \( T \geq 0 \) and \( \nabla = \nabla_v \) denotes the gradient operator with respect to the second variable of the function \( G \). It will be supposed throughout the thesis that

\[ J = (T, \infty) \]

and that the following hypotheses are satisfied:

(A1) \( G \in C^1(D \times \mathbb{R}^N, \mathbb{R}) \), where \( D \subset \mathbb{R}^N \) is an open set, \( G(u, \cdot) \) is strictly convex in \( \mathbb{R}^N \) for all \( u \in D \) such that \( u \neq 0 \), with \( G(u, 0) = 0 \) and \( \nabla G(u, 0) = 0 \);

(A2) there exists a non negative function \( F \in C^1(\mathcal{J} \times \mathbb{R}^N; \mathbb{R}) \), with \( F(t, 0) = 0 \) for all \( t \in \mathcal{J} \), such that

\[ \nabla_u F(t, u) = f(t, u) \quad \text{and} \quad F_t(t, u) \geq 0 \]

for all \( (t, u) \in \mathcal{J} \times \mathbb{R}^N; \)

(A3) \( Q \in C(\mathcal{J} \times D \times \mathbb{R}^N; \mathbb{R}^N) \) with

\[ \langle Q(t, u, v), v \rangle \geq 0 \quad \text{for all} \quad (t, u, v) \in \mathcal{J} \times D \times \mathbb{R}^N. \]

(1.3)

Here \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( \mathbb{R}^N \).
We say that \( u : [T, T + \tau) \to D \subset \mathbb{R}^N, \tau > 0, \) is a local solution of (1.1)–(1.2) if 

\[ \nabla G(u(t), u'(t)) \in C^1((T, T + \tau); \mathbb{R}^N), \]  

(1.4)

\( u \) satisfies the system (1.1) in \((T, T + \tau)\) and the initial conditions (1.2).

We say that a local solution \( u \) is a global solution of (1.1)–(1.2) if \( \tau = \infty \).

Note that we do not require the solution \( u \) to satisfy the system (1.1) at the initial time \( t = T \). This is due to the fact that in the applications to elliptic partial differential systems and equations in Chapters 4 and 5, the function \( Q(t, u, v) \) has the form

\[ Q(t, u, v) = \frac{n-1}{t} \nabla G(u, v) \]

and thus the system is singular when \( t = 0 \).

A canonical model is given by the initial value problem

\[ \left( |u|^\gamma |u'|^{m-2}u' \right)' - \frac{\gamma}{m} |u|^{-2} u |u'|^m + \frac{n-1}{t} |u|^\gamma |u'|^{m-2} u' = |u|^{p-2} u, \]

(1.5)

\[ u(0) = u_0 \neq 0, \quad u'(0) = 0, \]

with \( \gamma \in \mathbb{R}, m, p > 1 \). In this case, clearly

\[ G(u, v) = \frac{1}{m} |u|^\gamma |v|^m, \quad Q(t, u, v) = \frac{n-1}{t} |u|^\gamma |v|^{m-2} v, \quad F(t, u) = \frac{|u|^p}{p}, \]

and

\[ D = \begin{cases} \mathbb{R}^N & \text{if } \gamma > 0, \\ \mathbb{R}^N \setminus \{0\} & \text{if } \gamma \leq 0. \end{cases} \]

(1.6)

Note that in general the system may become singular or degenerate at points where \( u' = 0 \) or \( u = 0 \).

As we already mentioned in the introduction, the general system (1.1) may be considered as the motion equation for a holonomic dynamical system with \( N \) degrees of freedom, whose Lagrangian is defined by

\[ \mathcal{L}(t, u, v) = G(u, v) + F(t, u), \]

(1.7)
and whose dynamics are governed by a general nonlinear damping term \( Q = Q(t, u, v) \). The variables \( u_i \) represent appropriate Lagrangian or generalized coordinates, while the \( v_i \)'s are stand-in variables for the derivatives \( u'_i \). Consequently the initial value problem (1.1)–(1.2) can be written as follows

\[
(\nabla L(t, u, u'))' - \nabla_u L(t, u, u') + Q(t, u, u') = 0, \quad t > T, \\
u(T) = u_0, \quad u'(T) = v_0.
\]

(1.8)

To prove existence of local solutions it is customary to transform the initial value problem (1.8) into the Hamiltonian form

\[
u' = \nabla_w H(t, u, w), \\
w' = -\nabla_v H(t, u, w) - P(t, u, w), \quad t > T, \\
u(T) = u_0, \quad w(T) = w_0,
\]

(1.9)

where the Hamiltonian \( H \) is related to the Lagrangian \( L \) through the Legendre transform of \( L(t, u, \cdot) \), namely

\[
H(t, u, w) = H_0(t, u, (\nabla L(t, u, \cdot))^{-1}(w)),
\]

where

\[
H_0(t, u, v) = \langle \nabla L(t, u, v), v \rangle - L(t, u, v)
\]

and

\[
P(t, u, w) = Q(t, u, (\nabla L(t, u, \cdot))^{-1}(w)).
\]

In the next section we recall some basic notions in convex analysis, introduce the Legendre transform and study the problem of the invertibility of \( \nabla L(t, u, \cdot) \). In Section 3 we study the derivation of the Hamiltonian system (1.9), and, finally, in Section 4 we prove local existence of solutions of (1.8).
2. Some results on convex analysis

A set $A \subseteq \mathbb{R}^N$ is convex if

$$\lambda v_1 + (1 - \lambda)v_2 \in A$$

whenever $\lambda \in (0, 1)$ and $v_1, v_2 \in A$, $v_1 \neq v_2$.

A function $G : A \subseteq \mathbb{R}^N \to \mathbb{R}$, where $A$ is a convex set, is convex if

$$G(\lambda v_1 + (1 - \lambda)v_2) \leq \lambda G(v_1) + (1 - \lambda)G(v_2)$$

(2.1)

for all $\lambda \in (0, 1)$ and $v_1, v_2 \in A$, $v_1 \neq v_2$, while it is strictly convex if the strict inequality holds in (2.1).

Important examples of strictly convex functions are given by the $m$-Laplacian

$$G(v) = \frac{|v|^m}{m}, \quad m > 1,$$

and the mean curvature operator

$$G(v) = \sqrt{1 + |v|^2} - 1.$$

When the set $A$ is open but not convex and $G$ is differentiable we may introduce a weaker notion of convexity. More precisely, let $A \subseteq \mathbb{R}^N$ be open and consider a differentiable function $G : A \subseteq \mathbb{R}^N \to \mathbb{R}$. We say that $G$ is convex in the sense of Weierstrass if

$$G(v_2) - G(v_1) \geq \langle \nabla G(v_1), v_2 - v_1 \rangle$$

(2.2)

for all $v_1, v_2 \in A$, $v_1 \neq v_2$. If the strict inequality holds in (2.2), then we define $G$ to be strictly convex in the sense of Weierstrass.

**Theorem 1.** Let $A \subseteq \mathbb{R}^N$ be open and let $G : A \to \mathbb{R}$ be convex in the sense of Weierstrass. Then $G$ is convex on any convex subset $A_1 \subseteq A$. Conversely, if $A$ is convex and $G$ is convex and differentiable on $A$, then $G$ is convex in the sense of Weierstrass.
As first discovered by Pucci and Serrin in [32], strictly convex functions in the sense of Weierstrass play an important role for the invertibility of $\nabla G$ and the Legendre transform. Indeed, they proved the following result as a corollary of Theorem 1 in [32].

**Theorem 2.** Let $A \subseteq \mathbb{R}^N$ be open and let $G : A \rightarrow \mathbb{R}$ be strictly convex in the sense of Weierstrass. Then

$$\nabla G : A \rightarrow \mathbb{R}^N$$

is one-to-one, continuous and the set $\nabla G(A)$ is open.

Theorem 2 is rather sharp. Indeed the following example was given by Zampieri in [40]

$$G(v_1, v_2) = (v_1^2 + v_2^2) \left(1 + \frac{v_2^2}{(v_1 + \sqrt{v_1^2 + v_2^2})^2}\right)$$

defined in $A = \mathbb{R}^2 \setminus \{(v_1, 0) : v_1 \leq 0\}$. Note that the set $A$ is not convex. This function has a positive definite Hessian matrix at any point of $A$ and thus it is strictly convex on any convex subset of $A$. On the other hand, the level curves of $G$ can be represented by the simple polar equation $r = a\sqrt{1 + \cos \theta}$ and resemble cardioids. The gradient is given by

$$\nabla G(v_1, v_2) = \frac{2\sqrt{v_1^2 + v_2^2}}{v_1 + \sqrt{v_1^2 + v_2^2}} \left(2v_1^2 - v_2^2 + 2v_1\sqrt{v_1^2 + v_2^2}, 3v_1v_2 + 2v_2\sqrt{v_1^2 + v_2^2}\right).$$

Hence, the function $\nabla G : A \rightarrow \mathbb{R}^2$ is not one-to-one. Indeed, the picture (see Figure 1) suggests the existence of two points, on each level curve, with the same $v_1 < 0$ and opposite $v_2$, where the gradients are parallel to the $v_1$-axis and coincide. The equation $(\partial G/\partial v_2)(v_1, v_2) = 0$, $v_2 \neq 0$, gives $v_2 = \pm \sqrt{5}v_1/2$ and consequently, for any $v_1 < 0$ we have

$$\nabla G(v_1, \pm \sqrt{5}v_1/2) = (27v_1, 0),$$

that is $\nabla G$ is not one-to-one.
THEOREM 3. (Theorem 26.6, [36]) Let $G : \mathbb{R}^N \to \mathbb{R}$ be convex and differentiable. Then

$$\nabla G : \mathbb{R}^N \to \mathbb{R}^N$$

is one-to-one and onto if and only if $G$ is strictly convex and

$$\lim_{|v| \to \infty} |\nabla G| = \infty.$$

Next we introduce the notion of Legendre transform. Consider a differentiable function

$$G : A \subset \mathbb{R}^N \to \mathbb{R},$$

where $A$ is an open set. Define

$$H(v) = \langle \nabla G(v), v \rangle - G(v). \quad (2.3)$$

Assume that

$$\langle v_1, w \rangle - G(v_1) = \langle v_2, w \rangle - G(v_2), \quad (2.4)$$

whenever $\nabla G(v_1) = \nabla G(v_2) = w$ and for $v_1, v_2 \in A$, $w \in V$. Here $V = \nabla G(A)$. Then we may define a function

$$G^* : V \subset \mathbb{R}^N \to \mathbb{R},$$
called the *Legendre transform* as follows

\[ G^*(w) = H((\nabla G)^{-1}(w)) = \langle w, (\nabla G)^{-1}(w) \rangle - G((\nabla G)^{-1}(w)). \]  

(2.5)

The application

\[ (A, G) \rightarrow (V, G^*) \]

is called the *Legendre transformation*. Note that condition (2.4) is automatically satisfied if \( \nabla G : A \rightarrow \mathbb{R}^N \) is one-to-one.

The following result was proved by Pucci and Serrin as a corollary of Theorem 3 in [32].

**Theorem 4.** Let \( A \subseteq \mathbb{R}^N \) be open and let \( G : A \subseteq \mathbb{R}^N \rightarrow \mathbb{R} \) be strictly convex in the sense of Weierstrass. Then the Legendre transform

\[ G^* : V \rightarrow \mathbb{R} \]

is strictly convex in the sense of Weierstrass. Moreover

\[ (G^*)^* = G. \]

Note that, when

\[ G(v) = \frac{|v|^m}{m}, \quad m > 1, \quad A = \mathbb{R}^N, \]

then

\[ G^*(w) = \frac{m-1}{m} |w|^{m/(m-1)}, \quad w \in \mathbb{R}^N, \]

while when

\[ G(v) = \sqrt{1 + |v|^2} - 1, \quad A = \mathbb{R}^N, \]

then

\[ G^*(w) = 1 - \sqrt{1 - |w|^2}, \quad |w| < 1. \]

Finally when

\[ G(v) = |v|^2 + \frac{v_1^2}{v_2}, \quad A = \{v \in \mathbb{R}^2 : v_2 > 0\}, \]
then the domain of the Legendre transform $G^*(w)$ is
\[
\{ w \in \mathbb{R}^2 : w_1^2 + 4w_2 > 0 \},
\]
which is not convex and so $G^*$ is convex in the sense of Weierstrass but not convex.

3. Hamiltonian systems

As observed in Section 1, the general system (1.1)-(1.2), or equivalently the Lagrange system (1.8), governs a mechanical system with $N$ degrees of freedom, consequently it can customarily be transformed into Hamiltonian form. Standard derivations of the Hamilton equations in literature strongly rely on the fact that the Hessian matrix of the Lagrangian is non-singular. In [32], Pucci and Serrin obtained Hamilton equations when the Lagrangian is only of class $C^1$, so that the Hessian may be singular or not defined (note that convexity in itself forces the Hessian matrix only to be non-negative definite). This minimal condition allows one to consider singular Lagrangians, for example of the form
\[
\mathcal{L}(t, u, v) = \frac{1}{m} |v|^m + F(t, u), \quad m > 1.
\]
First note that the classical assumptions, namely that $\nabla \mathcal{L}$ is itself of class $C^1$ and its Hessian matrix
\[
\nabla^2 \mathcal{L} = \left( \frac{\partial^2 \mathcal{L}}{\partial v_i \partial v_j} \right)
\]
is non-singular, do not guarantee that the Hamiltonian is globally well defined and single valued on the complete domain of the conjugate variable. Indeed in [32], the following non-quadratic Lagrangian in $\mathbb{R}^2$ is considered
\[
\mathcal{L}(v_1, v_2) = e^{v_2} \sin v_1.
\]
In this case one checks that $|\det \nabla^2 \mathcal{L}| = e^{2v_2} > 0$, while the conjugate mapping
\[
w_1 = e^{v_2} \cos v_1, \quad w_2 = e^{v_2} \sin v_1
\]
is obviously not one-to-one in all of $\mathbb{R}^2$. 
In the classical theory, when the Lagrangian is of the form

\[ \mathcal{L}(t,u,v) = G(v) + F(t,u), \quad v \in A \subset \mathbb{R}^N, \]

the derivation of Hamiltonian equations can be given on the basis of work of Rockafellar \[36\] or of Mawhin & Willem \[20\] in the contest of convex analysis. In particular Mawhin & Willem’s derivation applies whenever \( A \) is convex and bounded and \( |\nabla G(v)| \to \infty \) as \( v \to v_0 \in \partial A \), while Rockafellar’s whenever \( A = \mathbb{R}^N \) and \( G(v)/|v| \to \infty \) as \( |v| \to \infty \). On the other hand in \[32\] there is an example in which both derivations cannot be applied, namely

\[ G(v) = (1 + |v_1|^m)^{1/m}/v_2, \quad m > 1, \quad m \neq 2, \]

with

\[ A = \{ v \in \mathbb{R}^2 : |v_1|^m < 2(m - 1), \quad v_2 > 0 \}. \]

Indeed, neither Mawhin & Willem’s condition nor Rockafellar’s is satisfied; moreover the function is strictly convex and of class \( C^1 \) in \( A \), it fails to be twice differentiable on the ray \( v_1 = 0, \quad v_2 > 0 \) if \( m < 2 \), while if \( m > 2 \) it is of class \( C^2 \) in \( A \) but \( \det \nabla^2 \mathcal{L} = 0 \) on the ray. In this case, the derivation of Pucci & Serrin can be applied.

**Theorem 5. (Theorem 1, \[32\])** Let \( \mathcal{L} : \Omega \times A \to \mathbb{R} \) be a function of class \( C^1 \), where \( \Omega \subseteq \mathbb{R} \times \mathbb{R}^N \) and \( A \subseteq \mathbb{R}^N \) are open sets. Assume that for every fixed \((t,u) \in \Omega \) the function \( \mathcal{L}(t,u,\cdot) \) is strictly convex in \( A \) in the sense of Weierstrass.

Then the function \( \nabla \mathcal{L}(t,u,\cdot) : A \to A_{t,u} \), for any fixed \((t,u) \in \Omega \), is an homeomorphism, where

\[ A_{t,u} = \{ w \in \mathbb{R}^N : w = \nabla \mathcal{L}(t,u,v), \quad \text{for some } v \in A \}. \]

Let

\[ \hat{D} = \{(t,u,w) : (t,u) \in \Omega, \quad w = \nabla \mathcal{L}(t,u,v) \quad \text{for some } v \in A \}. \]
Note that by Theorem 26.6 and Lemma 26.7 in [36], if
\[ |\nabla \mathcal{L}(t, u, v)| \to \infty \quad \text{as} \quad |v| \to \infty, \]
then \( \hat{D} = \Omega \times \mathbb{R}^N \).

**Theorem 6. (Theorem 2, [32])** Under the hypotheses of Theorem 5, the domain \( \hat{D} \) is open and
\[ (\nabla \mathcal{L}(t, u, \cdot))^{-1} : \hat{D} \to \mathbb{R}^N \quad \text{is continuous.} \]

Now consider the function
\[ H_0(t, u, v) = (\nabla \mathcal{L}(t, u, v), v) - \mathcal{L}(t, u, v). \]
Then we define \( \mathcal{H} : \hat{D} \to \mathbb{R} \) by
\[ \mathcal{H}(t, u, w) = H_0(t, u, (\nabla \mathcal{L}(t, u, \cdot))^{-1}(w)). \tag{3.1} \]
Clearly \( \mathcal{H} \in C(\hat{D}) \), being \( \mathcal{L} \) of class \( C^1 \). The following result holds

**Theorem 7. (Theorem 3, [32])** Under the hypotheses of Theorem 5, the function \( \mathcal{H} \) is continuously differentiable with respect to \( w \), and for \( (t, u, w) \in \hat{D} \) we have
\[ \nabla_w \mathcal{H}(t, u, w) = v. \]
Moreover, for every fixed \( (t, u) \in \Omega \) the function \( \mathcal{H}(t, u, \cdot) \) is strictly convex in the sense of Weierstrass.

In particular, in [32] it is proved the following general result, which complements Theorem 7 and which shows that in all cases \( \mathcal{H} \) is necessarily of class \( C^1(\hat{D}) \).

**Theorem 8. (Theorem 4, [32])** Under the hypotheses of Theorem 5, the function \( \mathcal{H} \) is continuously differentiable with respect to \( t, u \) and
\[ \mathcal{H}_t(t, u, w) = -\mathcal{L}_t(t, u, v), \]
\[ \nabla_w \mathcal{H}(t, u, w) = -\nabla_u \mathcal{L}(t, u, v). \]
Finally we recall that Pucci and Serrin in [32] understand a solution of Lagrangian system (1.8) to be a function $u = u(t)$ which is of class $C^1$ from some interval $I \subset \mathbb{R}$ into $\mathbb{R}^N$ and is such that

(a) $(t, u(t), u'(t)) \in \Omega \times A$ for all $t \in I$;
(b) the conjugate momentum $v(t) = \nabla \mathcal{L}(t, u(t), u'(t))$ is continuously differentiable in $I$;
(c) $v'(t) = \nabla_u \mathcal{L}(t, u(t), u'(t))$ for all $t \in I$.

