Existence of global solutions of elliptic systems

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Abstract

We study existence of entire solutions of elliptic systems whose prototype is given by
\[
\text{div} \left( |u|^{\gamma} |\nabla u|^{m-2} \nabla u \right) - \frac{\gamma}{m} |u|^{\gamma-2} |\nabla u|^{m} = |u|^{p-2} u, \quad u : \mathbb{R}^n \to \mathbb{R}^N,
\]
with \( m > 1, \gamma \in \mathbb{R}, \) and \( p > 1. \) In particular, we prove that, if \( \gamma \geq p - m, \) the system above admits a one parameter family of nontrivial entire radial solutions, \( u = u(|x|), \) such that \( \lim_{|x| \to \infty} |u(|x|)| = \infty. \)

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1. Introduction

In a previous paper [1], we study nonexistence of radial entire solutions \( u : \mathbb{R}^n \to \mathbb{R}^N \) of elliptic systems of the form
\[
\text{div} \left( g(u) A(|\nabla u|) \nabla u \right) - \nabla u g(u) A(|\nabla u|) = f(r, u), \quad r = |x|, \ x \in \mathbb{R}^n,
\]
where \( \nabla u \) denotes the Jacobian matrix and \( A(s) = \int_{0}^{s} A(\sigma) d\sigma, \ s > 0, \) with \( A \in C(\mathbb{R}^+; \mathbb{R}^+). \) Moreover, the diffusion \( g \) is of class \( C^1 \) in \( D, \) where either \( D = \mathbb{R}^N \) or \( D = \mathbb{R}^N \setminus \{0\}. \) Indeed, the model we have in mind for \( g \) is \( |u|^\gamma, \gamma \in \mathbb{R} \) (possibly negative), so that \( D = \mathbb{R}^N \) if \( \gamma \geq 0, \) otherwise \( D = \mathbb{R}^N \setminus \{0\} \) if \( \gamma < 0. \)

More specifically, the prototype for (1) we consider is
\[
\text{div} \left( |u|^{\gamma} |\nabla u|^{m-2} \nabla u \right) - \frac{\gamma}{m} |u|^{\gamma-2} |\nabla u|^{m} = |u|^{p-2} u, \quad m > 1, \gamma \in \mathbb{R}, \text{ and } p > 1,
\]
with \( m > 1, \gamma \in \mathbb{R}, \) and \( p > 1. \)

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In [1], for the vectorial case \( N > 1 \) we restrict our attention to radial solutions of (1) since in general maximum principle does not hold, for details see [1,2], while in the scalar case \( N = 1 \) we obtain nonexistence of all solutions of (1), radial or not, by using a technique developed by Naito and Usami in [6] (see also [7,15]) and by applying a weak comparison principle proved by Pucci, Serrin, and Zou in [14] (see also [13]).

In this paper we study the specular problem with respect to [1], namely the existence of entire solutions of (1). In particular, we look for global solutions of the ordinary differential system

\[
\begin{bmatrix}
g(u)A(|u'|)u'_r \n + \frac{n-1}{r}g(u)A(|u'|)u' - \nabla_u g(u)A(|u'|) = f(r,u), \quad r > 0, \\
u(0) = u_0 \neq 0, \quad u'(0) = 0,
\end{bmatrix}
\tag{3}
\]

where \( u' = du/dr \), \( r = |x| \), and \( A(s) = \int_0^s \sigma A(\sigma) d\sigma, s > 0 \). A canonical model for (3) is given by the initial value problem

\[
\begin{bmatrix}
|u|^\gamma |u'|^{m-2}u' \n - \gamma m |u|^\gamma - 2u|u'|^m + \frac{n-1}{r}|u|^\gamma |u'|^{m-2}u' = |u|^{p-2}u, \quad r > 0, \\
u(0) = u_0 \neq 0, \quad u'(0) = 0,
\end{bmatrix}
\tag{4}
\]

where \( \gamma \in \mathbb{R}, m, p > 1, n \geq 1 \). It results that the problem of existence and nonexistence of solutions of (4) is strictly related to the competition between the terms \( |u|^\gamma |u'|^{m-2}u' \) and \( |u|^{p-2}u \). Indeed, if we just consider the simple case when \( N = n = 1 \) and \( \gamma = 0 \) in (4), namely the initial value problem

\[
\begin{bmatrix}
|u'|^{m-2}u' \n = |u|^{p-2}u, \quad r > 0, \quad u(0) = u_0 > 0, \quad u'(0) = 0,
\end{bmatrix}
\tag{5}
\]

we obtain, by simple calculations (for details see Appendix A), that global solutions of (5) exist if and only if \( p \leq m \). This example in the case \( m = 2 \) and \( p > 2 \) was given in [10].

Here, we prove that the same relation between the exponents \( m \) and \( p \), when \( \gamma = 0 \), holds also for more general systems. Indeed, we obtain the following corollary.