This last conclusion is valid for every Lagrangian $\mathcal{L}$ without assuming that $\mathcal{L}$ is separable, namely $\mathcal{L}(t, u, v) = G(u, v) + F(t, u)$.

**Theorem 9.** (Theorem 6, [32]) Suppose that the hypotheses of Theorem 5 are satisfied. Let $u$ be a solution of the system

$$
\left( \nabla \mathcal{L}(t, u, v) \right)' - \nabla_u \mathcal{L}(t, u, v) + Q(t, u, v) = 0
$$
in some interval $I$. Then the pair

$$
u = u(t) \quad \text{and} \quad w = \nabla \mathcal{L}(t, u(t), u'(t))
$$
is a solution of the Hamiltonian system

$$
u' = \nabla_w \mathcal{H}(t, u, w)
$$

$$w' = -\nabla_u \mathcal{H}(t, u, w) - P(t, u, w), \quad t \in I,
$$

where

$$P(t, u, w) = Q(t, u, (\nabla \mathcal{L}(t, u, \cdot))^{-1}(w)), \quad (3.3)$$

and conversely.

### 4. Local existence

In this section we prove existence of local solutions of the initial value problem

$$
[\nabla G(u, u')]' - \nabla_u G(u, u') + Q(t, u, u') = f(t, u), \quad t > T,
$$

$$
u(T) = u_0 \neq 0, \quad u'(T) = 0, \quad T \geq 0.
$$

(4.1)
The main result of this section, Theorem 13 below, improve Theorem 1 of [14] and will appear in [11].

Note that local existence for solutions of problem (4.1) is extremely delicate since in general the damping term might be not defined for \( t = T \), indeed, as noted in Section 1, in the variational case we have \( T = 0 \) and \( Q(t, u, v) = \frac{n - 1}{t} \nabla G(u, v) \).

For this reason we assume, together with (A1), (A2) and (A3), also the following condition:

(A4) there exist two non-negative functions \( \delta(t) \in C(J) \) and \( \psi(u, v) \in C(D \times \mathbb{R}^N) \), with \( \psi(u, 0) = 0 \) for all \( u \in D \), such that

\[
|Q(t, u, v)| \leq \delta(t) \psi(u, v)
\]

for \((t, u) \in J \times D \) and \( v \) sufficiently small.

Our situation is the specular situation of [14] in which the term \( f(t, u) \) is a restoring force instead of a driving force and it depends only on \( u \).

Without loss of generality, and for simplicity in the notation, we may assume that \( T = 0 \) in (4.1), indeed if \( T > 0 \) it is enough to make a translation from \( t \) to \( t - T \).

**Theorem 10.** Suppose that conditions (A1)–(A4) hold, that there exists finite

\[
\lim_{t \to 0^+} e^{-\int_0^t \delta(s) ds} := c_0 \in \mathbb{R},
\]

and that the function

\[
h(t) = \begin{cases} 
  e^{-\int_0^t \delta(s) ds} - c_0, & t \in (0, 1], \\
  0, & t = 0,
\end{cases}
\]

is of class \( C^1 \) in \([0, 1]\).

Then the initial value problem (4.1), with \( T = 0 \), admits a local classical solution defined on \([0, \tau)\), for some \( 0 < \tau \leq 1 \). Furthermore \( |u(t)| > 0 \) for all \( t \in [0, \tau) \).
4. LOCAL EXISTENCE

Proof. First note that every solution of (4.1), with $T = 0$, is a solution of the following initial value problem for $t \in (0, 1)$

$$
\begin{align*}
[h(t) \nabla G(u, u')]' &= h(t)[\nabla_u G(u, u') + f(t, u)] + h'(t) \nabla G(u, u') - h(t)Q(t, u, u'), \\
u(0) &= u_0 \neq 0, \quad u'(0) = 0,
\end{align*}
$$

(4.4)

where $h$ is given in (4.3). As in the proof of ([14], Theorem 1), since $G(u, \cdot)$ is strictly convex, we use the result of Pucci & Serrin reported in Theorem 9 concerning the equivalence between (4.4), and the Hamiltonian problem

$$
\begin{align*}
u' &= \nabla_w H(t, u, w) \\
(h(t)w)' &= -h(t) \nabla_u H(t, u, w) + P_1(t, u, w), \quad t \in (0, 1), \\
u(0) &= u_0, \quad w(0) = 0,
\end{align*}
$$

(4.5)

where

$$
H(t, u, w) = H(u, (\nabla G(u, \cdot))^{-1}(w)) - F(t, u),
$$

(4.6)

and $H$ is defined by

$$
H(u, v) = \langle \nabla G(u, v), v \rangle - G(u, v),
$$

while

$$
P_1(t, u, w) = h'(t)w - h(t)Q(t, u, (\nabla G(u, \cdot))^{-1}(w)).
$$

Note that since

$$
\mathcal{L}(t, u, v) = G(u, v) + F(t, u),
$$

(4.7)

then $\nabla \mathcal{L} = \nabla G$. Thus the function $\mathcal{H}$, defined in (4.6), is the same as in (3.1). Moreover by Theorems 7, 8 and the fact that $F \in C^1[0, \infty)$ by (A2), the function $\mathcal{H}$ is of class $C^1$ over the set

$$
\{(t, u, w) \in [0, \infty) \times D \times \mathbb{R}^N : w = \nabla G(u, v) \text{ for some } v \in \mathbb{R}^N\}.
$$
By the fact that $\nabla_w \mathcal{H}(t, u, w) = (\nabla G(u, \cdot))^{-1}(w)$, we put in evidence the fact that $\nabla_w \mathcal{H}(t, u, w)$ does not depend on $t$. Now, let $B = B(u_0, |u_0|/2)$. By (A4) it results that

$$|Q(t, u, v)| \leq \delta(t)\psi(u, v) \quad \text{for } t \in (0, 1], \ u \in D, \ |v| \leq r_1,$$

(4.8)

for some $r_1 > 0$. Since $(\nabla G(u, \cdot))^{-1}(0) = 0$, we may find $r_2 > 0$ such that

$$|(\nabla G(u, \cdot))^{-1}(w)| \leq r_1 \quad \text{for all } u \in \mathbb{B} \quad \text{and } |w| \leq r_2.$$

Thus, since $h\delta = h'$ by virtue of (4.3), we get from (4.8)

$$|P_1(t, u, w)| \leq h'(t)S(u, w) \quad \text{for all } t \in (0, 1], \ u \in \mathbb{B} \quad \text{and } |w| \leq r_2,$$

(4.9)

where

$$S(u, w) = |w| + \psi(u, (\nabla G(u, \cdot))^{-1}(w)).$$

Since $\nabla_w \mathcal{H}(t, u, 0) = 0$ and $S(u, 0) = 0$, then for any $\sigma > 0$ there exists $r_3 < r_2$ such that

$$|\nabla_w \mathcal{H}(t, u, w)| \leq r_2 \quad \text{for all } t \in (0, 1], \ u \in \mathbb{B} \quad \text{and } |w| \leq r_3,$$

and

$$|P_1(t, u, w)| \leq \sigma h'(t) \quad \text{for all } t \in (0, 1], \ u \in \mathbb{B} \quad \text{and } |w| \leq r_3.$$

(4.10)

Now let $r = \min\{|u_0|/2, r_3\}$ and define, as in [14],

$$C = \{x = (u, w) \in C([0, \tau]; \mathbb{R}^{2N}) : \|(u, w) - (u_0, 0)\|_\infty \leq r\},$$

where $\tau \in (0, 1]$ is a sufficiently small number to be determined later. For every function $x \in C$ we have in particular that $\|u - u_0\|_\infty \leq r \leq |u_0|/2$, so that $|u_0|/2 \leq
\[ |u(t)| \leq 3|u_0|/2 \text{ for all } t \in [0, \tau]. \]

Now define \( T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \) as follows:

\[
T_1[x](t) = u_0 + \int_0^t \nabla_w \mathcal{H}(s, u, w) \, ds,
\]

\[
T_2[x](t) = \begin{cases} 
\frac{1}{h(t)} \int_0^t \left\{ -h(s) \nabla_u \mathcal{H}(s, u, w) + P_1(s, u, w) \right\} ds & \text{if } t \in (0, 1], \\
0 & \text{if } t = 0.
\end{cases}
\]

Note that \( T_2[x](t) \) is continuous for \( t = 0 \). Indeed let

\[
K = \max\{|\nabla_x \mathcal{H}(t, u, w)| : t \in [0, 1], |u - u_0| \leq r, |w| \leq r\}.
\]

Since \( h(0) = 0 \) by (4.3), then from (4.9) and using the fact that \( h(t) \) is non decreasing, we get that

\[
\lim_{t \to 0^+} |T_2[x](t)| \leq \lim_{t \to 0^+} \int_0^t \frac{h(s)}{h(t)} |\nabla_u \mathcal{H}(s, u, w)| ds + \lim_{t \to 0^+} \frac{1}{h(t)} \int_0^t |P_1(s, u, w)| ds
\]

\[
\leq \lim_{t \to 0^+} Kt + \lim_{t \to 0^+} \frac{b'(t) S(u(t), w(t))}{h'(t)} ds
\]

\[
= \lim_{t \to 0^+} S(u(t), w(t)) = 0,
\]

(4.11)

where we have used l’Hospital rule and the fact that \( w(t) = \nabla G(u(t), u'(t)) \), that \( u'(0) = 0 \) so that \( S(u, 0) = 0 \), hence

\[
\lim_{t \to 0^+} T_2[x](t) = 0
\]

for every \( x \in C \). For \( \tau \) sufficiently small we shall show that \( T(C) \subset C \) and \( T \) is compact, hence Schauder’s fixed point theorem holds. Fix \( \tau < r/K \), thus

\[
|T_1[x](t) - u_0| \leq \int_0^t |\nabla_w \mathcal{H}(s, u, w)| ds \leq K\tau < r.
\]

Now consider \( T_2 \). By (4.11) we have that

\[
|T_2[x](t)| \leq K\tau + \sigma.
\]
Hence it is enough to choose $\sigma < r$ and $\tau < (r - \sigma)/K$, to obtain that $|T_2[x](t)| \leq r$. Therefore $T(C) \subset C$.

As noted in [14], the family $\{T[x] : x \in C\}$ is equibounded by the proof above. $T$ is obviously continuous from $C$ to $C$. Thus $T$ is compact if $\{T[x] : x \in C\}$ is equicontinuous by Ascoli Arzelà’s lemma. Of course $\{T_1[x] : x \in C\}$ is equiLipschitz (uniformly), since $\||T_1[x]\'||_\infty \leq K$, and in turn equicontinuous.

Then, we shall prove the equicontinuity of $\{T_2[x] : x \in C\}$. First note that

$$|T_2[x](t_2) - T_2[x](t_1)| = \left| \frac{1}{h(t_2)} \int_0^{t_2} \{-h(s)\nabla u H(s, u, w) + P_1(s, u, w)\} \, ds \right|$$

$$+ \frac{1}{h(t_1)} \int_0^{t_1} \{-h(s)\nabla u H(s, u, w) + P_1(s, u, w)\} \, ds$$

$$\leq \left[ \frac{1}{h(t_1)} - \frac{1}{h(t_2)} \right] \int_0^{t_1} \{h(s)|\nabla u H(s, u, w)| + |P_1(s, u, w)|\} \, ds$$

$$+ \frac{1}{h(t_2)} \int_{t_1}^{t_2} \{h(s)|\nabla u H(s, u, w)| + |P_1(s, u, w)|\} \, ds.$$  (4.12)

Let $\varepsilon > 0$ be fixed, we want to find $\tau_\varepsilon = \tau_\varepsilon(\varepsilon) > 0$ such that for all $t_1, t_2 \in [0, \tau]$ with $0 < t_2 - t_1 \leq \tau_\varepsilon$, it results that

$$|T_2[x](t_2) - T_2[x](t_1)| \leq \varepsilon.$$

Now consider $0 < t_2 - t_1 \leq \varepsilon/3K$ and choose $\sigma$ in (4.10) such that $\sigma \leq \varepsilon/6$. There are now three cases. If $t_1 = 0$, $0 < t_2 \leq \varepsilon/3K$. Now from (4.11) we deduce that

$$|T_2[x](t_2) - T_2[x](t_1)| = |T_2[x](t_2)| \leq Kt_2 + \sigma \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{6} < \varepsilon.$$

If $0 < t_1 \leq \varepsilon/3K$ then we obtain that the right side of (4.12) is less than

$$\frac{h(t_2) - h(t_1)}{h(t_1)h(t_2)} \int_0^{t_1} \{Kh(s) + \sigma h'(s)\} \, ds + \frac{1}{h(t_2)} \int_{t_1}^{t_2} \{Kh(s) + \sigma h'(s)\} \, ds$$

$$\leq \frac{h(t_2) - h(t_1)}{h(t_2)} \int_0^{t_1} \frac{K}{h(t_1)} h(s) \, ds + 2\sigma \frac{h(t_2) - h(t_1)}{h(t_2)} + K(t_2 - t_1)$$

$$\leq Kt_1 + 2\sigma + K\varepsilon/3K \leq \varepsilon,$$

where we have used the fact that $h$ is non decreasing.
In the third case, namely when \( t_2 > t_1 \geq \frac{\varepsilon}{3K} \), by the mean value theorem
\[
K \frac{h(t_2) - h(t_1)}{h(t_1)h(t_2)} \int_0^{t_1} h(s)ds \leq \frac{|h'(\xi)|}{h(\varepsilon/3K)} K \eta,
\]
for some appropriate number \( \eta \in (t_1, t_2) \). Consequently, if we choose
\[
\tau_\varepsilon = \min \left\{ \frac{\varepsilon}{3K}, \frac{h(\varepsilon/3K)\varepsilon}{3K\tau \max_{[0,\tau]} |h'(t)| + 1} \right\},
\]
by (4.12) and (4.14) we immediately get
\[
|T_2[x](t_2) - T_2[x](t_1)| \leq \frac{\varepsilon}{3} + 2\sigma + K\tau \leq \varepsilon.
\]
This completes the proof of the claim.

By Schauder’s fixed point theorem there is a pair of continuous functions \((u, w)\) such that
\[
\begin{align*}
 u(t) &= u_0 + \int_0^t \nabla_w \mathcal{H}(s, u, w) \, ds, \\
 w(t) &= \frac{1}{h(t)} \int_0^t \left\{ -h(s)\nabla_u \mathcal{H}(s, u, w) + P_1(s, u, w) \right\} ds.
\end{align*}
\]
Finally, by the argument of [32], \( u \) is a classical solution. Moreover \(|u(t)| > 0\) by construction.

Now, we shall show that, the existence of local solutions of the initial value problem
\[
\begin{align*}
 [\nabla G(u, u')]' - \nabla_u G(u, u') + Q(t, u, u') &= f(t, u), \\
 u(T_1) = u_0 \neq 0, \ u'(T_1) = v_0, \ T_1 > T,
\end{align*}
\]
is extremely simple since can be immediately derived by the classical theory, without assuming \((A4)\). 

**Theorem 11.** Suppose that conditions \((A1)-(A3)\) hold. Then the initial value problem (4.16) admits at least a classical solution defined on \((T_1 - \tau, T_1 + \tau)\), for some \( \tau > 0 \). Furthermore \(|u(t)| > 0\) for all \( t \in (T_1 - \tau, T_1 + \tau)\).
Proof. Let \( \Omega \subseteq \mathbb{R} \times \mathbb{R}^N \)

\[ \Omega = \{(t, u) : t > 0, \ u \in D \setminus \{0\}\} \]

and let \( \mathcal{H} \) and \( \mathcal{L} \) be the functions given in (4.6) and (4.7) respectively, where \((t, u, w)\) vary in the set

\[ \widehat{D} = \{(t, u, w) : (t, u) \in \Omega, \ w = \nabla G(u, v) \text{ for some } v \in \mathbb{R}^N\}. \]

Again, as in the proof of the previous theorem, by hypothesis (A1), and Theorems 7 and 8, the function \( \mathcal{H} \) is well defined and of class \( C^1 \) over the open set \( \widehat{D} \). Consider now the corresponding Hamiltonian problem

\[
\begin{align*}
    u' &= \nabla_w \mathcal{H}(t, u, w), \\
    w' &= -\nabla_u \mathcal{H}(t, u, w) - Q(t, u, (\nabla G(u, \cdot)^{-1})(w)), \\
    u(T_1) &= u_0, \quad w(T_1) = w_0,
\end{align*}
\]

where

\[ w_0 = \nabla G(u_0, v_0). \]

Since \((u_0, w_0) \in \widehat{D} \) and \( \mathcal{H} \in C^1(\widehat{D}) \), then problem (4.17) admits a local solution \( u : (T_1 - \tau, T_1 + \tau) \rightarrow \mathbb{R}^N \), such that

\[ (t, u(t)) \in \widehat{D} \quad \text{for all} \quad t \in (T_1 - \tau, T_1 + \tau). \]

In particular \( u(t) \neq 0 \) for all \( t \in (T_1 - \tau, T_1 + \tau) \). By Theorem 9 the function \( u \) is a local solution of (4.16) in \((T_1 - \tau, T_1 + \tau)\). \( \square \)
CHAPTER 2

QUASIVARIATIONAL SYSTEMS: GLOBAL NON
EXISTENCE

1. Main hypotheses

The purpose of this chapter is to prove a global non-existence theorem for solution of the following initial value problem

\[ \begin{align*}
[\nabla G(u, u')]' - \nabla u G(u, u') + Q(t, u, u') &= f(t, u), \quad t > T, \\
u(T) &= u_0 \neq 0, \quad u'(T) = 0.
\end{align*} \]  

(1.1)

Here \( T \geq 0 \) and the functions \( F, G, Q \) satisfy conditions (A1)-(A4) of the previous chapter, with

\[ D = \mathbb{R}^N \quad \text{or} \quad D = \mathbb{R}^N \setminus \{0\}, \]  

(1.2)

together with the following additional structural conditions:

(S1) there exist functions \( \tilde{F} \in C(\mathbb{R}^N; \mathbb{R}^+_0) \) and \( \psi \in L^1(J; \mathbb{R}^+_0) \) such that

\[ \tilde{F}(0) = 0, \quad \tilde{F}(u) > 0 \quad \text{if} \ u \neq 0 \]

and

\[ 0 \leq F_t(t, u) \leq \psi(t) \tilde{F}(u) \quad \text{for all} \quad (t, u) \in J \times \mathbb{R}^N; \]  

(1.3)

for every \( U > 0 \) there exist an exponent \( p > 1 \) and three constants \( c_1, c_2, q > 0 \) such that

\[ c_1 F(t, u) \leq c_2 |u|^p \leq (f(t, u), u) - qF(t, u) \]  

(1.4)

for all \( (t, u) \in \overline{J} \times \mathbb{R}^N \), with \( |u| \geq U \);
(S2) for every \( U > 0 \) there exist two exponents \( l > 1, \gamma \in \mathbb{R} \), a function \( \phi \in C(D \times \mathbb{R}^N; \mathbb{R}) \) and two constants \( c_3, c_4 > 0 \) such that

\[
(q + 1)\langle \nabla G(u, v), v \rangle - qG(u, v) + \langle \nabla_u G(u, v), u \rangle \geq \phi(u, v)|u|\gamma|v|^l, \tag{1.5}
\]

\[
|\nabla G(u, v)| \leq c_3\phi(u, v)|u|\gamma|v|^{l-1}, \tag{1.6}
\]

\[
0 \leq \phi(u, v) \leq c_4 \tag{1.7}
\]

for all \((u, v) \in \mathbb{R}^N \times \mathbb{R}^N\) with \(|u| \geq U|;\)

(S3) for every \( U > 0 \) there exist exponents \( m > 1, \kappa \in \mathbb{R} \) and a non-negative function \( \delta \in L^\infty_{\text{loc}}(J; \mathbb{R}) \) such that

\[
|Q(t, u, v)| \leq \left[ \delta(t) \right]^{1/m}|u|^{\kappa/m}\langle Q(t, u, v), v \rangle^{1/m'}, \tag{1.8}
\]

for all \((t, u, v) \in J \times \mathbb{R}^N \times \mathbb{R}^N\) with \(|u| \geq U|, \text{where } m' \text{ denotes the } \text{older conjugate of } m.\)

**Remark 1.** Conditions (S1) and (S2) imply that

\[
\frac{-\gamma - l}{l - 1} \leq q < p \tag{1.9}
\]

and in turn that

\[
\gamma > -l - p(l - 1). \tag{1.10}
\]

Indeed, let \(|u| = 1, s \geq 1, t \geq T\). Then from (S1)

\[
(s^{-q}F(t, su))^\prime = s^{-q-1}(-qF(t, su) + \langle f(t, su), su \rangle) \geq c_2s^{-q+p-1}
\]

and upon integration, and another use of (S1) we get

\[
\frac{c_2}{c_1}s^p \geq F(t, su) \geq \begin{cases} 
\frac{s^p - s^q}{p - q} & \text{if } p \neq q, \\
 s^q \log s & \text{if } p = q.
\end{cases}
\]

Since the previous inequality is true for all \( s \geq 1 \) we obtain that necessarily \( p > q. \)
Similarly, let $|u| = |v| = 1$, $s \geq 1$. Then from (1.5)

$$(s^{-q}G(su, s^{q+1}v))' = s^{-q-1}[(q + 1)\langle \nabla G(su, s^{q+1}v), s^{q+1}v \rangle - qG(su, s^{q+1}v) + \langle \nabla_u G(su, s^{q+1}v), su \rangle] \geq 0. \quad (1.11)$$

Hence

$$G(su, s^{q+1}v) \geq s^q g_0,$$

where

$$g_0 = \min_{|u| = |v| = 1} G(u, v) > 0.$$

In turn, from (1.6)

$$c_3 c_4 s^{\gamma + (q+1)l} \geq (\nabla G(su, s^{q+1}v), s^{q+1}v) \geq G(su, s^{q+1}v) \geq s^q g_0$$

for all $s \geq 1$. Hence $\gamma + (q + 1)l \geq q$, or equivalently

$$q \geq \frac{-\gamma - l}{l - 1}.$$

We put in evidence that assumption (S3), when $\delta \in C(J)$, forces the existence of a local solution of (1.1), thanks to Theorem 10 of Chapter 1.

In the special case

$$(|u|^\gamma|u'|^{\gamma - 2}u')' - \frac{\gamma}{l}|u|^\gamma|u'|^l + \delta(t)|u|^\kappa|u'|^{m-2}u' = |u|^{p-2}u, \quad t > T,$$

$$u(T) = u_0 \neq 0, \quad u'(T) = 0, \quad T \geq 0,$$

with $m, l, p > 1$, $\gamma, \kappa \in \mathbb{R}$, $\delta \in L^\infty_{loc}(J)$, we have

$$G(u, v) = \frac{1}{l} |u|^\gamma |v|^l, \quad Q(t, u, v) = \delta(t)|u|^\kappa |v|^{m-2}v, \quad F(t, u) = \frac{1}{p} |u|^p.$$

If (1.10) holds, then (S1)–(S3) are trivially satisfied for any

$$q \in \left(\max \left\{0, \frac{-\gamma - l}{l - 1}\right\}, p\right),$$

with

$$c_1 = p - q, \quad c_2 = 1 - \frac{q}{p}, \quad \phi(u, v) = \frac{q}{p} + 1 + \frac{\gamma}{l} = L, \quad c_3 = \frac{1}{L}.$$
Another example concerning (S2) is given by

\[ G(u, v) = \left( 1 + \frac{1}{1 + |uv|^l} \right) \frac{|v|^l}{l}, \quad l \geq \frac{3}{2}. \]

Indeed in this case (1.5) and (1.6) hold with

\[ q \geq 1, \quad \gamma = 0, \quad \phi(u, v) = 1 + \frac{q}{l} - \frac{(1 + q/l|uv|^l)}{(1 + |uv|^l)^2}. \]

2. Variational identities

In this section we prove some variational identities, of Pohozaev–Pucci–Serrin type (see [27]) which will be instrumental in the proof of the main results.

By (A1) the function

\[ H(u, v) = \langle \nabla G(u, v), v \rangle - G(u, v) \tag{2.1} \]

is continuous, \( H(u, 0) = 0 \) and

\[ H(u, v) > 0 \quad \text{for all} \quad u \in \mathbb{R}^N \setminus \{0\}, \quad v \in \mathbb{R}^N \setminus \{0\}, \tag{2.2} \]

by the strict convexity of \( G(u, \cdot) \) when \( u \neq 0 \).

Let \( u : [T, T_1) \to \mathbb{R}^N, \ T_1 \leq \infty, \) be a local solution of (1.1). In spite of the fact that neither \( u' \) nor \( H \) need to be separately differentiable, the composite function \( H(u(t), u'(t)) \) is differentiable on \( (T, T_1) \) provided that \( u \) never vanishes on \([T, T_1)\).

Indeed, the following result holds (see [29] for the case \( G = G(v) \) and [32] for the general case \( G = G(u, v) \))

**Proposition 2.** Let \( u : [T, T_1) \to \mathbb{R}^N, \ T_1 \leq \infty, \) be a solution of (1.1) such that \( u(t) \neq 0 \) for all \( t \in [T, T_1) \). Then \( H(u(t), u'(t)) \) is continuously differentiable in \((T, T_1)\) and for all \( t \in (T, T_1) \) we have

\[ \{H(u(t), u'(t))\}' = -\langle Q(t, u(t), u'(t)), u'(t) \rangle + \langle f(t, u(t)), u'(t) \rangle. \tag{2.3} \]
2. VARIATIONAL IDENTITIES

Proof. The proof of this proposition was kindly given to me by Prof P. Pucci. For simplicity, we only give the proof of formula (2.3) under the additional hypothesis

\[ \nabla G(u, v) \text{ is continuously differentiable with respect to } u \text{ in } D \times \mathbb{R}^N, \quad (2.4) \]

which was used in [15]. The general case may be found in [32] and it is obtained by a mollification argument.

First, the differentiability of \( H \) along the solution \( u \) follows immediately from Theorems 7 and 8 of Chapter 1 and the fact that

\[ H(u(t), u'(t)) = G^*(u(t), w(t)), \quad (2.5) \]

where \( G^*(u, \cdot) \) is the Legendre transform of \( G(u, \cdot) \) and \( (u(t), w(t)) \) is a solution of the Hamiltonian system (3.2) of Chapter 1. To obtain (2.3) we write

\[
\{H(u(t), u'(t))\}' = \lim_{h \to 0} \frac{H(u(t + h), u'(t + h)) - H(u(t), u'(t))}{h} \\
= \lim_{h \to 0} \frac{H(u(t + h), u'(t + h)) - H(u(t), u'(t + h))}{h} \\
+ \lim_{h \to 0} \frac{H(u(t), u'(t + h)) - H(u(t), u'(t))}{h}.
\]

(2.6)

Now, since \( u \in C^1([T, T_1]) \) we can use the following approximation

\[ u(t + h) = u(t) + \tau(h) \quad \text{where} \quad \lim_{h \to 0} \frac{\tau(h)}{h} = u'(t). \]

Consequently, since \( H \) is differentiable with respect to \( u \) thanks to (2.4), by Lagrange’s Theorem we have, for \( \theta \in [0, 1] \),

\[
\lim_{h \to 0} \frac{H(u(t + h), u'(t + h)) - H(u(t), u'(t + h))}{h} = \\
\lim_{h \to 0} \langle \nabla_u H(u(t) + \theta \tau(h), u'(t + h)), \frac{\tau(h)}{h} \rangle = \langle \nabla_u H(u(t), u'(t)), u'(t) \rangle.
\]

(2.7)
On the other hand
\[
\lim_{h \to 0} \frac{H(u(t), u'(t + h)) - H(u(t), u'(t))}{h} = \lim_{h \to 0} \frac{\langle \nabla G(u(t), u'(t + h)) - \nabla G(u(t), u'(t)) \rangle}{h}
\]
where in the last equality we have used that \( G \in C^1 \), hence it is differentiable.

Now we can apply the same argument used in (2.7), with \( H \) replaced by every component of \( \nabla G \), namely we can apply Lagrange’s Theorem to every function \( \nabla G_i, i = 1, \ldots, N \), thanks to (2.4). Thus by virtue of (2.4), (2.6) and (2.7) we get
\[
\{ \nabla G_i(u(t), u'(t)) \}' = \langle \nabla_u [\nabla G_i(u(t), u'(t))] , u'(t) \rangle + \lim_{h \to 0} \frac{\nabla G_i(u(t), u'(t + h)) - \nabla G_i(u(t), u'(t))}{h}
\]
for \( i = 1, \ldots, N \). Consequently by (2.8) and (2.9) it follows
\[
\langle \{ \nabla G(u(t), u'(t)) \}' , u'(t) \rangle = \langle [\nabla_u \nabla G(u(t), u'(t)), u'(t)] , u'(t) \rangle + \lim_{h \to 0} \frac{H(u(t), u'(t + h)) - H(u(t), u'(t))}{h},
\]
where, omitting the dependence on \( t \) of the solution,
\[
\langle \nabla_u \nabla G(u, u') , u' \rangle = \langle \nabla_u \nabla G_i(u, u') , u' \rangle = \langle \nabla_u \nabla G_N(u, u') , u' \rangle = \nabla_u \langle \nabla G(u, u') , u' \rangle.
\]
Hence by (2.6), (2.7), (2.10) and (2.1), we get
\[
\{ H(u, u') \}' = \langle \nabla_u H(u, u'), u' \rangle + \{ \nabla G(u, u') \}' , u' \rangle - \langle \nabla_u \{ \nabla G(u, u') \}' , u' \rangle, \quad (2.12)
\]
Finally, we claim the required formula (2.3), by using the fact that \( u \) is a solution of (1.1).

From now on, for simplicity, the common notation where \( u = u(t) \) and \( u' = u'(t) \) denote the solution and its derivative. Hence if \( u : [T, T_1) \to \mathbb{R}^N \) is a solution of (1.1) on \((T, T_1)\) such that \( u \neq 0 \), we have thanks to (2.3) for \( t \in (T, T_1)\)

\[
\{H(u, u') - F(t, u)\}' = -\langle Q(t, u, u'), u' \rangle - F_i(t, u). \tag{2.13}
\]

Consequently, if we introduce the total energy of the vector field \( u \), defined by

\[
E(t) := H(u, u') - F(t, u), \tag{2.14}
\]

by (2.13), we can write that

\[
E'(t) = -\langle Q(t, u, u'), u' \rangle - F_i(t, u). \tag{2.15}
\]

We immediately note that by (A2) and (A3), the energy \( E \) is a non increasing function and the following conservation law holds, for any \( s \in (T, T_1) \), provided that \( u(t) \neq 0 \) for all \((T, T_1)\),

\[
E(t) = E(s) - \int_s^t \{\langle Q(\tau, u, u'), u' \rangle + F_i(\tau, u)\} \, d\tau \tag{2.16}
\]

for \( T < s \leq t < T_1 \).

Moreover, as noted in [31], since along a solution \( u : [T, T_1) \to \mathbb{R}^N \) the function \( H(u, u') \) is differentiable with respect to \( t \) (see Proposition 1), the following identity holds for solutions \( u \) of (1.1), again when \( u(t) \neq 0 \) for all \( t \in [T, T_1) \), and for any pair of scalar function \( \varphi, \omega \in C^1(J, \mathbb{R}) \):

\[
\{\omega[H(u, u') - F(t, u)] + \varphi(\nabla G(u, u'), u)\}'
\]

\[
= \omega'[H(u, u') - F(t, u)] + \varphi(f(t, u), u) - \omega F_i(t, u)
\]

\[
+ \varphi'\left(\langle \nabla G(u, u'), u' \rangle + \langle \nabla u G(u, u'), u \rangle\right) + \varphi'\langle \nabla G(u, u'), u \rangle
\]

\[
- \omega\langle Q(t, u, u'), u' \rangle - \varphi(Q(t, u, u'), u), \quad T < t < T_1. \tag{2.17}
\]

\]
This formula was originally discovered as a special case of the main identity in [27].

Note that (2.17) reduces to (2.13) when \( \omega = 1 \) and \( \varphi = 0 \).

The main results of this chapter will appear in [11].

### 3. A crucial property

In this chapter, from now on, we deal with solutions \( u : \left[T, T_1\right) \to \mathbb{R}^N \) of (1.1) having negative initial energy, namely we treat the case, in which the function

\[
E(t) = H(u, u') - F(t, u)
\]

is such that \( E(T) < 0 \). Moreover we recall that if the solution \( u \) is such that \( u(t) \neq 0 \) for all \( t \in (T, T_1) \) then it holds the following formula

\[
E'(t) = -(Q(t, u, u'), u') - F_t(t, u).
\]

In the sequel, to simplify the notation, we write

\[
\mathcal{E} = \mathcal{E}(t) = -E(t), \quad \mathcal{E}_0 = \mathcal{E}(T).
\]

We put in evidence that the required assumption \( E(T) < 0 \) is crucial for our analysis since it implies that every solution of (1.1) remains in norm far from zero. More specifically the following property holds

**Proposition 3.** Assume that (A1)–(A4) of Chapter 1 hold together with (1.3). Let \( u : \left[T, T_1\right) \to D, T_1 \leq \infty \) be a solution of (1.1) in \( (T, T_1) \). Then if \( F(T, u_0) > 0 \), there exists a positive constant \( U \) which depends only on \( F(T, u_0) \) such that

\[
|u(t)| \geq U \quad \text{for all } t \in [T, T_1).
\]

**Proof.** First note that the fact the \( F(T, u_0) > 0 \) implies that the solution \( u \) has negative initial energy, indeed

\[
E(T) = H(u_0, 0) - F(T, u_0) = -F(T, u_0) < 0.
\]
3. A CRUCIAL PROPERTY

By (A2) and (1.3) for all \( t \in J \) and \( u \in \mathbb{R}^N \) we have

\[
F(t, u) = F(T, u) + \int_T^t F_t(s, u) ds \\
\leq F(T, u) + \tilde{F}(u) \int_T^\infty \psi(s) ds \\
\leq F(T, u) + \text{Const.} \tilde{F}(u).
\] (3.5)

Let \( \mathcal{E}_0 = -E(T) > 0 \). Since \( F(T, 0) = 0 \) and \( \tilde{F}(0) = 0 \) there exists \( U > 0 \) such that

\[
F(T, u) + \text{Const.} \tilde{F}(u) < \frac{\mathcal{E}_0}{2}
\]

for all \( u \in \mathbb{R}^N \) with \(|u| < U\). In turn by (3.5)

\[
F(t, u) < \frac{\mathcal{E}_0}{2}
\] (3.6)

for all \( t \in \overline{J} \) and all \( u \in \mathbb{R}^N \) with \(|u| < U\).

We now divide the proof in two cases. We consider first the case \( D = \mathbb{R}^N \). Since \( u(T) = u_0 \neq 0 \), by continuity there exists \( t_2, \) with \( T < t_2 < T_1 \), such that

\[
u(t) \neq 0 \quad \text{for all} \quad t \in [T, t_2).
\] (3.7)

Without loss of generality we may assume that \([T, t_2)\) is the maximal interval in which (3.7) holds. By (3.1)–(3.3), (A2) and (A3) for \( t \in [T, t_2) \) we have

\[
\mathcal{E}'(t) \geq 0,
\]

and so

\[
F(t, u(t)) \geq -H(u(t), u'(t)) + F(t, u(t)) = \mathcal{E}(t) \geq \mathcal{E}(T) = \mathcal{E}_0 > 0.
\]

Thus by (3.6) we get

\[
|u(t)| \geq U \quad \text{for all} \quad t \in (T, t_2).
\]

In turn, by the maximality of \( t_2 \), this implies that \( t_2 = T_1 \) and the proof is complete in the case \( D = \mathbb{R}^N \).
2. QUASIVARIATIONAL SYSTEMS: GLOBAL NON EXISTENCE

When \( D = \mathbb{R}^N \setminus \{0\} \) then (3.7) holds with \( t_2 = T_1 \), since \( u(t) \in \mathbb{R}^N \setminus \{0\} \) for all \( t \in [T, T_1) \) by the definition of solution. We may now continue as before. \( \square \)

**Remarks.** In the scalar case, \( N = 1 \), radial solutions are strictly monotone functions (see Section 1 of Chapter 5), thus we immediately obtain that they remain far from zero.

Finally we point out that the condition \( E(T) < 0 \) is trivially satisfied by the prototype considered in the applications, see Chapters 4 and 5.

4. Preliminary Lemmas

**Lemma 1.** Assume that for every \( U > 0 \) the exponents in \((S1)-(S3)\) satisfy the further restrictions

\[ 1 < m < p, \quad \kappa < p - m, \quad (4.1) \]

and denote by \( \overline{\alpha} \) and \( c \) the following constants

\[ \overline{\alpha} = \frac{1}{m} - \frac{\kappa}{mp} - \frac{1}{p} \quad \text{and} \quad c = \left( \frac{c_2}{c_1} \right)^{\frac{\overline{\alpha}}{\overline{\alpha}}} \quad (4.2) \]

Let \( u \) be a solution of \((1.1)\) on \( J \). Then, for every \( \epsilon > 0 \), the following inequality holds for the damping term \( Q \)

\[ \langle Q(t, u, u') \rangle \leq C_1 (\epsilon^m |u|^p + \epsilon^{-m'} \delta^{1/(m-1)} \mathcal{H}^{-\alpha} \mathcal{E}') \quad (4.3) \]

for all \( \alpha \) such that \( 0 < \alpha < \overline{\alpha} \) and where \( C_1 = \mathcal{H}^{-\overline{\alpha}} \max \{ \epsilon^m, \mathcal{E}'_0 \} \).