**Corollary 1.** Let \( m > 1, \gamma \in \mathbb{R}, \) and \( p > 1 \). The elliptic system

\[
\text{div}(|u|^\gamma |\nabla u|^{m-2}\nabla u) - \frac{\gamma}{m}|u|^\gamma - 2u|\nabla u|^m = |u|^{p-2}u, \tag{6}
\]

(i) admits a local radial solution;
(ii) does not admit nontrivial entire radial solutions if

\[1 < m < p, \quad -m - p(m - 1) < \gamma < p - m;\]
(iii) admits entire radial solutions, namely every local radial solution of (6) can be continued to all of \( \mathbb{R}^n \), if \( \gamma \geq p - m \).

We point out that (i) and (ii) are proved in [1]. Furthermore, we remind that, if \( \gamma = 0 \) in (iii), condition (ii) becomes \( p \leq m \) and it is equivalent to the divergence of the Keller Osserman integral, namely

\[
\int_0^\infty \frac{ds}{H^{-1}(F(s))} = \infty, \tag{7}
\]
where $F(u) = \int_0^u f(s) \, ds$ and $H$ is the Legendre transform of $A$. Indeed, if (6) is considered in the scalar case $N = 1$, then $H(s) = (m - 1)s^m / m$, $s \geq 0$, and $F(u) = u^p / p$ for $u \geq 0$ and so $H^{-1}(F(u)) = cu^{p/m}$, for $u \geq 0$ and for some constant $c$. Finally, note that (7) is a necessary and sufficient condition for the existence of global solutions (for details see [2,3,8]).

Next we consider the generalized mean curvature operator and deduce the same result obtained for the $m$-Laplacian, namely the following corollary.

**Corollary 2.** Let $1 < m \leq 2$, $\gamma \in \mathbb{R}$, and $p > 1$. The elliptic system
\[
div (|u|^\gamma (1 + |\nabla u|^2)^{m/2 - 1} \nabla u) - \frac{\gamma}{m} |u|^{\gamma - 2} u [(1 + |\nabla u|^2)^{m/2} - 1] = |u|^{p - 2} u, \tag{8}
\]
(i) admits a local radial solution;
(ii) does not admit nontrivial entire radial solutions if
\[1 < m < p, \quad -m - p(m - 1) < \gamma < p - m;
\]
(iii) admits entire radial solutions, namely every local solution of (8) can be continued to all of $\mathbb{R}^n$, if $\gamma \geq p - m$.

In this paper, for the scalar case $N = 1$, we extend some results of Naito and Usami obtained in [6], in particular, we obtain the following corollary.

**Corollary 3.** The elliptic inequality
\[
div (|u|^\gamma |\nabla u|^{m-2} \nabla u) - \frac{\gamma}{m} |u|^{\gamma - 2} |\nabla u|^m \geq |u|^{p - 2} u, \quad p, m > 1, \tag{9}
\]
admits entire solutions if and only if $\gamma \geq p - m$.

Moreover, as noted in [1], in the scalar case $N = 1$ it is easy to see that if $u$ is any local nonnegative radial solution of (3), with $uf(r, u) > 0$ for $u \neq 0$ and $r \geq 0$, then $u = u(|x|) = u(r)$ is increasing and so admits limit at infinity when it is entire.

The situation is significantly more complicated in the vectorial case $N > 1$ as no monotonicity is available. Nevertheless, we still prove the following result.

**Corollary 4.** Let
\[
m > 1, \quad p > 1, \quad \text{and} \quad \gamma \geq p - m \quad \text{in (6)},
\]
\[1 < m \leq 2, \quad p > 1, \quad \text{and} \quad \gamma \geq p - m \quad \text{in (8)},
\]
then the elliptic systems (6) and (8) admit a one parameter family of nontrivial entire radial solutions $u : \mathbb{R}^n \to \mathbb{R}^N \setminus \{0\}$, $u = u(|x|)$, such that $\lim_{|x| \to \infty} |u(|x|)| = \infty$, that is (6) and (8) admit entire radial solutions.

The paper is organized as follows. Section 2 contains some preliminaries and a local existence theorem proved in [1] by using a technique of Leoni given in [4]. Section 3 is devoted to global existence of solutions of quasivariational systems whose particular case is
represented by (3). In particular, in Section 3 we give the main theorems of the paper concerning the two situations in which either the damping term or the action energy dominates over the driving force of the quasivariational system considered. In the first case, analyzed in Theorem 4, we obtain continuation while in the latter, investigated in Theorem 5, we prove either continuation or the fact that every local solution stops to exist at a finite point in which it reaches the zero value. The technique used is based on a result of Levine et al. [5] and of Leoni [4]. In Section 4 we apply the results of the previous section to elliptic systems obtaining existence of radial entire solutions of system (1). Furthermore, in Theorem 7, we establish the fact that every global solution of (1) explodes in norm at infinity. This fact, as noted above, is far from trivial in the vectorial case, since we have no monotonicity.