**Proof.** First note that reasoning as in Remark 1 it follows from (1.4) that \( F(T, u_0) > 0 \), hence the solution \( u \) has negative initial energy \( E(T) < 0 \).

Moreover from (4.1) it follows that \( \overline{\alpha} > 0 \). Now, by Cauchy-Schwarz inequality and (1.8) we get

\[ \langle Q(t, u, u') , u \rangle \leq |Q(t, u, u')| \cdot |u| \leq [\delta(t)]^{1/m} |u|^{\kappa/m} |\langle Q(t, u, u') , u \rangle|^{1/m'} |u|. \quad (4.4) \]
From (3.2) and (A2) it follows that
\[ E'(t) \geq \langle Q(t, u, u'), u' \rangle, \quad t \in J, \]
yielding, by virtue of (4.4),
\[ \langle Q(t, u, u'), u \rangle \leq [\delta(t)]^{1/m} |u|^{1+c/m} (E')^{1/m'} \]
\[ = \frac{\delta(t)}{|u|^p} |u|^{1-p/m} |u|^{-p(1/m-\kappa/m^{1/p})} (E')^{1/m'} \]
\[ = \frac{\delta(t)}{|u|^p} (|u|^{-p})^{(1/m-1)} \]
By (2.2) and (1.4) it follows
\[ \frac{c_2}{c_1} |u|^p \geq F(t, u) \geq E(t), \]
then applying Young's inequality in (4.5) we get for all \( \varepsilon > 0 \)
\[ \langle Q(t, u, u'), u \rangle \leq \varepsilon [\delta(t)]^{1/m} |u|^1 \frac{1}{\varepsilon} ([\delta(t)]^{1/(m-1)} E')^{1/m'} [F(t, u)]^{-1} \]
\[ \leq [\varepsilon^{m} |u|^p + \varepsilon^{-m'} [\delta(t)]^{1/(m-1)} E'] E^{-\alpha}. \]
Now consider \( \alpha \) such that \( 0 < \alpha < \bar{\alpha} \), from (3.1)–(3.3), (A2) and (A3) of Chapter 1, we obtain
\[ E(t) \geq E_0 = -E(T) > 0, \quad E^{-\alpha} = E^{\alpha-\bar{\alpha}} \geq E_0^{\alpha-\bar{\alpha}}, \]
hence (4.6) implies
\[ \langle Q(t, u, u'), u \rangle \leq (\varepsilon^{m} |u|^p E_0^{-\alpha} + \varepsilon^{-m'} [\delta(t)]^{1/(m-1)} E_0^{\alpha-\bar{\alpha}} \bar{H}^{-\alpha} E' \]
and (4.3) is proved.
Lemma 2. Assume that for every $U > 0$ the exponents in $\text{(S1)}$--$\text{(S3)}$ satisfy the further restrictions

$$1 < l < p, \quad \gamma < p - l.$$  \hspace{1cm} (4.8)

Let $u$ be a solution of (1.1) on $J$. Then

(i) \( \frac{d}{dt} \langle \nabla G(u, u'), u \rangle \geq \phi(u, u')|u|\gamma|u'|^l + c_2|u|^p + q\mathcal{E}(t) - \langle Q(t, u, u'), u \rangle; \)

(ii) \( |\langle \nabla G(u, u'), u \rangle| \leq \phi(u, u')|u|^\gamma|u'|^l + C_2|u|^p, \quad C_2 = c_3c_4^{\gamma+l-p}; \)

(iii) \( |u|^r \cdot |\nabla G(u, u')|^r \leq \phi(u, u')|u|^\gamma|u'|^l + C_3|u|^{r(\gamma+l)/(l-r(l-1))}, \)

\[ C_3 = (c_3c_4)^{r/l-r(l-1)}, \text{ for all } 1 < r < l'. \]

Proof. First note that reasoning as in Remark 1 it follows from (1.4) that $F(T, u_0) > 0$, hence the solution $u$ has negative initial energy $E(T) < 0$.

(i) Using (1.1), we have

\[
\frac{d}{dt} \langle \nabla G(u, u'), u \rangle = \langle \{ \nabla G(u, u') \}^t, u \rangle + \langle \nabla G(u, u'), u' \rangle \\
= \langle \nabla G(u, u'), u' \rangle + \langle \nabla_u G(u, u') - Q(t, u, u') + f(t, u), u \rangle. \hspace{1cm} (4.9)
\]

Now, since by (3.3),

\[
H(u, u') - F(t, u) + \mathcal{E}(t) = 0,
\]

we obtain from (4.9) that

\[
\frac{d}{dt} \langle \nabla G(u, u'), u \rangle = \langle \nabla G(u, u'), u' \rangle + \langle \nabla_u G(u, u') - Q(t, u, u') + f(t, u), u \rangle \\
+ q[H(u, u') - F(t, u) + \mathcal{E}(t)] \\
= (q + 1)\langle \nabla G(u, u'), u' \rangle - qG(u, u') + \langle \nabla_u G(u, u'), u \rangle \\
+ \langle f(t, u), u \rangle - qF(t, u) - \langle Q(t, u, u'), u \rangle + q\mathcal{E}(t) \\
\geq \phi(u, u')|u|^\gamma|u'|^l + c_2|u|^p + q\mathcal{E}(t) - \langle Q(t, u, u'), u \rangle,
\]

where $q$ is the exponent given in (S1) and in the last inequality we have used (1.4) and (1.5) to obtain the required inequality.
(ii) From (1.6) and Young’s inequality
\[
|\langle \nabla G(u, u'), u \rangle| \leq c_3 \phi(u, u')|u|^{\gamma} |u'|^{l-1} |u| = \phi(u, u') |u|^{\gamma/l} |u'|^{l-1} |u|^{1+\gamma/l} \\
\leq c_3 \phi(u, u') \left\{ (|u|^{\gamma/l} |u'|^{l-1})^{l'} + |u|^{(1+\gamma/l)l} \right\} \\
= c_3 \phi(u, u') (|u|^{\gamma} |u'|^{l} + |u|^{\gamma+l}).
\]
(4.11)

Now, by (4.8) and Proposition 3 it results that
\[
|u|^{\gamma+l} = |u|^p |u|^{\gamma+l-p} \leq \tau_0^{\gamma+l-p} |u|^p.
\]

Thus from the above inequality together with (4.11) and (S2)
\[
|\langle \nabla G(u, u'), u \rangle| \leq \phi(u, u') |u|^{\gamma} |u'|^{l} + c_3 c_4 |u|^p |u|^{\gamma+l-p} \\
\leq \phi(u, u') |u|^{\gamma} |u'|^{l} + c_3 c_4 \tau_0^{\gamma+l-p} |u|^p,
\]
(4.12)

namely (ii).

(iii) From (1.6) and Young’s inequality, it follows
\[
|u|^r \cdot |\nabla G(u, u')|^r \leq c_5^r [\phi(u, u')]^r |u|^{r+\gamma} |u'|^{(l-1)r} \\
= \left\{ [\phi(u, u')]^{1/l} |u|^{\gamma/l} |u'|^{l-1} \right\}^r \cdot \left\{ c_3 [\phi(u, u')]^{1/l} |u|^{\gamma/l+1} \right\}^r \\
\leq \left\{ [\phi(u, u')]^{1/l} |u|^{\gamma/l} |u'|^{l-1} \right\}^{r\mu} + \left\{ c_3 [\phi(u, u')]^{1/l} |u|^{\gamma/l+1} \right\}^{r\mu},
\]
(4.13)

where \(1 < \mu < \infty\). Now choose \(\mu\) such that
\[
\mu r = l' = \frac{l}{l-1},
\]
then
\[
r\mu' = \frac{r\mu}{\mu - 1} = \frac{1}{r - \frac{l}{l-1}}.
\]

Consequently, since
\[
\frac{r\mu'}{l} = \frac{r}{l - r(l-1)}.
\]

then (4.13) can be written as
\[
|u|^r \cdot |\nabla G(u, u')|^r \leq \phi(u, u') |u|^\gamma |u'|^{l} + (c_5^r [\phi(u, u')]^{1/l} |u|^{\gamma+l})^{r/l-r(l-1)},
\]
and thus the claim is proved by using the boundness of \(\phi\).
5. Main result

THEOREM 12. Assume that conditions (A1)–(A3) of Chapter 1 are satisfied and that conditions (S1)–(S3) hold for every $U > 0$ with

$$1 < l < p, \quad 1 < m < p, \quad \kappa < p - m, \quad -l - p(l - 1) < \gamma < p - l. \quad (5.1)$$

Assume that there exist two positive functions $\rho, k \in W^{1,1}_{loc}(J; \mathbb{R}^+)$, $k' \geq 0$ on $J$ and $\rho'(t) = o(\rho(t))$ as $t \to \infty$, such that

$$\delta(t) \leq \left[ \frac{k(t)}{\rho(t)} \right]^{m-1} \quad \text{in } J \quad (5.2)$$

$$\int_T^\infty \rho(t) \left[ \max(k(t), \rho(t)) \right]^{-1 \theta} \, dt = \infty \quad (5.3)$$

for some constant $\theta \in (0, \theta_0)$, where

$$\theta_0 = \frac{\alpha_0}{1 - \alpha_0}, \quad \alpha_0 = \min \left\{ \frac{1}{m} - \frac{1}{p} - \frac{\kappa}{m p}, \frac{1}{l} - \frac{1}{p} - \frac{\gamma}{l p}, 1 \right\}. \quad (5.4)$$

Then no global solution of (1.1) can exist.

Remark. If $\gamma \geq -l$ then

$$\frac{1}{l} - \frac{1}{p} - \frac{\gamma}{l p} \leq \frac{1}{l},$$

hence

$$\alpha_0 = \left\{ \begin{array}{ll}
\frac{1}{m} - \frac{1}{p} - \frac{\kappa}{m p} & \text{if } \frac{\kappa}{p} + \frac{m}{l} - \frac{m \gamma}{l p} \geq 1, \\
\frac{1}{l} - \frac{1}{p} - \frac{\gamma}{l p} & \text{if } \frac{\kappa}{p} + \frac{m}{l} - \frac{m \gamma}{l p} < 1.
\end{array} \right.$$ 

Now we shall prove Theorem 12.

PROOF. The proof of the theorem is based on an argument of Levine and Serrin [18], even if it is adapted to problem (1.1) instead of the abstract evolution problem studied in [18].

First note that reasoning as in Remark 1 it follows from (1.4) that $F(T, u_0) > 0$, hence the solution $u$ has negative initial energy $E(T) < 0$. 

Assume for contradiction that there is a global solution $u$ of (1.1). Let $\alpha$ be a constant such that

$$0 < \alpha < \overline{\alpha}, \quad \overline{\alpha} = \min\{\overline{\alpha}, 1\},$$

where $\overline{\alpha}$ is given in (4.2). Define the following auxiliary function

$$Z(t) = \lambda k(t)\mathcal{E}^{1-\alpha} + \rho(t)\langle \nabla G(u, u'), u \rangle,$$  \hspace{1cm} (5.5)

where $\lambda$ is a positive constant to be fixed later. As noted in [18], $Z$ is absolutely continuous in $J$ and a.e. it results

$$Z'(t) = \lambda k(t)(1 - \alpha)\mathcal{E}^{-\alpha} \mathcal{E}' + \lambda k'(t)\mathcal{E}^{1-\alpha} + \rho(t)\frac{d}{dt}\langle \nabla G(u, u'), u \rangle + \rho'(t)\langle \nabla G(u, u'), u \rangle,$$  \hspace{1cm} (5.6)

Note that this expression of $Z'$ can be obtained from the variational identity (2.17) by choosing

$$\omega(t) = \lambda k(t)(1 - \alpha)\mathcal{E}^{-\alpha}(t), \quad \varphi(t) = \rho(t).$$

By Lemmas 1 and 2 and using the fact that $k'(t) \geq 0$ we obtain the following estimation for $Z'$

$$Z' \geq \lambda k(t)(1 - \alpha)\mathcal{E}^{-\alpha} \mathcal{E}' + \rho(t)\{\phi(u, u')|u|^{\gamma}|u'|^{\ell} + c_2|u|^p + q\mathcal{E}(t)
- \langle Q(t, u, u'), u \rangle \} - |\rho'(t)| \cdot \langle \nabla G(u, u'), u \rangle]
\geq \lambda k(t)(1 - \alpha)\mathcal{E}^{-\alpha} \mathcal{E}' + \rho(t)\{\phi(u, u')|u|^{\gamma}|u'|^{\ell} + c_2|u|^p + q\mathcal{E}(t) - C_1\varepsilon^m|u|^p
- C_1\varepsilon^{-m'}\delta^{1/(m-1)}\mathcal{E}^{-\alpha} \mathcal{E}'\} - |\rho'(t)|\{\phi(u, u')|u|^{\gamma}|u'|^{\ell} + C_2|u|^p\}
\geq [(1 - \alpha)\lambda k - \rho\varepsilon^{-m'}\delta^{1/(m-1)}C_1] \mathcal{E}^{-\alpha} \mathcal{E}' + q\mathcal{E}
+ \rho \left(1 - c_3 \frac{\rho'}{\rho}\right) \phi(u, u')|u|^{\gamma}|u'|^{\ell} + \rho \left[c_2 - C_2 \frac{\rho'}{\rho} - C_1\varepsilon^m\right]|u|^p.$$  \hspace{1cm} (5.7)

Since $\rho' = o(\rho)$ as $t \to \infty$, for $t$ sufficiently large, say $t \geq t_0$, and $\varepsilon$ sufficiently small, we have

$$c_2 - C_2 \frac{\rho'}{\rho} - C_1\varepsilon^m \geq \frac{1}{2}c_2 \quad \text{and} \quad 1 - c_3 \frac{\rho'}{\rho} \geq \frac{1}{2}.$$  \hspace{1cm} (5.8)
Hence for $t$ sufficiently large it holds
\[ Z' \geq [(1 - \alpha)\lambda k - \rho \varepsilon^{-m'}\delta^{1/(m-1)}C_1]E^{-\alpha}E' + q\rho E \]
\[ + \frac{1}{2}\rho\phi(u, u')|u'|^l + \frac{c_2}{2}\rho|u|^p. \] (5.9)

Now by (5.2)
\[ \rho(t)\delta^{1/(m-1)}(t) \leq k(t), \]
thus
\[ (1 - \alpha)\lambda k(t) - \rho(t)\varepsilon^{-m'}\delta^{1/(m-1)}(t)C_1 \geq [(1 - \alpha)\lambda - \varepsilon^{-m'}C_1]k(t). \]

Consequently it is enough to choose $\lambda > C_1\varepsilon^{-m'}/(1 - \alpha)$ to obtain from (5.9) that
\[ Z' \geq C\rho(\mathcal{E} + \phi(u, u')|u|^\gamma|u'|^l + |u|^p) \] (5.10)
for all $t \geq t_0$ and with $C = \min\{c_2, 1, q\}/2$. Since $k(t_0) > 0$ and $\mathcal{E}(t_0) > 0$ we may choose $\lambda$ even larger to ensure also that $Z(t_0) > 0$ and $\lambda \geq 1$. Therefore
\[ Z(t) \geq Z(t_0) > 0 \quad \text{for} \quad t \geq t_0. \]

Now we want to find an upper bound for $Z^r$, $r > 1$. Applying the following inequality
\[ (a + b)^r \leq 2^{r-1}(a^r + b^r) \quad \text{for all} \quad a, b > 0, \ r > 1, \] (5.11)
we have
\[ Z^r \leq 2^{r-1}\{\max[\lambda k, \rho]\}^r[\mathcal{E}^{(1-\alpha)r} + |u|^r|\nabla G(u, u')|^r]. \] (5.12)

Now, if we choose $\alpha$ even smaller, namely
\[ 0 < \alpha < \min\{\overline{\alpha}, \frac{1}{l}\}, \]
then $r = 1/(1 - \alpha)$ is such that
\[ 1 < r < \frac{l}{l - 1} = l'. \] (5.13)

Hence we can apply (iii) of Lemma 2 obtaining
\[ |u|^r \cdot |\nabla G(u, u')|^r \leq \phi(u, u')|u|^\gamma|u'|^l + C_3|u|^{(\gamma+\ell)/(1-\alpha)}. \]
Finally, suppose that \( \alpha \) satisfies the stronger condition \( \alpha \in (0, \alpha_0) \), where \( \alpha_0 \) is given in (5.4), then

\[
\frac{l + \gamma}{1 - l\alpha} < p,
\]

and by (3.4)

\[
|u|^{(l+\gamma)/(1-l\alpha)} \leq C_4 |u|^p, \quad C_4 = \tau_0^{(l+\gamma)/(1-l\alpha)-p}. \tag{5.14}
\]

Hence from (5.12) and (5.14) there results

\[
\mathcal{Z}^r \leq C_5 \{ \max(\lambda k, \rho) \}^r (\mathcal{E} + \phi(u, u')|u|^\gamma |u'|^l + |u|^p), \tag{5.15}
\]

where \( C_5 = 2^{r-1}(1 + C_3 C_4) \). In turn, by (5.10) and (5.15) and the fact that \( \lambda \geq 1 \) we get for \( t \geq t_0 \)

\[
\mathcal{Z}' \geq \rho C_6 \{ \max(\lambda k, \rho) \}^{-r} \mathcal{Z}^r \geq \rho C_6 \lambda^{-r} \{ \max(k, \rho) \}^{-r} \mathcal{Z}^r, \tag{5.16}
\]

where \( C_6 = C/C_5 \). Finally, integrating from \( t_0 \) to \( t \) (5.16) and since \( r > 1 \) we get

\[
\frac{[\mathcal{Z}(t_0)]^{-r+1}}{r-1} \geq \frac{[\mathcal{Z}(t)]^{-r+1}}{r-1} - \frac{[\mathcal{Z}(t_0)]^{-r+1}}{r-1} \geq C_6 \lambda^{-r} \int_{t_0}^{t} \rho(s) \{ \max(k(s), \rho(s)) \}^{-r} ds.
\]

Consequently, the non integrability hypothesis (5.3) with \( r = 1 + \theta \) forces that \( \mathcal{Z} \) cannot be defined for \( t \) large, namely \( \mathcal{Z} \) cannot be global. Clearly \( \theta = \alpha/(1-\alpha) \), thus \( 0 < \theta < \theta_0 \), where \( \theta_0 \) is given by (5.4). This completes the proof of the theorem. \( \square \)

**Remark.** In the case \( \delta(t) = (n-1)/t \) we can take

\[
\rho(t) = t^{1/(m-1)} \quad \text{and} \quad k(t) = (n-1)^{1/(m-1)}.
\]

Consequently the integral in (5.3) reduces to

\[
\int_{\delta}^{\infty} t^{-\theta/(m-1)} dt
\]
for $S$ sufficiently large. This integral is divergent if and only if $\theta \leq m - 1$, hence by (5.4) it results $\theta < \min\{\theta_0, m - 1\}$. For example, in the prototype case (1.12), with $m = l$ and $\gamma = \kappa$, we have that

$$\theta_0 = \frac{p - m - \gamma}{pm - (p - m - \gamma)},$$

(5.17)

and if $m \geq 2$ it immediately follows that $\theta_0 \leq m - 1$, thus condition (5.3) holds automatically for every $\theta < \theta_0$. 
CHAPTER 3

QUASIVARIATIONAL SYSTEMS: GLOBAL EXISTENCE

1. Main hypotheses

In this chapter we shall be concerned with the existence of global solutions of the initial value problem

\[
\begin{aligned}
[\nabla G(u, u')]' - \nabla_a G(u, u') + Q(t, u, u') &= f(t, u), \quad t > T, \\
u(T) &= u_0 \neq 0, \quad u'(T) = 0, \quad T \geq 0.
\end{aligned}
\]

(1.1)

As in the previous chapter we assume that the functions \(F, G\) and \(Q\) satisfy the standing hypotheses (A1){(A4) of Chapter 1 with \(D = \mathbb{R}^N\) or \(D = \mathbb{R}^N \setminus \{0\}\).

Moreover we will assume the following additional structural hypotheses

\[(R1)\] there exist a positive constant \(C > 0\) and an exponent \(p > 1\) such that

\[|f(t, u)| \leq C |u|^{p-1} \quad \text{for all } t \in J \quad \text{and } u \in \mathbb{R}^N;\]

\[(R2)\] \(H(u, v) \to \infty\) as \(|v| \to \infty\) uniformly for \(u\) in compact sets which do not contain the origin, where we recall that

\[H(u, v) = \langle \nabla G(u, v), v \rangle - G(u, v).\]

(1.3)

Since the technique used to prove continuation theorems of Section 2 below is based on Gronwall’s inequality, we recall for sake of completeness both the differential form and the integral form of this celebrated inequality.
Theorem 13. (Gronwall’s inequality (differential form), [8]) Let $\psi$ be a non negative, absolutely continuous function on $I = [t_0, t_1]$, $0 \leq t_0 < t_1 < \infty$, which satisfies for a.e. $t$ the differential inequality

$$\psi'(t) \leq \beta(t)\psi(t) + \alpha(t),$$

where $\beta, \alpha$ are non negative, summable functions on $I$. Then

$$\psi(t) \leq e^{\int_{t_0}^{t} \beta(s)ds} \left[ \psi(t_0) + \int_{t_0}^{t} \alpha(s)ds \right]$$

for a.e. $t \in I$.

Theorem 14. (Gronwall’s inequality (integral form), [8]) Let $\xi$ be a non negative, summable function on $I = [t_0, t_1]$, $0 \leq t_0 < t_1 < \infty$, which satisfies for a.e. $t$ the integral inequality

$$\xi(t) \leq C_1 \int_{t_0}^{t} \xi(s)ds + C_2,$$

for constants $C_1, C_2 \geq 0$. Then

$$\xi(t) \leq C_2 \left( 1 + C_1 te^{C_1 t} \right)$$

for a.e. $t \in I$.

2. Global existence

In this section we prove under suitable assumptions that all local solutions of (1.1), which exist by Theorem 10 of Chapter 1, can be continued in the entire $[T, \infty)$. As we explained in the Introduction global existence holds in general if either the damping term $Q$ or the action energy $G$ dominates over the driving force $F$. These two cases will be treated in Theorems 18 and 19 respectively. The proof use ideas of [14], [17] and [30].
**Theorem 15.** Assume that conditions \((A1)-(A4)\) of Chapter 1 hold with \(D\) as in (1.2). Suppose also that assumption (1.3) of Chapter 2 and the structure conditions \((R1)\) and \((R2)\) are satisfied with

\[
p - m \leq \kappa.
\]

Finally assume that there exists a measurable function \(\delta_1 : J \rightarrow \mathbb{R}_0^+\) such that

\[
\langle Q(t,u,v), v \rangle \geq \delta_1(t)|u|^n|v|^m
\]

for all \((t, u, v) \in J \times D \setminus \{0\} \times \mathbb{R}^N\) and

\[
\delta_1^{-1/(m-1)} \in L^1_{\text{loc}}(J).
\]

Then every local solution \(u : [T, T + \tau) \rightarrow D\) of (1.1) can be continued in the entire \([T, \infty)\).

**Proof.** Let \(u\) be a local solution of (1.1), whose existence is guaranteed by Theorem 10 of Chapter 1. Denote by \(J_T = [T, T_1)\) the maximal interval of existence of \(u\).

We claim that there exists \(U_0 > 0\) such that

\[
|u(t)| \geq U_0 \quad \text{for all} \quad t \in J_T.
\]

To prove this assume for contradiction that

\[
E(t) \equiv 0 \quad \text{for all} \quad t \in J_T.
\]

Then

\[
E'(t) \equiv 0 \quad \text{for all} \quad t \in (T, T_1)
\]

and thus from (3.2) of Chapter 2 we obtain

\[
\langle Q(t, u, u'), u' \rangle + F_i(t, u) \equiv 0 \quad \text{for all} \quad t \in (T, T_1).
\]

In turn, from (A2) and (A3) we get

\[
F_i(t, u) \equiv 0, \quad \langle Q(t, u, u'), u' \rangle \equiv 0 \quad \text{for all} \quad t \in (T, T_1).
\]
By (2.2) and (2.3) this implies that
\[ u(t) \equiv u_0 \quad \text{in} \quad J_T. \]
Indeed by (2.3) we obtain that
\[ \delta(t) > 0 \quad \text{for a.e.} \quad t \in J_T. \]
By (2.2) and since \( u(T) = u_0 \neq 0 \) it follows
\[ |u|^\kappa |u'|^m \equiv 0 \quad (2.7) \]
as long as \( u(t) \neq 0 \) (and thus at least for \( t \) near \( T \)). But (2.7) clearly implies
\[ u'(t) \equiv 0 \]
and so necessarily \( u(t) = u_0 \) in \( J_T \). This in turn contradicts the fact that \( J_T \) is the maximal interval of existence of \( u \), as one can consider the initial value problem
\[ [\nabla G(u, u')]' - \nabla_u G(u, u') + Q(t, u, u') = f(t, u), \quad t > T, \]
\[ u(T_1) = u_0 \neq 0, \quad u'(T_1) = 0, \]
and continue \( u \) by Theorem 11 of Chapter 1.

Hence (2.5) is false and so there exists \( T_2 \in [T, T_1) \) such that \( E(T_2) \neq 0 \). But since
\[ E(T) = H(u_0, 0) - F(T, u_0) = -F(T, u_0) \leq 0 \]
by (A2) and from the fact that \( E'(t) \leq 0 \) it follows that \( E(T_2) < 0 \) and thus we may apply Proposition 3 in Chapter 2 to obtain (2.4).

Next we claim that \( T_1 = \infty \). Thus assume by contradiction that \( T_1 < \infty \).

Consider the auxiliary function
\[ \psi(t) = H(u, u') + |u|^p, \quad (2.8) \]
where \( H \) is the function given in (1.3). We observe that by (2.2)
\[ \psi(t) \geq |u|^p \quad \text{for all} \quad t \in J_T, \quad (2.9) \]
moreover $\psi$ can be written as follows

$$\psi(t) = F(t, u) + |u|^p + E(t).$$

Since $|u(t)| > 0$ on $J_T$ by (2.4), we can calculate the derivative of $\psi$ by using Proposition 2, namely from (2.3) we obtain

$$\psi'(t) = \langle f(t, u), u' \rangle + p|u|^{p-2}\langle u, u' \rangle - \langle Q(t, u, u'), u' \rangle, \quad t \in (T, T_1). \quad (2.10)$$

We claim that $\psi$ is bounded on $J_T$ independently of $T_1$. First observe that from (R1) it follows that, for $t \in (T, T_1),$

$$|f(t, u)||u'| + p|u|^{p-1}|u'| \leq (C + p)|u|^{p-1}|u'|$$

$$= [\delta_1(t)]^{1/m}|u|^{\sigma/m}|u'| (C + p)|u|^{(p-m-\kappa)/m} [\delta_1(t)]^{-1/m}|u|^{p/m} \quad (2.11)$$

$$\leq \delta_1(t)|u|^{\sigma}|u'|^m + (C + p)^m|u|^{(p-m-\kappa)/(m-1)} \delta_1(t)^{-1/(m-1)}|u|^p,$$

where we have used Young’s inequality with exponents $m$ and $m' = m/(m - 1)$. Consequently from (2.1) and (2.4) we derive

$$|u(t)|^{(p-m-\kappa)/(m-1)} \leq U_0^{(p-m-\kappa)/(m-1)} \quad \text{for all} \quad t \in (T, T_1),$$

thus inequality (2.11) becomes

$$|f(t, u)||u'| + p|u|^{p-1}|u'| \leq \delta_1(t)|u|^\sigma|u'|^m + C_2[\delta_1(t)]^{-1/(m-1)}|u|^p, \quad (2.12)$$

with $C_2 = (C + p)^m U_0^{(p-m-\kappa)/(m-1)}$. Hence by (2.10), (2.2) and (2.9) the inequality above implies that for $t \in (T, T_1)$

$$\psi'(t) \leq \delta_1(t)|u|^\sigma|u'|^m + C_2[\delta_1(t)]^{-1/(m-1)}|u|^p - \langle Q(t, u, u'), u' \rangle$$

$$\leq C_2[\delta_1(t)]^{-1/(m-1)}|u|^p \leq C_2[\delta_1(t)]^{-1/(m-1)}\psi(t). \quad (2.13)$$

Finally, by (2.3), we can apply Gronwall’s inequality with

$$\beta(t) = C_2[\delta_1(t)]^{-1/(m-1)}$$

to obtain that

$$\psi(t) \leq \psi(s)e^{\int_s^t \beta(r)dr} \quad \text{for all} \quad T < s < t < T_1.$$
Hence we conclude that $\psi$ is bounded in $J_T$ independently of $T_1$. This implies that both $H(u, u')$ and $|u|$ are bounded on $J_T$. Furthermore by (R2) we derive that also $|u'|$ is bounded on $J_T$. Hence, by Cauchy’s Theorem, there exists $u_1 \in D$ with $u_1 \neq 0$, thanks to (2.4), such that
\[
\lim_{t \to T_1^-} u(t) = u_1.
\] (2.14)
Now, from the equation in (1.1) we derive that for $T < s \leq t < T_1$
\[
\nabla G(u, u') - \int_s^t \nabla_a G(u, u')d\tau + \int_s^t Q(\tau, u, u')d\tau - \int_s^t f(\tau, u)d\tau = \nabla G(u(s), u'(s)),
\]
and note that $\nabla_a G(u, u')$, $Q(t, u, u')$ and $f(t, u)$ are bounded on $[s, T_1]$ by continuity and by the boundedness of $t$, $|u|$ and $|u'|$. Hence, there exists $v_1 \in \mathbb{R}^N$ such that
\[
\lim_{t \to T_1^-} \nabla G(u, u') = v_1.
\] (2.15)
Therefore, since $u'$ is bounded and $\nabla G(u_1, \cdot)$ is one-to-one by (A1), we obtain
\[
\lim_{t \to T_1^-} u'(t) = v_1 = (\nabla G(u_1, \cdot))^{-1}(v_1).
\]
Now, if we consider the initial value problem
\[
[\nabla G(u, u')]' - \nabla_a G(u, u') + Q(t, u, u') = f(t, u),
\]
\[
u(T_1) = u_1 \neq 0, \quad u'(T_1) = v_1,
\] (2.16)
we obtain that $u$ can be continued to the right beyond $T_1$, thanks to Theorem 11. Hence $J_T$ is not the maximal interval of existence of the solution. \hfill \Box

Next we consider the case when the action energy dominates over the driving force $F$.

**Theorem 16.** Assume that conditions (A1)–(A4) of Chapter 1 hold with $D$ as in (1.2) and that also (1.3) of Chapter 2 holds. Suppose that condition (R1) is verified. Finally assume that, for every $V > 0$ there exist exponents $\gamma \in \mathbb{R}$ and $l > 1$, with
\[
p - l \leq \gamma,
\] (2.17)
and a constant $\Theta > 0$ such that assumption (R2) is strengthened as follows

$$H(u, v) \geq \Theta|u|^\gamma|v|^l$$

for all $(u, v) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|u|, |v| \geq V$.

If $F(T, u_0) > 0$, then every local solution $u : [T, T + \tau) \to \mathbb{R}^N$ of problem (1.1) can be continued in the entire $[T, \infty)$.

**Proof.** Thanks to (1.3) of Chapter 2 and the fact that $F(T, u_0) > 0$, we can apply Proposition 3 of Chapter 2 obtaining that

$$|u(t)| \geq U \text{ in } J_T$$

for some $U > 0$ and where, as in the previous theorem, $J_T = [T, T_1)$ is the maximal interval of existence of $u$. Let $\gamma, l, \Theta$ be the corresponding numbers in (2.18). Consider the same auxiliary function defined in (2.8) and observe that not only (2.9) holds but also

$$\psi(t) \geq H(u, u') \text{ for all } t \in J_T.$$ (2.19)

Now by (2.10), (R1) and (A3) of Chapter 1, we get that

$$\psi'(t) = \langle f(t, u), u' \rangle + p|u|^{p-2}\langle u, u' \rangle - \langle Q(t, u, u'), u' \rangle \leq (C + p)|u|^{p-1}|u'|,$$ (2.20)

where we have used Cauchy Schwarz inequality.

Let $J_1 = \{t \in J_T : |u'(t)| \leq |u(t)|\}$ and $J_2 = \{t \in J_T : |u'(t)| \geq |u(t)|\}$. If $t \in J_1$, from (2.20) and (2.9), we immediately deduce that

$$\psi'(t) \leq (C + p)\psi(t).$$

In the other case when $t \in J_2$, we check that

$$|u|^{p-1}|u'| = |u|^{l-1}|u|^{p-l}|u'| \leq |u'|^{l-1}|u|^{p-l}|u'|$$

$$= |u|^{p-l}|u'|^l = |u|^{p-\gamma}|u|^\gamma|u'|^l \leq U^{p-\gamma}|u|^\gamma|u'|^l,$$ (2.21)
where we have used (2.17). Furthermore, when \( t \in J_2 \), we have that \( |u'| \geq |u| \geq U \), hence by (2.18) and using (2.21) and (2.19) we get
\[
|u|^{p-1}|u'| \leq U^{p-l-\gamma}H(u, u') \leq U^{p-l-\gamma}\psi(t), \quad t \in J_2.
\]
Thus, in both cases, we get, for all \( t \in J_T \),
\[
\psi'(t) \leq \beta \psi(t), \quad \beta = \min \left\{ (C + p), \frac{(C + p)U^{p-l-\gamma}}{\Theta} \right\}.
\]
Thus, by Gronwall’s inequality for all \( t \in J_T \) we have
\[
\psi(t) \leq \psi(s)e^{(t-s)\beta} \quad \text{for all} \quad T < s \leq t < T_1.
\]
Now the proof proceed as in Theorem 15 since assumption (2.18) forces the validity of (R2). \( \square \)
CHAPTER 4

APPLICATIONS TO ELLIPTIC SYSTEMS

1. Global non existence of radial solutions

In this chapter we apply the results of Chapters 1–3 to the study of existence and non existence of radial solutions of elliptic systems of the general form

\[
\text{div}(g(u)A(|\nabla u|)\nabla u) - \nabla_u g(u)A(|\nabla u|) = f(|x|, u), \quad x \in \mathbb{R}^n, \quad (1.1)
\]

where

\[
A(s) = \int_0^s \sigma A(\sigma) d\sigma, \quad s > 0,
\]

and the functions \(A, f\) and \(g\) satisfy

(C1) \(A : \mathbb{R}^+ \to \mathbb{R}^+\) is continuous, \(s \mapsto sA(s)\) is strictly increasing in \(\mathbb{R}^+\) and

\[
\lim_{s \to 0^+} sA(s) = 0;
\]

(C2) \(g : D \to \mathbb{R}^+_0\) is of class \(C^1\), where \(D = \mathbb{R}^N\) or \(D = \mathbb{R}^N \setminus \{0\}\);

(C3) there exists a non negative function \(F \in C^1(\mathbb{R}^+_0 \times \mathbb{R}^N; \mathbb{R})\), with \(F(r, 0) = 0\) for all \(r \geq 0\), such that

\[
\nabla_u F(r, u) = f(r, u) \quad \text{and} \quad F_r(r, u) \geq 0
\]

for all \((r, u) \in \mathbb{R}^+_0 \times \mathbb{R}^N\);

(C4) there exist two functions \(\tilde{F} \in C(\mathbb{R}^N; \mathbb{R}^+_0)\) and \(\psi \in L^1(\mathbb{R}^+; \mathbb{R}^+_0)\) such that

\[
\tilde{F}(0) = 0, \quad \tilde{F}(u) > 0 \quad \text{if} \ u \neq 0
\]

and

\[
0 \leq F_r(r, u) \leq \psi(r)\tilde{F}(u) \quad \text{for all} \quad (r, u) \in \mathbb{R}^+ \times \mathbb{R}^N.
\]
Note that $A \in C^1(\mathbb{R}_0^+; \mathbb{R})$ by (C1).

An important special case of (1.1) is given when the operator $A$ is the $m$-Laplacian

$$A(s) = s^{m-2}, \quad s > 0, \quad m > 1.$$ 

In this case (1.1) becomes

$$\text{div}(g(u)|\nabla u|^{m-2}\nabla u) - \frac{1}{m} \nabla_u g(u)|\nabla u|^m = f(|x|, u) \quad \text{in} \quad \mathbb{R}^n.$$ 

Other examples of operators $A$ that we analyze here are the mean curvature operator

$$A(s) = \frac{1}{\sqrt{1 + s^2}}, \quad s \geq 0,$$

and the generalized mean curvature operator

$$A(s) = m(1 + s^2)^{m/2 - 1}, \quad s \geq 0, \quad 1 < m \leq 2.$$ 

The canonical model for $g$ is given by the function

$$g(u) = |u|^\gamma, \quad \gamma \in \mathbb{R}.$$ 

**Theorem 17.** Assume that conditions (C1)-(C3) are satisfied and that there exist a function $\varphi \in C(\mathbb{R}_0^+; \mathbb{R})$, an exponent $m > 1$, and three positive constants $d_1, d_2 > 1$ and $M_1$ such that

$$\varphi(s) s^m \leq d_1 A(s) \leq s^2 A(s) \leq d_2 \varphi(s) s^m, \quad (1.2)$$

$$0 < \varphi(s) \leq M_1 \quad (1.3)$$

for all $s > 0$. Suppose also that for every $U > 0$ there exist a function $\psi \in C(D; \mathbb{R})$, an exponent $\gamma \in \mathbb{R}$, and three positive constants $d_3, d_4, M_2$ such that

$$d_3 g(u) + \langle u, \nabla_u g(u) \rangle \geq \psi(u)|u|^\gamma, \quad (1.4)$$

$$g(u) \leq d_4 \psi(u)|u|^\gamma, \quad (1.5)$$

$$0 < \psi(u) \leq M_2 \quad (1.6)$$

for all $u \in \mathbb{R}^N$ with $|u| \geq U$. 