2. Preliminaries

Consider the quasi-variational ordinary differential system

\[
\nabla G(u, u') - \nabla_u G(u', u) + Q(t, u, u') = f(t, u)
\]

on \( J = (T, \infty), \ T \geq 0, \) where \( \nabla = \nabla_u \) denotes the gradient operator with respect to the second variable of the function \( G. \) It will be supposed throughout the paper that

(A1) \( G \in C^1(D \times \mathbb{R}^N; \mathbb{R}), \) where \( D \subset \mathbb{R}^N \) is an open set, \( G(u, \cdot) \) is strictly convex in \( \mathbb{R}^N \) for all \( u \in D \) such that \( u \neq 0, \) with \( G(u, 0) = 0, \nabla G(u, 0) = 0; \)

(A2) there exists a nonnegative function \( F \in C^1(\bar{J} \times \mathbb{R}^N; \mathbb{R}) \) with \( F(t, 0) = 0 \) for all \( t \in \bar{J}, \) such that

\[
\nabla_u F(t, u) = f(t, u) \quad \text{and} \quad F_t(t, u) \geq 0 \quad \text{for all } (t, u) \in \bar{J} \times \mathbb{R}^N;
\]

(A3) \( Q \in C(J \times D \times \mathbb{R}^N; \mathbb{R}^N) \) and \( \langle Q(t, u, v), v \rangle \geq 0 \) for all \( (t, u, v) \in J \times D \times \mathbb{R}^N. \)

Here \( \langle \cdot, \cdot \rangle \) represents the inner product in \( \mathbb{R}^N. \)

Let \( \mathcal{H}(u, \cdot) \) be the Legendre transform of \( G(u, \cdot), \) namely

\[
\mathcal{H}(u, v) = \langle \nabla G(u, v), v \rangle - G(u, v). \quad (11)
\]

Clearly, as already noted in [1],

\[
\mathcal{H}(u, 0) = 0 \quad \text{and} \quad \mathcal{H}(u, v) > 0 \quad \text{for all } u \in D \setminus \{0\}, \ v \in \mathbb{R}^N \setminus \{0\}, \quad (12)
\]
as a standard consequence of the strict convexity of \( G(u, \cdot) \) when \( u \neq 0. \)

We now turn to system (10) and say that \( u = (u_1, \ldots, u_N) \) is a local solution of (10) if \( u \) is a vector function \( u : I \to D \subset \mathbb{R}^N \) in some interval \( I = [T, T + \tau) , \tau > 0, \) such that \( u \) is of class \( C^1(I), \nabla G(u(t), u'(t)) \in C^1((T, T + \tau); \mathbb{R}^N) \) and \( u \) satisfies the system (10) in \( (T, T + \tau). \)

We say that a local solution \( u \) is a global solution of (10) if \( \tau = \infty. \)

In spite of the fact that neither \( u' \) nor \( \mathcal{H} \) need be separately differentiable, the composite function \( \mathcal{H}(u(t), u'(t)) \) is differentiable on \( (T, T + \tau), \) provided that \( u \) never vanishes on \( (T, T + \tau), \) and, as proved in [12] (for details, see also [9,11]), the following identity holds on \( (T, T + \tau): \)

\[
\{ \mathcal{H}(u(t), u'(t)) \} = \langle Q(t, u(t), u'(t)), u'(t) \rangle + \{ f(t, u(t)) \}, u'(t) \). \quad (13)
From now on, for simplicity, the common notation where \( u = u(t) \) and \( u' = u'(t) \) denote the solution and its derivative. Hence, if \( u \) is a solution of (10) on \( (T, T + \tau) \) such that \( u(t) \neq 0 \) for all \( t \in (T, T + \tau) \), we have thanks to (13),

\[
\{ \mathcal{H}(u, u') - F(t, u) \}' = -\langle Q(t, u, u'), u' \rangle - F_t(t, u).
\] (14)

Consequently, if we introduce the total energy of the vector field \( u \), defined by

\[
E(t) := \mathcal{H}(u, u') - F(t, u),
\] (15)

by (14) the following conservation law holds, for any \( s \in (T, T + \tau) \), provided that \( u(t) \neq 0 \) for all \( t \in (T, T + \tau) \):

\[
E(t) = E(s) - \int_s^t \{ \langle Q(\sigma, u, u'), u' \rangle + F_t(\sigma, u) \} d\sigma, \quad T < s \leq t < T + \tau.
\] (16)

Concerning the existence of local solutions of the initial value problem

\[
\left[ \nabla G(u, u') \right]' - \nabla u G(u, u') + Q(t, u, u') = f(t, u), \quad t > T,
\]

\[
u(T) = u_0 \neq 0, \quad u'(T) = 0, \quad T \geq 0,
\] (17)

we recall that the problem is extremely delicate since in general the damping term might be not defined for \( t = T \), indeed, for instance, in the variational case we have \( T = 0 \) and \( Q(t, u, v) = (n - 1)\nabla G(u, v)/t \).

However in [1] it is proved the following result.

**Theorem 1** [1, Theorem 1]. Assume that

(A4) there exist two nonnegative functions \( \delta \in C(J) \) and \( \varphi \in C(D \times \mathbb{R}^N) \), with \( \varphi(u, 0) = 0 \) for all \( u \in D \), such that \( |Q(t, u, v)| \leq \delta(t) \varphi(u, v) \) for \( (t, u) \in J \times D \) and \( v \) sufficiently small.

Suppose that there exists finite limit \( \lim_{t \to T^+} e^{-\int_t^{T+1} \delta(s) ds} := c_0 \in \mathbb{R} \), and that the function

\[
h(t) = e^{-\int_t^{T+1} \delta(s) ds} - c_0, \quad t \in (T, T + 1], \quad h(T) = 0,
\]

is of class \( C^1[J, T + 1] \).

Then the initial value problem (17) admits a local solution defined on \( [T, T + \tau) \), for some \( 0 < \tau \leq 1 \). Furthermore, \( |u(t)| > 0 \) for all \( t \in [T, T + \tau) \).

On the other hand, if we consider \( T_1 > T \) as the initial point of (17), namely

\[
\left[ \nabla G(u, u') \right]' - \nabla u G(u, u') + Q(t, u, u') = f(t, u),
\]

\[
u(T_1) = u_0 \neq 0, \quad u'(T_1) = 0, \quad T_1 > T.
\] (18)

we immediately obtain, by using the classical Cauchy theorem, the following theorem.

**Theorem 2** [1, Theorem 2]. The initial value problem (18) admits at least a solution defined on \( (T_1 - \tau, T_1 + \tau) \), for some \( \tau > 0 \). Furthermore, \( |u(t)| > 0 \) for all \( t \in (T_1 - \tau, T_1 + \tau) \).
Finally, we remind a crucial result proved in [1] which asserts that every local solution of (17), having negative initial energy, that is $E(T) < 0$, remains in norm far from zero.

More specifically, if we require the additional condition

\[(A2)' \quad \text{there exist functions } \tilde{F} \in C(\mathbb{R}^N; [0, +\infty)) \text{ and } \phi \in L^1(J; [0, +\infty)) \text{ such that } \tilde{F}(0) = 0, \]
\[\tilde{F}(u) > 0 \text{ if } u \neq 0 \text{ and } 0 \leq F_t(t, u) \leq \phi(t) \tilde{F}(u) \text{ for all } (t, u) \in J \times \mathbb{R}^N; \]

then the following result holds.

**Proposition 1.** Assume that $(A2)'$ and $(A4)$ hold. Let $u : [T, T_1) \to D$, $T \leq \infty$ be a solution of (17) in $(T, T_1)$. Then if $F(T, u_0) > 0$, there exists a positive constant $U$ which depends only on $F(T, u_0)$ such that

$$|u(t)| \geq U \text{ for all } t \in [T, T_1).$$

(19)

**Remark.** Note that $F(T, u_0) > 0$ implies that a solution $u$ of (17) has negative initial energy, indeed by (12) and (15), we have $E(T) = H(u_0, 0) - F(T, u_0) = -F(T, u_0) < 0$.

### 3. Global existence

In this section we shall be concerned with the existence of **global solutions** of the initial value problem (17). In particular, we prove, under suitable assumptions, that all local solutions of (17), which exist by Theorem 1, can be continued in the entire $[T, \infty)$. We assume from now on that the functions $F, G,$ and $Q$ satisfy the standing hypotheses $(A1), (A2)'$, $(A3)$, and $(A4)$ with $D = \mathbb{R}^N$ or $D = \mathbb{R}^N \setminus \{0\}$. Moreover, we shall assume from now on the following additional structural hypotheses:

(R1) there exist a constant $C > 0$ and an exponent $p > 1$ such that

$$|f(t, u)| \leq C|u|^{p-1} \quad \text{for all } t \in J \text{ and } u \in \mathbb{R}^N;$$

(R2) $\mathcal{H}(u, v) \to \infty$ as $|v| \to \infty$ uniformly for $u$ in compact sets which do not contain the origin, where $\mathcal{H}$ is defined in (11).

Since the technique used to prove continuation theorems below is based on Gronwall’s inequality, we remind it for the sake of completeness.

**Theorem 3** (Gronwall’s inequality). Let $\psi$ be a nonnegative, absolutely continuous function on $I = [t_0, t_1]$, $0 \leq t_0 < t_1 < \infty$, which satisfies for a.e. $t$ the differential inequality

$$\psi'(t) \leq \beta(t)\psi(t) + \alpha(t),$$

where $\beta, \alpha$ are nonnegative, summable functions on $I$. Then

$$\psi(t) \leq e^{\int_{t_0}^{t} \beta(s) \, ds} \left[ \psi(t_0) + \int_{t_0}^{t} \alpha(s) \, ds \right] \quad \text{for a.e. } t \in I.$$
Finally, we point out that, as we explained in the Introduction, global existence holds in general if either the damping term $Q$ or the action energy $G$ dominates over the driving force $F$. These two cases will be treated in Theorems 4 and 5 below, respectively. The proof uses ideas of [4,5,10].