Finally assume that, for every $U > 0$, there exist an exponent $p > 1$ and three constants $c_1, c_2, q > 0$ such that
\[
c_1 F(r, u) \leq c_2 |u|^p \leq \langle f(r, u), u \rangle - q F(r, u),
\] (1.7)
whenever $r \geq 0$ and for all $u \in \mathbb{R}^N$ with $|u| \geq U$. If, for every $U > 0$,
\[
1 < m < p, \quad -m - p(m - 1) < \gamma < p - m,
\] (1.8)
\[
0 < d_3 \leq d_1(q + 1) - q,
\] (1.9)
then problem (1.1) does not admit non trivial entire radial solutions.

**Proof.** Radial solutions $u = u(r)$, $r = |x|$, of (1.1) satisfy the initial value problem
\[
\left[ g(u) A(|u'|) u' \right]' + \frac{n-1}{r} g(u) A(|u'|) u' - g'(u) A(|u'|) = f(r, u), \quad r > 0,
\] (1.10)
\[
u(0) = u_0, \quad u'(0) = 0.
\]
Problem (1.10) is a special case of (1.1) of Chapter 2 with
\[
G(u, v) = g(u) A(|v|), \quad Q(r, u, v) = \frac{n-1}{r} g(u) A(|v|) v.
\]
Hence to prove non existence of entire radial solutions, we only need to verify that the hypotheses of Theorem 12 of Chapter 2 are satisfied.

Conditions (A1)–(A4) of Chapter 1 follow immediately from hypotheses (C1)–(C3). Assumption (S1) of Chapter 2 is given by (C4) and (1.7). Also (5.1) of Chapter 2 holds with $l = m$ and $\kappa = \gamma$ thanks to (1.8). To verify (S2) of Chapter 2 fix $U > 0$. By (1.2) and (1.4) we get
\[
(q + 1)g(u) A(s) s^2 - q g(u) A(s) + \langle u, \nabla_u g(u) \rangle A(s)
\geq A(s) \left\{ \left[ (q + 1) d_1 - q g(u) \right] + \langle u, \nabla_u g(u) \rangle \right\}
\geq A(s) \left\{ d_3 g(u) + \langle u, \nabla_u g(u) \rangle \right\} \geq \varphi(s) \psi(s) s^m |u|^\gamma
\]
for all $s > 0$ and $|u| \geq U$, where we have used the fact that $(q + 1)d_1 - q \geq d_3$. Thus (1.5) in (S2) of Chapter 2 is verified with $l = m$ and

$$
\phi(u, v) = \psi(u)\varphi(|v|), \quad c_4 = M_1M_2.
$$

By (1.2) and (1.5)

$$
0 \leq g(u)sA(s) \leq d_2d_4\varphi(s)\psi(u)|u|^\gamma s^{m-1}
$$

(1.11)

for all $s > 0$ and $|u| \geq U$, which proves (1.6) of (S2) with $c_3 = d_2d_4$.

To verify (S3) of Chapter 2 note that

$$
0 \leq \frac{n-1}{r}g(u)sA(s) = \left(\frac{n-1}{r}g(u)sA(s)\right)^{1/m} s^{-1/m'} \left(\frac{n-1}{r}g(u)s^2A(s)\right)^{1/m'}
$$

$$
\leq \left(\frac{n-1}{r}d_2d_4M_1M_2\right)^{1/m} |u|^\gamma |s|^{m-1} \left(\frac{n-1}{r}g(u)s^2A(s)\right)^{1/m'}
$$

for all $s > 0$ and $|u| \geq U$, and where we have used (1.11), (1.6) and (1.2). This implies the validity of (S3) of Chapter 2 with

$$
\delta(r) = \frac{n-1}{r}d_2d_4M_1M_2, \quad \kappa = \gamma.
$$

Finally to verify (5.2) and (5.3) of Chapter 2 take

$$
\rho(r) = r^{1/(m-1)}, \quad k(r) = (n-1)^{1/(m-1)},
$$

and let $R > 0$ so large that

$$
r^{1/(m-1)} \geq (n-1)^{1/(m-1)} \quad \text{for all} \quad r \geq R.
$$

Then

$$
\int_R^\infty \rho(r) [\max\{k(r), \rho(r)\}]^{-(1+\theta)} dr = \int_R^\infty r^{-\theta/(m-1)} dr = \infty,
$$

where we have used the fact that $\theta \in (0, \theta_0)$, with

$$
\theta_0 = \min\left\{ m - 1, \frac{p - m - \gamma}{pm - (p - m - \gamma)} \right\}.
$$
Corollary 8. Assume that

\[ 1 < m < p, \quad -m - p(m - 1) < \gamma < p - m. \]  

Then the elliptic system

\[ \text{div}(|u|^{\gamma}|\nabla u|^{m-2}\nabla u) - \frac{\gamma}{m}|u|^{\gamma-2}u|\nabla u|^m = |u|^{p-2}u, \]

does not admit nontrivial entire radial solutions, but it admits a one-parameter family of solutions \( u : B(0, R) \to \mathbb{R}^N \setminus \{0\} \) such that

\[ \lim_{|x| \to R^-} |u(|x|)| = \infty. \]

Proof. We claim that all the hypotheses of Theorem 17 are satisfied. Indeed assumptions (1.2)–(1.3) hold with

\[ A(s) = s^{m-2}, \quad \varphi(s) = \frac{1}{m}, \quad d_1 = d_2 = m. \]

Also (1.4), (1.5) and (1.6) hold with for any \( d_3 > -\gamma \) with

\[ g(u) = |u|^{\gamma}, \quad \psi(u) = d_3 + \gamma, \quad d_4 = \frac{1}{d_3 + \gamma}. \]

Furthermore, since \( f(r, u) = |u|^{p-2}u \), then (C4) holds with \( \psi(r) \equiv 1 \) and \( \tilde{F}(u) = F(u) = |u|^p/p \), while (1.7) is satisfied for any \( q \in (0, p) \) with \( c_1 = p - q \) and \( c_2 = (p - q)/p \). If we now fix \( d_3 \) such that

\[ \max\{0, -\gamma\} < d_3 < -p(m - 1) - m, \]

we have that

\[ \frac{d_3 - m}{m - 1} < p, \]

and so we can choose

\[ q \in \left[ \frac{d_3 - m}{m - 1}, p \right), \]

so that \( d_3 > -\gamma, q < p \) and (1.9) holds. This proves the first part of the theorem.
To prove the second part note that system (1.13) admits a one parameter family of radial solutions \( u = u(r), \ r = |x| \). Indeed for any \( u_0 \neq 0 \) every radial solution of (1.13) satisfies the initial value problem

\[
\left( |u|^{\gamma} |u'|^{m-2} u' \right)' - \frac{\gamma}{m} |u|^{\gamma-2} u |u'|^m + \frac{n-1}{r} |u|^{\gamma} |u'|^{m-2} u' = |u|^{p-2} u, \quad r > 0, \]

\[
u(0) = u_0 \neq 0, \quad u'(0) = 0.
\]

(1.15)

Hence, thanks to Theorem 10 of Chapter 1 applied with

\[
\delta(r) = \frac{n-1}{r}, \quad \phi(u, v) = |u|^\gamma |v|^{m-1},
\]

(1.15) admits a local solution \( u : I \to \mathbb{R}^N \setminus \{0\} \), where \( I = [0, R), \ R > 0 \). Without loss of generality we may assume that \( I \) is the maximal interval of existence of \( u \). Note that necessarily \( R < \infty \) by Theorem 17.

We first claim that \( u \) is unbounded on \( I \). For contradiction assume that \( u \) is bounded. Since along the solution \( u \) the expression of the energy is

\[
E(r) = \frac{m-1}{m} |u|^{\gamma} |u'|^m - \frac{|u|^p}{p} \leq E(0) < 0, \quad r \in I,
\]

(1.16)

where we have used the fact the \( E \) is non increasing by (3.2) of Chapter 2 and by (A2) and (A3) of Chapter 1, it follows that

\[
|u'|^m < \frac{m}{p(m-1)} |u|^{p-\gamma}, \quad r \in I.
\]

(1.17)

Hence \( u' \) is bounded on \( I \) and by Cauchy’s Theorem it follows that

\[
\lim_{r \to R^-} u(r) = u_1.
\]

In particular we can assert that \( u_1 \in \mathbb{R}^N \setminus \{0\} \), indeed the fact that the initial energy is negative guarantees that \( u_1 \) cannot be zero thanks to Proposition 3 of Chapter 2.

Now, note the system (1.15) can be written in the equivalent form

\[
\left( r^{n-1} |u|^{\gamma} |u'|^{m-2} u' \right)' = \frac{\gamma}{m} r^{n-1} |u|^{\gamma-2} u |u'|^m + r^{n-1} |u|^{p-2} u, \quad r \in (0, R),
\]

(1.18)
consequently, since $|u| \geq U > 0$ and by the fact that $u$ is bounded by contradiction, we have that $|u|$ belongs to a compact set which do not contain the origin. Furthermore, also $|u'|$ is bounded thanks to (1.17). Hence $(r^{n-1}|u|^\gamma |u'|^{m-2}u')'$ is bounded in $I$ also by (1.18), and so

$$r^{n-1}|u|^\gamma |u'|^{m-2}u' \in \text{Lip}(I).$$

Consequently, since $u_1 \neq 0$,

$$\lim_{r \to R^-} u'(t) = v_1$$

for some $v_1 \in \mathbb{R}^N$.

Now we can consider the initial value problem

$$\begin{align*}
(|u|^\gamma |u'|^{m-2}u')' - \frac{\gamma}{m} |u|^\gamma - 1 u|u'|^m + \frac{n-1}{r} |u|^\gamma |u'|^{m-2}u' &= |u|^{p-2}u, \quad r > R, \\
u(R) &= u_1, \quad u'(R) = v_1, \quad R > 0.
\end{align*}$$

(1.19)

Theorem 11 of Chapter 1 can be applied with $T_1 = R$, so that (1.19) admits a local solution defined on a neighborhood of $R$, namely $u$ can be continued to the right beyond $R$. This contradicts the maximality of $R$ and proves the claim.

In conclusion $u$ cannot be bounded, that is

$$\limsup_{r \to R^-} |u(r)| = \infty.$$  

(1.20)

Now, to prove (1.14), we use the same argument in [14, 0Lemma 5.3]. Thus assume for contradiction that

$$\liminf_{r \to R^-} |u(r)| = \ell_1 \in (0, \infty).$$  

(1.21)

By (1.20) and (1.21), we can find two sequences $(a_n)_n$ and $(b_n)_n$ approaching $R$ from below, such that $a_n < b_n < a_{n+1}$ and

$$\lim_{n \to \infty} |u(a_n)| = \infty, \quad \lim_{n \to \infty} |u(b_n)| = \ell_1.$$  

(1.22)

Let $v = |u|$, by (1.17) and Schwartz inequality, we get

$$v' = \left( \frac{u}{|u|} u' \right) \leq \left[ \frac{m}{p(m-1)} \right]^{1/m} v^{(p-\gamma)/m}.$$
and by integration from $b_n$ to $a_{n+1}$ it results
\[
\frac{[v(a_{n+1})]^{1-(p-\gamma)/m}}{1-(p-\gamma)/m} - \frac{[v(b_n)]^{1-(p-\gamma)/m}}{1-(p-\gamma)/m} \leq \left[\frac{m}{p(m-1)}\right]^{1/m} (a_{n+1} - b_n).
\]
(1.23)

Now, letting $n \to \infty$ in (1.23) and using (1.22) and the fact that $(p-\gamma)/m > 1$ and $R < \infty$, we obtain that
\[
0 \leq -\ell_1^{1-(p-\gamma)/m} \leq 0,
\]
which is a contradiction by virtue of (1.21). Hence, since by Proposition 3 of Chapter 2 $\ell_1 \geq U > 0$, then $\ell_1 = \infty$ and this completes the proof of (1.14).

\begin{corollary}
Assume that
\[
1 < m \leq 2, \quad 1 < m < p, \quad -m - p(m-1) < \gamma < p - m,
\]
then the elliptic system
\[
\text{div}(|u|^\gamma(1 + |\nabla u|^2)^{m/2-1}\nabla u) - \frac{\gamma}{m}|u|^{\gamma-2}u[(1 + |\nabla u|^2)^{m/2} - 1] = |u|^{p-2}u,
\]
(1.25)
does not admit non trivial entire radial solutions.
\end{corollary}

\begin{proof}
First note that any radial solution of (1.25) is a solution of the ordinary differential system
\[
(|u|^\gamma(1 + |u'|^2)^{m/2-1}u')' - \frac{\gamma}{m}|u|^{\gamma-2}u[(1 + |u'|^2)^{m/2} - 1] + \frac{n-1}{r} |u|^\gamma(1 + |u'|^2)^{m/2-1}u' = |u|^{p-2}u,
\]
(1.26)
which is the special case of (1.1) when
\[
A(s) = (1 + s^2)^{m/2-1}, \quad g(u) = |u|^\gamma, \quad f(r, u) = |u|^{p-2}u.
\]
Hence (1.4) and (1.5) hold for any $d_3 > -\gamma$ with
\[
\psi(u) = d_3 + \gamma \quad \text{and} \quad d_4 = 1/(d_3 + \gamma).
\]
As in the previous corollary (1.7) is trivially satisfied for any $q \in (0, p)$ with $c_1 = p - q$
and $c_2 = (p - q)/p$. 

To verify (1.2) observe that the second inequality in (1.2) is immediately verified with $d_1 = m$, indeed

$$\mathcal{A}(s) = \frac{1}{m}[(1 + s^2)^{m/2} - 1], \quad s^2 \mathcal{A}(s) = (1 + s^2)^{m/2-1}s^2,$$

so that inequality

$$m\mathcal{A}(s) \leq s^2 \mathcal{A}(s)$$

becomes

$$(1 + s^2)^{m/2} - 1 \leq (1 + s^2)^{m/2-1}s^2,$$

namely, by multiplying for $(1 + s^2)^{-m/2+1}$,

$$1 \leq (1 + s^2)^{-m/2+1}$$

which is verified for any $s \geq 0$ being $-m/2 + 1 \geq 0$ since $m \leq 2$ by (1.24). To find the function $\varphi$ in (1.2) such that all the inequalities in (1.2) hold, note that

$$\mathcal{A}(s) \sim s^m \quad \text{as} \quad s \to \infty,$$

while

$$\mathcal{A}(s) \sim \frac{m}{2}s^2 \quad \text{as} \quad s \to 0.$$

On the other hand

$$s^2 \mathcal{A}(s) \sim ms^m \quad \text{as} \quad s \to \infty,$$

$$s^2 \mathcal{A}(s) \sim ms^2 \quad \text{as} \quad s \to 0.$$

Consequently, there exist $s_0 \in (0, 1)$ and $s_1 > 1$ such that

$$\mathcal{A}(s) \geq \begin{cases} 
\frac{m}{4}s^2, & 0 < s \leq s_0 < 1, \\
\frac{m}{4}s^m, & s \geq s_1 > 1,
\end{cases}$$

(1.28)
and

\[
s^2 A(s) \leq \begin{cases} 
2ms^2, & 0 < s \leq s_0 < 1, \\
2ms^m, & s \geq s_1 > 1.
\end{cases}
\] (1.29)

Now, let

\[
\varphi_1(s) = \begin{cases} 
\frac{m}{4}s^{2-m}, & 0 < s \leq s_0 < 0, \\
\frac{m(1 - s_0^{2-m})}{4(s_1 - s_0)}(s - s_0) + \frac{m}{4}s_0^{2-m}, & s_0 < s < s_1, \\
\frac{m}{4}, & s \geq s_1 > 1.
\end{cases}
\] (1.30)

Then, if we choose \(\varepsilon\) so small such that

\[\mathcal{A}(s_0) \geq \varepsilon \varphi_1(s_1)s_1^m,\] (1.31)

the function \(\varphi(s) = \varepsilon \varphi_1(s)\) satisfies (1.2), provided that \(d_2\) is taken so large that

\[
\frac{1}{m}s_1^2 A(s_1) \leq d_2 \varepsilon \varphi_1(s_0)s_0^m \quad \text{and} \quad d_2 \geq \frac{8}{m}.
\] (1.32)

Indeed, if \(s \geq s_1\), from (1.28), (1.27) and (1.29) we immediately deduce that

\[
\varphi(s)s^m = \varepsilon \frac{m}{4}s^m \leq \varepsilon A(s) \leq \frac{\varepsilon}{m}s^2 A(s) \leq \frac{\varepsilon}{m}2ms^m \leq d_2 \varepsilon \frac{m}{4}s^m = d_2 \varphi(s)s^m,
\]

where in the last inequality we have used the fact that \(d_2 \geq 8/m\).

If \(s \leq s_0\), again from (1.28), (1.27) and (1.29) we obtain

\[
\varphi(s)s^m = \varepsilon \frac{m}{4}s^2 \leq \varepsilon A(s) \leq \frac{\varepsilon}{m}s^2 A(s) \leq \frac{\varepsilon}{m}2ms^2 \leq d_2 \varepsilon \frac{m}{4}s^{2-m}s^m = d_2 \varphi(s)s^m,
\]

where, as above, in the last inequality we have used the fact that \(d_2 \geq 8/m\).
Finally, if \( s_0 \leq s \leq s_1 \), then, using the fact that all the functions involved in (1.2) are non decreasing, we get
\[
\varphi(s)s^m \leq \varphi_1(s_1)s_1^m \leq A(s_0) \leq A(s) \leq \frac{1}{m}s^2A(s) \leq \frac{1}{m_1}s_1^2A(s_1) \leq d_2\varphi_1(s_0)s_0^m \leq d_2\varphi_1(s)s^m = d_2\varphi(s)s^m,
\]
where we have used (1.31) and (1.32). Hence (1.2) is proved.

The next corollary deals with the mean curvature operator
\[
A(s) = \frac{1}{\sqrt{1 + s^2}}, \quad s \geq 0.
\]
This operator does not satisfy condition (1.2), thus must be treated separately.