**Theorem 4.** Assume that there exist exponents $m > 1$, $k \in \mathbb{R}$, and a measurable function $\delta_1 : J \to \mathbb{R}_0^+$ such that

$$\left\langle Q(t, u, v), v \right\rangle \geq \delta_1(t) |u|^k |v|^m$$  \hspace{1cm} \text{for all } (t, u, v) \in J \times D \setminus \{0\} \times \mathbb{R}^N,  \hspace{1cm} (20)$$

$$\delta_1^{-1/(m-1)} \in L^1_{\text{loc}}(J).  \hspace{1cm} (21)$$

If $p - m \leq \kappa$, then every local solution $u : [T, T + \tau) \to D$ of (17) can be continued in the entire $[T, \infty)$. 

**Proof.** Let $u$ be a local solution of (17), whose existence is guaranteed by Theorem 1. Denote by $J_T = [T, T_1)$ the maximal interval of existence of $u$.

**Step 1.** We claim that there exists $U_0 > 0$ such that

$$|u(t)| \geq U_0 \text{ for all } t \in J_T.  \hspace{1cm} (22)$$

To prove this we use Proposition 1. Hence, we need to find a point $T_2 \geq T$ such that $F(T_2, u(T_2)) > 0$. Assume for contradiction that $E(t) \equiv 0$ for all $t \in J_T$. Then $E'(t) \equiv 0$ for all $t \in (T, T_1)$ and thus again from (14) we obtain

$$\left\langle Q(t, u, u'), u' \right\rangle + F_t(t, u) \equiv 0 \text{ for all } t \in (T, T_1).$$

In turn, from (A2)' and (A3) we get

$$F_t(t, u) \equiv 0, \quad \left\langle Q(t, u, u'), u' \right\rangle \equiv 0 \text{ for all } t \in (T, T_1).  \hspace{1cm} (23)$$

By (20) and (21) this implies that $u(t) \equiv u_0$ in $J_T$. Indeed, by (21) we obtain that $\delta_1(t) > 0$ for a.e. $t \in J_T$. By (20) and since $u(T) = u_0 \neq 0$, it follows

$$|u|^k |u'|^m \equiv 0$$  \hspace{1cm} (24)

as long as $u(t) \neq 0$ (and thus at least for $t$ near $T$). But (24) clearly implies $u'(t) \equiv 0$ and so necessarily $u(t) \equiv u_0$ in $J_T$. This in turn contradicts the fact that $J_T$ is the maximal interval of existence of $u$, as one can consider the initial value problem (18) with $v_0 = 0$ and continue $u$ by Theorem 2.

Hence there exists $T_2 \in [T, T_1)$ such that $E(T_2) \neq 0$. But since

$$E(T) = H(u_0, 0) - F(T, u_0) = -F(T, u_0) \leq 0,$$

by (A2) and from the fact that $E'(t) \leq 0$ it follows that $E(T_2) < 0$, consequently by (12) and (15) we obtain $F(T_2, u(T_2)) \geq -E(T_2) > 0$. Thus we may apply Proposition 1 to obtain (22).
Step 2. We claim that $T_1 = \infty$. Thus assume by contradiction that $T_1 < \infty$.

Consider the auxiliary function

$$
\psi(t) = H(u, u') + |u|^p,
$$

(25)

where $H$ is the function given in (11). We observe that $\psi$ can be written as follows: $\psi(t) = F(t, u) + |u|^p + E(t)$. Since $|u(t)| \geq U_0 > 0$ on $J_T$ by (22), we can calculate the derivative of $\psi$ by using (13), obtaining

$$
\psi'(t) = \left[ f(t, u) \cdot u' + p|u|^{p-1}u' \right] + \left[ Q(t, u, u') - \langle Q(t, u, u'), u' \rangle \right], \quad t \in (T, T_1).
$$

(26)

We claim that $\psi$ is bounded on $J_T$ independently of $T_1$. First, observe that from (R1) it follows that, for $t \in (T, T_1)$,

$$
\left| f(t, u) \cdot u' \right| + p|u|^{p-1}|u'| \leq (C + p)|u|^{p-1}|u'|
$$

$$
= \left[ \delta_1(t) \right]^{1/m}|u|^{m}|u'|^{m}(C + p)|u|^{(p-m-\kappa)}|\delta_1(t)|^{-1/m}|u|^{p/m'}
$$

$$
\leq \delta_1(t)|u|^{m} + (C + p)^{m'}|u|^{(p-m-\kappa)/(m-1)}\left[ \delta_1(t) \right]^{-1/(m-1)}|u|^p,
$$

(27)

by Young’s inequality with exponents $m$ and $m' = m/(m - 1)$. Consequently, from (22) and since $p - m \leq \kappa$, we obtain

$$
|u(t)|^{(p-m-\kappa)/(m-1)} \leq U_0^{(p-m-\kappa)/(m-1)} \quad \text{for all } t \in (T, T_1),
$$

thus inequality (27) becomes

$$
\left| f(t, u) \cdot u' \right| + p|u|^{p-1}|u'| \leq \delta_1(t)|u|^m + C_2\left[ \delta_1(t) \right]^{-1/(m-1)}|u|^p,
$$

(28)

with $C_2 = (C + p)^{m'}U_0^{(p-m-\kappa)/(m-1)}$. Therefore, by (26), (20), and the fact that, by (12), $\psi(t) \geq |u|^p$ in $J_T$, the inequality above implies that for $t \in (T, T_1)$,

$$
\psi'(t) \leq \delta_1(t)|u|^m + C_2\left[ \delta_1(t) \right]^{-1/(m-1)}|u|^p - \langle Q(t, u, u'), u' \rangle
$$

$$
\leq C_2\left[ \delta_1(t) \right]^{-1/(m-1)}|u|^p \leq C_2\left[ \delta_1(t) \right]^{-1/(m-1)}\psi(t).
$$

(29)

Finally, by (21), we can apply Gronwall’s inequality with $\beta(t) = C_2\left[ \delta_1(t) \right]^{-1/(m-1)}$ and $\alpha(t) \equiv 0$, to obtain that

$$
\psi(t) \leq \psi(s)e^{\int_s^t \beta(\tau)\,d\tau} \quad \text{for all } T < s \leq t < T_1.
$$

(30)

Hence we conclude that $\psi$ is bounded in $J_T$ independently of $T_1$. This implies that both $H(u, u')$ and $|u|$ are bounded on $J_T$. Furthermore, by (R2) we derive that also $|u'|$ is bounded on $J_T$. Now, by Cauchy’s theorem, there exists $u_1 \in D$, with $u_1 \neq 0$, thanks to (22), such that

$$
\lim_{t \to T_1^-} u(t) = u_1.
$$

(31)

By (17), for $T < s \leq t < T_1$,

$$
\nabla G(u, u') = \int_s^t \nabla_u G(u, u')\,d\tau + \int_s^t Q(\tau, u, u')\,d\tau - \int_s^t f(\tau, u)\,d\tau
$$

$$
= \nabla G(u(s), u'(s)),
$$

(32)
and so $\nabla_u G(u, u')$, $Q(t, u, u')$, and $f(t, u)$ are bounded on $[s, T_1]$ by continuity and by the boundness of $t$, $|u|$, and $|u'|$. Hence, there exists $v_1 \in \mathbb{R}^N$ such that

$$\lim_{t \to T_1^-} \nabla G(u, u') = v_1. \quad (32)$$

Therefore, since $u'$ is bounded and $\nabla G(u_1, \cdot)$ is one-to-one by (A1), we obtain that

$$\lim_{t \to T_1^-} u'(t) = v_1 = \left(\nabla G(u_1, \cdot)\right)^{-1}(v_1). \quad (33)$$

Now, if we consider the initial value problem (18) with $u_0 = u_1$ and $v_0 = v_1$, we obtain that $u$ can be continued to the right beyond $T_1$, thanks to Theorem 2. Hence $J_T$ is not the maximal interval of existence of the solution and this proves the claim. □

Next we treat the case when the action energy dominates over the driving force $F$.

**Theorem 5.** Assume that, for every $V > 0$ there exist exponents $\gamma \in \mathbb{R}$ and $l > 1$, with

$$p - l \leq \gamma, \quad (33)$$

and a constant $\Theta > 0$ such that assumption (R2) is strengthened as follows:

$$H(u, v) \geq \Theta |u|^\gamma |v|^l \quad (34)$$

for all $(u, v) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|u|, |v| \geq V$.

Let $u : [T, T + \tau) \to \mathbb{R}^N$ be a local solution of problem (17). If $0 \in D$, namely $D = \mathbb{R}^N$, then $u$ can be continued for all $t \in J$, while if $0 \notin D$, namely $D = \mathbb{R}^N \setminus \{0\}$, then either $u$ can be continued for all $t \in J$ or there exists $T_1 > T$ such that

$$\lim_{t \to T_1^-} u(t) = 0, \quad (35)$$

so that the maximal interval of existence of the solution is $[T, T_1)$.

**Proof.** Let $u$ be a local solution of (17). As in the previous theorem, denote by $J_T = [T, T_1)$ the maximal interval of existence of $u$. Now consider two different situations.