**Corollary 10.** Assume that \( p > 1 \). Then the system
\[
\text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = |u|^{p-2}u \quad \text{in} \quad \mathbb{R}^n
\]
does not admit non trivial entire radial solution \( u : \mathbb{R}^n \to \mathbb{R}^N \).

**Proof.** First note that any radial solution \( u \) of (1.34) satisfies the initial value problem
\[
\left( \frac{u'}{\sqrt{1 + |u'|^2}} \right)' + \frac{n-1}{r} \frac{u'}{\sqrt{1 + |u'|^2}} = |u|^{p-2}u, \quad r > 0,
\]
\[
u(0) = u_0 \neq 0, \quad u'(0) = 0.
\]

Consider the auxiliary function \( Z \) defined in (5.5) of Chapter 2 with \( k(r) = 1 \) and \( \rho(r) = 1 \), namely
\[
Z(r) = \lambda \mathcal{E}^{1-\alpha}(r) + \left( \frac{u'}{\sqrt{1 + |u'|^2}}, u \right),
\]
where $\lambda > 0$ will be fixed later and $0 < \alpha < \bar{\alpha} = 1 - \frac{1}{p}$. Hence we get

$$Z'(r) = \lambda(1 - \alpha)\mathcal{E}^{-\alpha}(r)\mathcal{E}'(r) + \langle \left( \frac{u'}{\sqrt{1 + |u'|^2}} \right)', u \rangle + \langle \frac{u'}{\sqrt{1 + |u'|^2}}, u' \rangle$$

$$\geq \langle \left( \frac{u'}{\sqrt{1 + |u'|^2}} \right)', u \rangle + \langle \frac{u'}{\sqrt{1 + |u'|^2}}, u' \rangle + \mathcal{E}(r) + H(u') - \frac{|u|^p}{p},$$

where we have used the fact that $\mathcal{E}$ is positive and increasing and that here

$$\mathcal{E}(r) = -H(u') + \frac{|u(r)|^p}{p}.$$

Now using (1.35)$_1$ we obtain

$$Z'(r) \geq -\frac{n - 1}{r} \langle \frac{u'}{\sqrt{1 + |u'|^2}}, u \rangle + \langle |u|^{p-2}u, u \rangle - \frac{|u|^p}{p}$$

$$+ \langle \frac{u'}{\sqrt{1 + |u'|^2}}, u' \rangle + H(u') + \mathcal{E}(r) \geq (1 - \frac{1}{p})|u|^p - \frac{n - 1}{r} \frac{|u'||u|}{\sqrt{1 + |u'|^2}} + \mathcal{E}(r).$$

Now from Proposition 3 of Chapter 2, which continue to hold in this setting,

$$|u(r)| \geq U > 0 \quad \text{for} \quad r \geq 0$$

hence

$$|u(r)| = |u(r)|^{1-p}|u(r)|^p \leq U^{1-p}|u(r)|^p,$$

and since $|u'|/\sqrt{1 + |u'|^2} \leq 1$, we have

$$Z'(r) \geq [1 - \frac{1}{p} - \frac{n - 1}{r} U^{1-p}]|u|^p + \mathcal{E}(r) \geq C||u|^p + \mathcal{E}(r)|, \quad C > 0,$$

for $r$ sufficiently large. On the other hand, as in (5.12) of Chapter 2

$$Z^a \leq C_1 \left[ a^{\alpha(1-\alpha)} + |u|^a \left( \frac{|u'|}{\sqrt{1 + |u'|^2}} \right)^a \right] \leq C_1 \left[ a^{\alpha(1-\alpha)} + |u|^a \right], \quad C_1 > 0,$$
with $1 < a < \bar{a} = p$. Consequently, as in the proof of Theorem 12 of Chapter 2, we can choose $a = 1/(1 - \alpha)$ in order to obtain that

$$\mathcal{Z}^u \leq C_1[\mathcal{E} + U^{a-p}|u|^p] \leq C_2[\mathcal{E} + |u|^p], \quad C_2 > 0.$$ 

Hence, for $r$ sufficiently large,

$$\mathcal{Z}'(r) \geq C_3 \mathcal{Z}^u(r), \quad C_3 > 0.$$ 

We can now argue as in the last part of the proof of Theorem 12 of Chapter 2 to obtain the conclusion.

\[\square\]

2. Global existence of radial solutions

In this section we apply the results of Chapters 1 and 3, to obtain global existence of radial solutions of problem (1.1)

**Theorem 18.** Assume that (C1)–(C4) hold and that there exist a positive constant $C > 0$ and an exponent $p > 1$ such that

$$|f(r, u)| \leq C |u|^{p-1} \quad \text{for all } r \in \mathbb{R}_0^+ \quad \text{and} \quad u \in \mathbb{R}^N. \quad (2.1)$$

Suppose that there exist an exponent $m > 1$ and a positive constant $b_1$ such that

$$s^2 A(s) \geq b_1 s^m \quad \text{for all} \quad s > 0, \quad (2.2)$$

and

$$\lim_{s \to \infty} [s^2 A(s) - A(s)] = \infty. \quad (2.3)$$

Finally assume that for every $U > 0$ there exist an exponent $\gamma \in \mathbb{R}$ and a positive constant $b_2$ such that

$$g(u) \geq b_2 |u|^\gamma \quad (2.4)$$

for all $u \in \mathbb{R}^N$ with $|u| \geq U$.

If for every $U > 0$

$$\gamma \geq p - m, \quad (2.5)$$

then (1.1) admits a one parameter family of non trivial entire radial solutions.
PROOF. As noted at the beginning of the proof of Theorem 17, every radial solution of (1.1) satisfies the initial value problem

\[
\left[ g(u)A(|u'|)u' \right]' + \frac{n-1}{r} g(u)A(|u'|)u' - g'(u)A(|u'|) = f(r, u), \quad r > 0,
\]

\[ u(0) = u_0 \neq 0, \quad u'(0) = 0, \]

which is a special case of (1.1) of Chapter 3. Conditions (A1)–(A4) of Chapter 1 follow immediately from hypotheses (C1)–(C3). In particular to satisfy (A4) we take

\[ \delta(r) = \frac{n-1}{r} \quad \text{and} \quad \phi(u, v) = g(u)A(|v|)|v|. \]

By Theorem 10 of Chapter 1 the initial value problem (2.6) admits a local solution \( u : [0, R) \rightarrow \mathbb{R}^N \setminus \{0\}, R > 0 \). Without loss of generality we may assume that \([0, R)\) is the maximal interval of existence of the solution.

To prove that \( R = \infty \) we need to verify the hypotheses of Theorem 15 of Chapter 3. Assumption (R2) follows immediately from (2.3) and (2.4), indeed let \( K \subset \mathbb{R}^N \setminus \{0\} \) be a compact set. Then

\[
H(u, v) = g(u)\{v^2A(|v|) - A(|v|)\}
\geq b_2|u|^\gamma\{v^2A(|v|) - A(|v|)\}
\geq \min_{u \in K} b_2|u|^\gamma\{v^2A(|v|) - A(|v|)\}
\]

for all \( u \in K \) and \( v \in \mathbb{R}^N \). In turn, by virtue of (2.3), (R2) is satisfied.

To verify (2.2) of Chapter 3, fix \( U > 0 \), then by using (2.2) and (2.4) it follows that

\[
\langle Q(r, u, v), v \rangle = \frac{n-1}{r} g(u)A(|v|)|v|^2 \geq \frac{n-1}{r} b_1 b_2 |u| |v|^m
\]

for all \( r > 0, u \in \mathbb{R}^N \) with \( |u| \geq U \) and all \( v \in \mathbb{R}^N \). So that (2.2) of Chapter 3 holds with

\[ \kappa = \gamma \quad \text{and} \quad \delta_1(r) = \frac{n-1}{r} b_1 b_2. \]
Note that also (2.3) of Chapter 3 holds since
\[
[\delta_1(r)]^{-1/(m-1)} = \left( \frac{1}{(n-1)b_1b_2} \right)^{1/(m-1)} r^{1/(m-1)}
\]
hence \( \delta_1 \in L^1_{loc}(\mathbb{R}^+) \). Consequently Theorem 15 of Chapter 3 can be applied to (2.6).

We now study the asymptotic behavior of global radial solution of (1.1). In particular we obtain the following result.

**Theorem 19.** Assume that all the hypotheses of Theorem 18 hold. Suppose in addition that there exists a constant \(a_1 > 1\) such that
\[
s^2 A(s) \geq a_1 A(s) \quad \text{for all } s > 0, \tag{2.8}
\]
and
\[
a_1 g(u) + \langle \nabla_u g(u), u \rangle \geq 0 \quad \text{for all } u \in \mathbb{R}^N \setminus \{0\}. \tag{2.9}
\]
Finally, suppose also that for every \(U > 0\) there exist a positive constant \(a_2\) such that
\[
\langle f(r, u), u \rangle \geq a_2 \tag{2.10}
\]
for all \((r, u) \in \mathbb{R}^+_0 \times \mathbb{R}^N\) with \(|u| \geq U\).

Let \(u\) be a radial global solution of (1.1), then
\[
\lim_{|x| \to \infty} |u(|x|)| = \infty. \tag{2.11}
\]

**Proof.** Consider the auxiliary function, defined in \(\mathbb{R}^+_0\),
\[
Z(r) = g(u)A(|u'|)(u', u). \tag{2.12}
\]
We claim that
\[
Z(r) > 0 \quad \text{for all } r > 0. \tag{2.13}
\]
Indeed, by using the equation in (2.6), we can calculate
\[
Z'(r) = \langle \{g(u)A(|u'|)u'\}', u \rangle + g(u)A(|u'|)|u'|^2
\]
\[
= g(u)A(|u'|)|u'|^2 + \langle \nabla_u g(u), A(|u'|) \rangle - \frac{n-1}{r} g(u)A(|u'|)u' + f(r, u), u \rangle
\]
\[
= g(u)A(|u'|)|u'|^2 + \langle \nabla_u g(u), A(|u'|) \rangle + \langle f(r, u), u \rangle - \frac{n-1}{r} Z(r).
\] (2.14)

Since for any \( r \geq 0 \) and \( u \in \mathbb{R}^N \) by (C3)
\[
F(r, u) = \int_0^1 \langle f(r, tu), u \rangle dt,
\]
it follows from (2.10) that
\[
F(r, u) > 0 \quad \text{for all} \quad r \geq 0 \quad \text{and} \quad u \in \mathbb{R}^N \setminus \{0\}.
\]

Indeed note that (2.10) implies that
\[
\langle f(r, u), u \rangle > 0 \quad \text{for all} \quad r \geq 0 \quad \text{and} \quad u \in \mathbb{R}^N \setminus \{0\}.
\]

In particular \( F(0, u_0) > 0 \) and so
\[
E(0) = -F(0, u_0) < 0.
\]

Hence by Proposition 3 of Chapter 2, there exists \( U > 0 \) such that
\[
|u(r)| \geq U \quad \text{for all} \quad r > 0.
\] (2.15)

Hence from (2.14), (2.8), (2.9) and (2.10) we obtain
\[
Z'(r) + \frac{n-1}{r} Z(r) = g(u)A(|u'|)|u'|^2 + \langle \nabla_u g(u), A(|u'|) \rangle + \langle f(r, u), u \rangle
\]
\[ \geq \left[ a_1 g(u) + \langle \nabla_u g(u), u \rangle \right] A(|u'|) + a_2 \geq a_2, \] (2.16)
or equivalently
\[
\left( r^{n-1} Z(r) \right)' \geq C_1 r^{n-1} > 0 \quad \text{for all} \quad r > 0.
\]

Thus \( r^{n-1} Z(r) \) is strictly increasing for all \( r > 0 \). Since \( Z(0) = 0 \), we have \( Z(r) > 0 \) if \( r > 0 \), and the claim (2.13) is proved.
Now, to prove (2.11), first we will show that
\[ \limsup_{r \to \infty} |u(r)| = \infty. \]  
(2.17)

Hence assume for contradiction that
\[ \limsup_{r \to \infty} |u(r)| < \infty. \]  
(2.18)

Consequently there exists \( U_1 > 0 \) such that \( |u(r)| \leq U_1 \) for all \( r > 0 \), thus, by (2.15), it follows that
\[ |u(r)| \in [U, U_1] \quad \text{for all } r > 0. \]

This implies, by integrating (2.1), that \( |F(r, u)| \leq |u|^p/p \leq U_1^p/p \). Consequently, from the fact that
\[ E(0) \geq E(r) = H(u, u') - F(r, u) \geq -F(r, u) \geq -|u|^p/p \geq -U_1^p/p, \]
we deduce that, since \( F \) is bounded, then also \( H \) must be bounded along the solutions, hence by (R2) of Chapter 3, which holds in this setting thanks to (2.3) and (2.4) as showed in (2.7), there exists \( L_1 > 0 \) such that
\[ |u'(r)| \leq L_1 \quad \text{for all } r \geq 0. \]

Consequently, the boundness of both \( |u| \) and \( |u'| \) implies, by Cauchy-Schwarz inequality and by (C1), that for all \( r > 0 \)
\[ Z(r) \leq |A(|u'|)|u'|u| \leq C, \quad C > 0. \]

Thus
\[ \lim_{r \to \infty} \frac{n - 1}{r} Z(r) = 0. \]

Consequently from (2.16) and (2.18) we obtain
\[ \liminf_{r \to \infty} Z'(r) \geq C_1, \]

namely
\[ Z'(r) \geq C_2, \quad r \geq r_2; \]
for \( r_2 \) sufficiently large. Consequently, by integration and thanks to (2.13),

\[
Z(r) \geq C_2(r - r_2) + Z(r_2) > C_1(r - r_2), \quad r \geq r_2,
\]

which implies that \( Z(r) \to \infty \) as \( r \to \infty \). This contradicts the boundness of \( Z \), thus (2.18) cannot occur and this proves (2.17).

Finally, by virtue of (C1), (C2), (2.12) and (2.13) we deduce

\[
\frac{d}{dr} \left( \frac{|u|^2}{2} \right) = \langle u, u' \rangle > 0
\]

for all \( r > 0 \). This implies that the function \(|u|^2\) is increasing, thus \(|u|\) tends to a limit at infinity, namely

\[
\lim_{r \to \infty} |u(r)| = \ell \in (0, \infty].
\]

By (2.17), we immediately deduce that \( \ell = \infty \) and consequently (2.11) is proved.

Now we apply the results above to two model elliptic cases. Precisely we obtain the following

**Corollary 11.** Assume that

\[
m > 1, \quad p > 1, \quad \gamma \geq p - m
\]

and consider the system

\[
\text{div} \left( |u|^\gamma |\nabla u|^{m-2} \nabla u \right) - \frac{\gamma}{m} |u|^{\gamma-2} u |\nabla u|^{m} = |u|^{p-2} u.
\]

Then (2.20) admits a one parameter family of non trivial entire radial solutions \( u : \mathbb{R}^+_0 \to \mathbb{R}^N \setminus \{0\} \) such that

\[
\lim_{|x| \to \infty} |u(|x|)| = \infty,
\]

namely (2.21) admits entire radial solutions.
Proof. It is enough to apply Theorems 18 and 19 with
\[ A(s) = s^{m-2} \quad \text{and} \quad g(u) = |u|^{\gamma}. \]
Indeed (2.2)–(2.4) are trivially satisfied with \( b_1 = b_2 = 1 \), while \( a_1 = m \) in (2.8) of Theorem 19 so that (2.9) holds since \( \gamma + m \geq p > 1 \) by (2.5).

\[
\text{Corollary 12. Assume that} \quad 1 < m \leq 2, \quad p > 1, \quad \gamma \geq p - m, \tag{2.23}
\]
then the elliptic system
\[
\text{div}(|u|^{\gamma}(1 + |\nabla u|^2)^{m/2 - 1}\nabla u) - \frac{\gamma}{m} |u|^{\gamma - 2} u[(1 + |\nabla u|^2)^{m/2} - 1] = |u|^{p-2} u, \tag{2.24}
\]
Then (2.24) admits a one parameter family of non trivial entire radial solutions \( u : \mathbb{R}^+ \to \mathbb{R}^N \setminus \{0\} \) such that
\[
\lim_{|x| \to \infty} |u(|x|)| = \infty, \tag{2.25}
\]
\text{namely (2.24) admits entire radial solutions.}

Proof. As in the corollary above, we apply Theorems 18 and 19 with
\[
A(s) = (1 + s^2)^{m/2 - 1} \quad \text{and} \quad g(u) = |u|^{\gamma}.
\]
Then \( b_1 = b_2 = 1 \) in (2.2) and (2.4) respectively in Theorem 18, while as showed in the proof of Corollary 9, see (1.27), \( a_1 = m \) in (2.8) of Theorem 19 so that (2.9) holds since \( \gamma + m \geq p > 1 \) by (2.5).
CHAPTER 5

APPLICATIONS TO ELLIPTIC EQUATIONS

1. Non existence of entire solutions

In this chapter we study the existence and non existence of entire solutions of the elliptic equation

$$\text{div}(g(u)A(|\nabla u|)\nabla u) - g'(u)A(|\nabla u|) = f(|x|, u), \quad x \in \mathbb{R}^n, \quad (1.1)$$

where the functions $A, f, g$ satisfy conditions (C1)--(C3) of the previous chapter with $N = 1$.

Radial solutions $u = u(|x|)$ of (1.1) satisfy the initial value problem

$$[g(u)A(|u'|)u']' + \frac{n-1}{r}g(u)A(|u'|)u' - g'(u)A(|u'|) = f(r, u), \quad r = |x| > 0, \quad (1.2)$$

$$u(0) = u_0, \quad u'(0) = 0.$$ 

We begin by proving that when $u_0 \neq 0$ solutions of (1.2) are monotone. For simplicity we consider only the case $u_0 > 0$, the case $u_0 < 0$ being analogous.

**Theorem 20.** Assume that conditions (C1)--(C3) of Chapter 4 hold with $N = 1$ and that

$$uf(r, u) > 0 \quad \text{for all} \quad r \geq 0 \quad \text{and} \quad u > 0. \quad (1.3)$$

Let $u : [0, R) \to \mathbb{R}$ be a local solution of (1.2). If $u_0 > 0$, then

$$u'(r) > 0 \quad \text{for all} \quad r \in (0, R). \quad (1.4)$$

**Proof.** Rewrite equation (1.2) as follows

$$[r^{n-1}A(|u'|)u']' = r^{n-1}[g'(u)A(|u'|) + f(r, u)]. \quad (1.5)$$
Since $u'(0) = 0$, $A(0) = 0$ and $g \in C^1(D; \mathbb{R})$ we have

$$[g'(u(r))A(|u'(r)|) + f(r, u(r))]_{r=0} = f(0, u_0) > 0,$$

by (1.3). Hence there exists $r_0 > 0$ such that

$$g'(u)A(|u'(r)|) + f(r, u(r)) > 0 \quad \text{in } (0, r_0],$$

consequently

$$[r^{n-1}g(u)A(|u'|)u']' > 0 \quad \text{in } (0, r_0].$$

Thus $r^{n-1}g(u)A(|u'|)u'$ is an increasing function in $(0, r_0]$ and so by (C1)

$$r^{n-1}g(u)A(|u'|)u' > 0 \quad \text{in } (0, r_0].$$

By (C1) this yields $u'(r) > 0$ in $(0, r_0]$. Now assume for contradiction that there exists $r_1 > r_0$ such that $u'(r) > 0$ for $r \in (0, r_1)$ and $u'(r_1) = 0$. Applying the same argument above we get that there is $\epsilon > 0$ such that

$$r^{n-1}g(u)A(|u'|)u' < 0 \quad \text{in } [r_1 - \epsilon, r_1),$$

so that $u'(r_1 - \epsilon) < 0$ which is a contradiction. Thus the claim (1.4) is proved. $\square$

The following theorem was first proved by Naito and Usami in [21] under stronger regularity assumptions on the operator $A$.

**Theorem 21.** Consider the initial value problem

$$
\left( A(|v'|)v' \right)' + \frac{n-1}{r} A(|v'|)v' = f(v), \quad r > 0,
$$

$$
v(0) = v_0 > 0, \quad v'(0) = 0,
$$

(1.6)

where the function $A$ satisfies condition (C1) of Chapter 4 and the function $f \in C(\mathbb{R}^+_0; \mathbb{R})$ is such that

$$f(0) = 0, \quad f \quad \text{is non decreasing.}
$$

If

$$\lim_{s \to \infty} sA(s) = \infty$$
and
\[
\int_{t}^{\infty} \left( \Phi \left( \frac{1}{n} \int_{s}^{t} f(\tau) \, d\tau \right) \right)^{-1} \, dt < \infty, \tag{1.7}
\]
where $\Phi$ is the inverse function of
\[
H(s) = s^2 A(s) - A(s), \quad s > 0, \tag{1.8}
\]
then the initial value problem (1.6) does not admit any global positive solution.

**Proof.** The proof follows closely that of Naito and Usami, we present it here for the convenience of the reader.

Suppose to the contrary that there exists a positive solution $v$ of (1.6) on the interval $\mathbb{R}_0^+$ with $v(0) > 0$ and $v'(0) = 0$. Moreover, by Theorem 20, $v' > 0$ in $\mathbb{R}^+$.

Now, rewrite (1.6) as follows
\[
\left( r^{n-1} A(|v'|) v' \right)' = r^{n-1} f(v), \quad r > 0,
\]
and upon an integration we obtain that
\[
A(|v'(r)|) v'(r) = r^{1-n} \int_{0}^{r} \tau^{n-1} f(v(\tau)) \, d\tau.
\]
Since $f$ is non-decreasing and $v' > 0$, from the inequality above, we deduce that
\[
A(|v'(r)|) v'(r) \leq f(v(r)) r^{1-n} \int_{0}^{r} \tau^{n-1} \, d\tau = f(v(r)) \frac{r}{n}, \quad r > 0. \tag{1.9}
\]
On the other hand, by substituting (1.9) in (1.6) we obtain
\[
\left( A(|v'|) v' \right)' \geq \frac{1}{n} f(v). \tag{1.10}
\]
Now, integrate (1.10) from 0 to $r$ and use again the monotonicity of $f$ and $v$ to obtain
\[
A(|v'(r)|) v'(r) \geq \frac{1}{n} \int_{0}^{r} f(v(\tau)) \, d\tau \geq \frac{f(v(0))}{n} r, \quad r > 0.
\]
Letting $r \to \infty$ we have $\lim_{r \to \infty} v'(r) = \infty$ and therefore
\[
\lim_{r \to \infty} v(r) = \infty. \tag{1.11}
\]
Multiplying (1.10) by $v'$ and integrating form 0 to $r$, we have

$$H(v'(r)) \geq \frac{1}{n} \int_{v(0)}^{v(r)} f(\tau) d\tau,$$

where $H$ is the function defined in (1.8). Using the fact that $H(s)$ is strictly increasing if $s > 0$ and so also its inverse $\Phi$, we obtain

$$v'(r) \geq \Phi \left( \frac{1}{n} \int_{v(0)}^{v(r)} f(\tau) d\tau \right),$$

namely

$$\left( \Phi \left( \frac{1}{n} \int_{v(0)}^{v(r)} f(\tau) d\tau \right) \right)^{-1} v'(r) \geq 1, \quad r > 0.$$

Now, an integration from 1 to $r$ of the inequality above, gives

$$\int_{v(1)}^{v(r)} \left( \Phi \left( \frac{1}{n} \int_{v(0)}^{t} f(\tau) d\tau \right) \right)^{-1} dt \geq r - 1, \quad r > 1. \tag{1.12}$$

Letting $r \to \infty$ in (1.12) and using (1.11), we have

$$\int_{v(1)}^{\infty} \left( \Phi \left( \frac{1}{n} \int_{v(0)}^{t} f(\tau) d\tau \right) \right)^{-1} dt = \infty,$$

which contradicts (1.7). Hence (1.6) does not admit any global positive solution. \(\Box\)

In the sequel we will also need the weak comparison principle which is due to Pucci, Serrin and Zou (see [34]), which we state and proof only when the domain $\Omega \subset \mathbb{R}^n$ is bounded.

**Theorem 22.** (Lemma 3, [34]) Let $u$ and $v$ be respective solutions of the differential inequalities

$$\text{div}(A(|\nabla u|) \nabla u) - f(u) \geq 0, \quad u \geq 0, \tag{1.13}$$

$$\text{div}(A(|\nabla v|) \nabla v) - f(v) \leq 0, \quad v \geq 0, \tag{1.14}$$

in a bounded domain $\Omega$ of $\mathbb{R}^n$, $n \geq 2$. Assume that the function $A$ satisfies condition (C1) of Chapter 4 and that the function $f \in C(\mathbb{R}_0^+; \mathbb{R})$ is such that

$$f(0) = 0, \quad f \text{ is non decreasing in } [0, \delta), \quad 0 < \delta \leq \infty.$$
If \( u \) and \( v \) are continuous in \( \Omega \), with \( u < \delta \) in \( \Omega \) and
\[
v \geq u \quad \text{on} \quad \partial \Omega,
\]
then
\[
v \geq u \quad \text{in} \quad D.
\]

**Proof.** Let \( w = v - u \) in \( \Omega \). If the conclusion fails, then there exists a point \( x_1 \in \Omega \) such that \( w(x_1) < 0 \). Now fix \( \varepsilon > 0 \) so small that \( w(x_1) + \varepsilon < 0 \). Consequently, since by hypothesis \( w \geq 0 \) on \( \partial \Omega \) it follows that the function \( w_\varepsilon = \min\{w + \varepsilon, 0\} \) is non positive and has compact support in \( \Omega \). Take the Lipschitzian function \( w_\varepsilon \) as a test function in the distributional meaning of solutions, namely from (1.13) and (1.14)
\[
\int_{\Omega} \{A(|\nabla v|)\nabla v - A(|\nabla u|)\nabla u\} \nabla w_\varepsilon \leq \int_{\Omega} \{f(u) - f(v)\} w_\varepsilon.
\] (1.15)
The left hand side of (1.15) is positive, due to the strict monotonicity of \( sA(s) \) and the fact that \( \nabla w_\varepsilon \equiv \nabla w \neq 0 \) when \( w + \varepsilon < 0 \), while otherwise \( \nabla w_\varepsilon = 0 \) (a.e.).

Moreover, when \( w + \varepsilon < 0 \), there holds \( 0 \leq v < u - \varepsilon(\varepsilon < \delta) \); hence \( f(v) - f(u) \geq 0 \), since \( f \) is non decreasing in \( (0, \delta) \). Thus the right side of (1.15) is non positive, which is a contradiction. This complete the proof of the theorem. \( \square \)

Under stronger regularity assumptions on the operator \( A \), Theorem 22 has been proved by Naito and Usami in \([21]\).

We are now ready to prove the first global non existence result for entire solutions (not necessarily radial) of (1.1).

**Theorem 23.** Consider the elliptic equation
\[
\text{div}(g(u)A(|\nabla u|)\nabla u) - g'(u)A(|\nabla u|) = f(|x|, u) \quad \text{in} \quad \mathbb{R}^n, \tag{1.16}
\]
where the functions \( A, f, g \) satisfy conditions (C1)–(C3) of Chapter 4 with \( N = 1 \). Assume that
\[
g \quad \text{is non increasing} \tag{1.17}
\]
and that there exists a non decreasing function \( \varphi \in C(\mathbb{R}_0^+; \mathbb{R}) \), with \( \varphi(0) = 0 \), such that
\[
\frac{f(r,u)}{g(u)} \geq \varphi(u) \quad \text{for all } r \geq 0 \text{ and all } u > 0.
\] (1.18)

If
\[
\lim_{s \to \infty} sA(s) = \infty
\]
and \( \varphi \) satisfies the condition
\[
\int_{-\infty}^{\infty} \left( \Phi \left( \frac{1}{n} \int_{-\infty}^{\tau} \varphi(\tau)d\tau \right) \right)^{-1} dt < \infty,
\] (1.19)
where we recall that \( \Phi \) is the inverse function of
\[ H(s) = s^2A(s) - A(s), \quad s > 0, \]
then equation (1.16) does not admit any entire positive solution.

**Proof.** Assume by contradiction that (1.16) admits a positive entire solution \( u : \mathbb{R}^n \to \mathbb{R}^+ \). Since \( A(s) \leq s^2A(s) \) for \( s > 0 \) by (C1) and \( g'(u) \leq 0 \) for all \( u \geq 0 \), by (1.16) and (1.18), we have
\[
\text{div} \left( A(|\nabla u|)\nabla u \right) = \frac{g'(u)}{g(u)} \left[ A(|\nabla u|) - |\nabla u|^2A(|\nabla u|) \right] + \frac{f(|x|,u)}{g(u)} \geq \varphi(u).
\] (1.20)

Consider now the initial value problem
\[
\left( r^{n-1}A(|v'|)v' \right)' = r^{n-1}\varphi(v),
\]
\[
v(0) = a \in (0,u(0)), \quad v'(0) = 0.
\] (1.21)

By Theorem 10 of Chapter 1 the initial value problem (1.21) admits a local solution \( v : [0,R) \to \mathbb{R}^+ \), where \( R > 0 \). Without loss of generality we may assume that \([0,R)\) is the maximal interval of existence of \( v \). By Theorem 21 we have that \( R < \infty \). As noted in the proof of Theorem 21 it results that \( v'(r) > 0 \) for \( r \in (0,R) \). Hence either
\[
\lim_{r \to R^-} v(r) = \infty \quad \text{or} \quad \lim_{r \to R^-} v'(r) = \infty.
\]
1. NON EXISTENCE OF ENTIRE SOLUTIONS

Case 1:

$$\lim_{r \to R^-} v(r) = \infty.$$ 

In this case, we can take $R_1 \in (0, R)$ so that

$$v(R_1) \geq \max\{u(x) : |x| = R_1\}. \quad (1.22)$$

Define $B_1 = \{x \in \mathbb{R}^n : |x| < R_1\}$. Then the function $v = v(|x|)$ is such that

$$\text{div}(A(|\nabla v|)\nabla v) = \varphi(v) \quad \text{in} \quad B_1$$

and we have that $v \geq u$ on $\partial B_1$. Hence, by Theorem 22 applied with $\Omega = B_1$, it results that $u \leq v$ in $B_1$ and this is a contradiction since $v(0) = a < u(0)$.

Case 2:

$$\lim_{r \to R^-} v'(r) = \infty.$$ 

First note that if we can find an $R_1 \in (0, R)$ such that (1.22) holds, then we have the same contradiction as above and the theorem is proved. Otherwise $v(r) < \max\{u(x) : |x| = r\}$ for $0 < r < R$. We can take $R_1 \in (0, R)$ so that

$$\frac{\partial v}{\partial \nu}(R_1) > \max\left\{\frac{\partial u}{\partial \nu}(x) : |x| = R_1\right\}, \quad (1.23)$$

where $\nu$ is the unit outer normal of $\partial B_1$. Define $\delta = \max\{u(x) - v(x) : |x| = R_1\}$ and let $w(x) = v(x) + \delta$. Then $w(x) \geq u(x)$ on $|x| = R_1$ and for some $x^*$, with $|x^*| = R_1$ it results $w(x^*) = u(x^*)$. From (1.23) we can find that

$$w(x_0) < u(x_0) \quad \text{for some} \quad x_0, \quad \text{with} \quad |x_0| < R_1. \quad (1.24)$$

Again we have that

$$\text{div}(A(|\nabla w|)\nabla w) \leq \varphi(w) \quad \text{in} \quad B_1$$

and again we have that $w \geq u$ on $\partial B_1$. Then again by Theorem 22 with $\Omega = B_1$, we obtain $w \geq u$ in $B_1$ which contradicts (1.24). Therefore the proof of the claim is complete.
Corollary 13. The elliptic equation
\[
\text{div}(u^\gamma \nabla |u|^{m-2} \nabla u) - \frac{\gamma}{m} u^{\gamma-1} |\nabla u|^m = u^{p-1} \quad \text{in} \quad \mathbb{R}^n,
\]
with
\[
p > m > 1, \quad \gamma \leq 0,
\]
does not admit any positive entire solutions.

**Proof.** It is enough to apply Theorem 23 with
\[
A(s) = s^{m-2}, \quad g(u) = u^\gamma, \quad f(r, u) = f(u) = u^{p-1}
\]
for all \(s > 0\) and \(u > 0\). In particular in (1.18) we have that \(\varphi(u) = u^{p-1-\gamma}, u > 0\), which is increasing thanks to (1.25). Moreover
\[
H(s) = \frac{m-1}{m} s^m \quad \text{and} \quad \Phi(t) = \left(\frac{m}{m-1}\right)^{1/m} t^{1/m}
\]
and consequently the integral in (1.19) is convergent because \(p > m\) in (1.25) \(\square\)

Next, we consider the special case of (1.1) in which \(A(s) = s^{m-2}, s > 0, m > 1,\) and the function \(g\) does not satisfy (1.17), namely \(g\) is non decreasing. For instance we refer to the case \(g(u) = u^\gamma, u > 0, \gamma > 0\). To obtain our result we make use of the following corollary obtained by Naito and Usami in [21].

**Corollary 14.** Let
\[
m > 1 \quad \text{and} \quad p > 1.
\]
Then the inequality
\[
\text{div}(|\nabla u|^{m-2} \nabla u) \geq u^{p-1}, \quad x \in \mathbb{R}^n,
\]
has non negative entire solutions \(u \neq 0\), if and only if
\[
p \leq m.
\]

We shall use the above corollary to obtain the following result.
Corollary 15. Consider the elliptic equation

\[ \text{div}(g(u)|\nabla u|^{m-2}\nabla u) - \frac{1}{m}g'(u)|\nabla u|^m = |u|^{p-2}u, \]  

(1.26)

where \( g \) is a non decreasing function satisfying (C2) of Chapter 4 with \( N = 1 \) and such that

\[ g(u) \leq Cu^\gamma, \quad u > 0. \]  

(1.27)

If

\[ p > m > 1, \quad 0 < \gamma < p - m \]  

(1.28)

then (1.26) does not admit positive entire solutions.

Proof. Assume for contradiction that (1.26) admits a positive entire solution \( u \). Consider the following change of variables \( v(x) = h(u(x)) \), where

\[ h(u) = \int_0^u [g(s)]^{1/(m-1)}ds \quad \text{for} \quad u \geq 0. \]

We have that

\[ \nabla v(x) = h'(u(x))\nabla u(x) = [g(u(x))]^{1/(m-1)}\nabla u(x), \]

hence

\[ |\nabla v(x)|^{m-2}\nabla v(x) = [g(u(x))]^{(m-2)/(m-1)}|\nabla u(x)|^{m-2}[g(u(x))]^{1/(m-1)}\nabla u \]

\[ = g(u(x))|\nabla u(x)|^{m-2}\nabla u(x). \]

Now consider the function \( u^{p-1} \) and observe that by the fact that \( g \) is increasing it results that

\[ v(x) = h(u(x)) = \int_0^{u(x)} [g(s)]^{1/(m-1)}ds \leq u(x)[g(u(x))]^{1/(m-1)}, \]

so that, by (1.27) we get

\[ v(x) \leq C^{1/(m-1)}[u(x)]^{\gamma/(m-1)+1}, \]

namely

\[ [v(x)]^{m-1} \leq C[u(x)]^{m-1+\gamma}, \]  

(1.29)
This implies that
\[ [u(x)]^{p-1} \geq C^{(p-1)/(m-1+\gamma)} [v(x)]^{(p-1)(m-1)/(m-1+\gamma)}. \]

Thus \( v \) is positive solution of
\[
\text{div}(|v|^{m-2} v) - \frac{1}{m} g'(h^{-1}(v)) [g(h^{-1}(v))]^{-m/(m-1)} |\nabla v|^m \geq C v^{(p-1)(m-1)/(m-1+\gamma)},
\]

namely, since \( g' \geq 0 \), \( v \) solves in \( \mathbb{R}^n \) the inequality
\[
\text{div}(|v|^{m-2} v) \geq C v^{(p-1)(m-1)/(m-1+\gamma)}. \tag{1.30}
\]

This is a contradiction since by Corollary 14, (1.30) does not admit entire solutions being
\[
\frac{(p-1)(m-1)}{m-1+\gamma} > m-1,
\]

by virtue of (1.28) \( \square \)

**Corollary 16.** The elliptic equation
\[
\text{div}(u^\gamma |\nabla u|^{m-2} \nabla u) - \frac{\gamma}{m} u^{\gamma-1} |\nabla u|^m = u^{p-1} \quad \text{in} \quad \mathbb{R}^n,
\]

with
\[
p > m > 1 \quad 0 < \gamma < p - m, \tag{1.31}
\]

does not admit any positive entire solutions.

**Proof.** It is enough to apply the above corollary with \( g(u) = u^\gamma \), \( u > 0 \), consequently the opportune change of variable is
\[
h(t) = t^{a+1}, \quad a = \frac{\gamma}{m-1}.
\]

\( \square \)

Hence if we combine Corollaries 13, 16 with Corollary 11 of Chapter 4 with \( N = 1 \) we obtain the following general result
Corollary 17. The elliptic equation
\[
\text{div}(u^\gamma |\nabla u|^{m-2}\nabla u) - \frac{\gamma}{m} u^{\gamma-1} |\nabla u|^m = u^{\gamma-1} \quad \text{in} \quad \mathbb{R}^n, \quad p, m > 1,
\]
does not admit any positive entire solutions if
\[
p > m \quad \text{and} \quad \gamma < p - m,
\]
while it admits entire solutions if
\[
\gamma \geq p - m.
\]
Bibliography