**Case 1.** If

$$\lim_{t \to T_1^-} |u(t)| > 0,$$

then we get that there exists $U > 0$ such that $|u(t)| \geq U$ for $t$ near $T_1$ and by regularity we can deduce that

$$|u(t)| \geq U \quad \text{for all } t \in J_T. \quad (36)$$

Let $\gamma, l, \Theta$ be the corresponding numbers in (33) and (34). Consider the same auxiliary function defined in (25) and observe that here

$$\psi(t) \geq H(u, u') \quad \text{for all } t \in J_T. \quad (37)$$

Now by (26), (R1), and (A3), we get that

$$\psi'(t) = \left[f(t, u, u') + p|u|^{p-2}(u, u') - \left(Q(t, u, u'), u'\right)\right] \leq (C + p)|u|^{p-1}|u'|, \quad (38)$$

by the Cauchy–Schwarz inequality.
Let $J_1 = \{ t \in J_T : |u'(t)| \leq |u(t)| \}$ and $J_2 = \{ t \in J_T : |u'(t)| \geq |u(t)| \}$. If $t \in J_1$, from (38) and the fact that $\psi(t) \geq |u|^p$ in $J_T$, we immediately deduce that $\psi'(t) \leq (C + p)\psi(t)$. In the other case when $t \in J_2$, we check that
\[
|u|^p - |u'| \leq |u|^{p-1} |u'| = |u|^{p-1-y} |u'|^y \leq U^{p-l-y} |u|^y |u'|^l,
\]
by (33) and (36). Furthermore, $|u'| \geq |u| \geq U$ in $J_2$, hence by (34), and using (39) and (37), we get
\[
|u|^p - |u'| \leq C_1 \mathcal{H}(u, u') \leq C_1 \psi(t), \quad t \in J_2, \quad C_1 = U^{p-l-y}/\Theta.
\]
Thus, in both cases, we get $\psi'(t) \leq \beta \psi(t)$, $\beta = (C + p) \min\{1, U^{p-l-y}/\Theta\}$, $t \in J_T$. In turn, by Gronwall’s inequality for all $t \in J_T$, we have $\psi(t) \leq \psi(s) e^{(t-s)/\beta}$ for all $T < s \leq t < T_1$, namely the analogous inequality given in (30) of Theorem 4. Now the proof proceeds as in Theorem 4 since assumption (34) forces the validity of (R2).

**Case 2.** If
\[
\liminf_{t \to T_1^-} |u(t)| = 0,
\]
then we shall show that (35) holds. To obtain this it is enough to apply the same argument used in [4, Lemma 5.3]. More specifically, assume for contradiction that
\[
\limsup_{t \to T_1^-} |u(t)| > 0,
\]
consequently by (40) it is possible to find a sequence of intervals $I_n = (a_n, b_n) \subset [T, T_1)$, with $a_n < b_n \leq a_{n+1}$ and $b_n \to T_1^-$, and a positive number $\epsilon$ such that
\[
|u(a_n)| = \frac{\epsilon}{2}, \quad |u(b_n)| = \epsilon, \quad \text{and} \quad \frac{\epsilon}{2} \leq |u(t)| \leq \epsilon, \quad t \in I_n.
\]
Now, by (A2), (A3), (14), and (15) we obtain that
\[
\lim_{t \to T_1^-} E(t) = \inf_{t \in [T, T_1)} E(t) = \ell \in (\infty, E(T)).
\]
Indeed, we can exclude the case $\ell = \infty$, since $E(t) \geq -F(t, u) \geq -\text{Const.} \epsilon^p$, by virtue of (12) and (R1). In turn, since $H = E + F$, we get that $H$ is bounded on $I_n$, thus by (R2) there exists $L = L(\epsilon) > 0$ such that $|u'(t)| \leq L$ for all $t \in I_n$ and $n \in \mathbb{N}$. Hence
\[
0 < \frac{\epsilon}{2} \leq |u(b_n)| - |u(a_n)| = \int_{a_n}^{b_n} |u'(t)| \, dt \leq L(b_n - a_n),
\]
from which we derive that $T - T_1 \geq \sum_{n=1}^{\infty} \epsilon/2L = \infty$ which is a contradiction and completes the proof.

Consequently we have two possibilities, namely if $0 \in D$, then the solution $u$ can be continued to all $J$ and it remains zero for all $t \geq T_1$, otherwise if $0 \notin D$, then the solution stops to exist at the finite time $t = T_1$. \qed
Corollary 5. Assume that assumptions (33) and (34) hold. If \( F(T, u_0) > 0 \), then every local solution \( u : [T, T + \tau) \to \mathbb{R}^N \) of problem (17) can be continued in the entire \([T, \infty)\).

Proof. It is enough to note that, by Proposition 1, when \( F(t, u_0) > 0 \) every local solution \( u \) satisfies condition (36). Consequently, Case 1 of the proof of Theorem 5 occurs and thus the corollary follows at once.

4. Global existence of radial solutions

We now apply the results of the previous Section 3 to obtain the existence of radial entire solutions of elliptic systems of the general form

\[
\text{div}(g(u)A(|\nabla u|)\nabla u) - \nabla u g(u)A(|\nabla u|) = f(|x|, u), \quad x \in \mathbb{R}^n, \quad (44)
\]

where the functions \( A, f, \) and \( g \) satisfy throughout the section the following assumptions:

(C1) \( A : \mathbb{R}^+ \to \mathbb{R}^+ \) is continuous, \( s \mapsto sA(s) \) is strictly increasing in \( \mathbb{R}^+ \), and \( \lim_{s \to 0^+} sA(s) = 0 \);

(C2) \( g : D \to \mathbb{R}^+_0 \) is of class \( C^1 \), where \( D = \mathbb{R}^N \) or \( D = \mathbb{R}^N \setminus \{0\} \);

(C3) there exists \( F \in C^1(\mathbb{R}^+_0 \times \mathbb{R}^N; \mathbb{R}^+_0) \), with \( F(r, 0) = 0 \) for all \( r \geq 0 \), such that \( \nabla_u F(r, u) = F(r, u) \) and \( F_r(r, u) \geq 0 \) for all \( (r, u) \in \mathbb{R}^+_0 \times \mathbb{R}^N \);

(C4) there exist \( \tilde{F} \in C(\mathbb{R}^N; \mathbb{R}^+_0) \) and \( \phi \in L^1(\mathbb{R}^+_0; \mathbb{R}^+_0) \) such that \( \tilde{F}(0) = 0 \), \( \tilde{F}(u) > 0 \) if \( u \neq 0 \) and \( 0 \leq F_r(r, u) \leq \phi(r)\tilde{F}(u) \) for all \( (r, u) \in \mathbb{R}^+_0 \times \mathbb{R}^N \).

Note that since \( A(s) = \int_0^s \sigma A(\sigma) d\sigma \), then \( A \in C^1(\mathbb{R}^+_0; \mathbb{R}) \) by (C1).

Theorem 6. Assume that (R1) holds in the new setting, namely that there exist a positive constant \( C > 0 \) and an exponent \( p > 1 \) such that

\[
|f(r, u)| \leq C|u|^{p-1} \quad \text{for all } r \in \mathbb{R}^+_0 \text{ and } u \in \mathbb{R}^N. \quad (45)
\]

Suppose that there exist an exponent \( m > 1 \) and a positive constant \( b_1 \) such that

\[
s^2A(s) \geq b_1s^m \quad \text{for all } s > 0. \quad (46)
\]

Finally, assume that there exist an exponent \( \gamma \in \mathbb{R} \) and a positive constant \( b_2 \) such that

\[
g(u) \geq b_2|u|^\gamma \quad \text{for all } u \in \mathbb{R}^N \setminus \{0\}. \quad (47)
\]

If

\[
\gamma \geq p - m, \quad (48)
\]

then (44) admits a one parameter family of nontrivial entire radial solutions.

Remark. As showed by Naito and Usami in [6], the following inequality holds:

\[
H(s) + \int_0^1 sA(s) ds = s^2A(s) - \int_1^s \sigma A(\sigma) d\sigma \geq sA(s), \quad s > 1, \quad (49)
\]
where \( H(s) = s^2A(s) - A(s) \), \( s > 0 \), is the Legendre transform of \( A(s) = \int_0^s \sigma A(\sigma) d\sigma \).
Thus, if \( sA(s) \to \infty \) as \( s \to \infty \) then, by (49),
\[
\lim_{s \to \infty} H(s) = \infty. \tag{50}
\]
Consequently we get that if \( s \to \infty \), then (46) implies (50).

**Proof.** As noted in the Introduction, every radial solution of (44) satisfies the initial value problem
\[
\left[ g(u)A(|u'|)u' \right]' + \frac{n-1}{r} g(u)A(|u'|)u' - g'(u)A(|u'|) = f(r,u), \quad r > 0,
\]
\( u(0) = u_0 \neq 0, \quad u'(0) = 0, \tag{51} \]
which is a special case of (17). Conditions (A1), (A2)', (A3), and (A4) follow immediately from hypotheses (C1)--(C3). In particular, to satisfy (A4) we take
\[
\delta(r) = \frac{(n - 1)b_1b_2}{r}
\]
and
\[
\phi(u,v) = g(u)A(|v|). \tag{52}
\]
By Theorem 1, the initial value problem (51) admits a local solution
\( u : [0, R) \to \mathbb{R}^N \setminus \{0\}, \quad R > 0. \)

To prove that \( R = \infty \), we shall show that the hypotheses of Theorem 4 are verified.
Assumption (R2) follows immediately from (46), (50), and (47). Indeed, if \( K \subset \mathbb{R}^N \setminus \{0\} \) is a compact set, then
\[
H(u,v) = g(u)H(|v|) \geq \min_{u \in K} b_2 |u|^\gamma |v|^m
\]
for all \( u \in K \) and \( v \in \mathbb{R}^N \). So that (20) holds with
\[
\kappa = \gamma \quad \text{and} \quad \delta_1(r) = \frac{(n - 1)b_1b_2}{r^{1/(m-1)}},
\]
hence \( \delta_1 \in L^1_{\text{loc}}(\mathbb{R}^+) \). Consequently, Theorem 4 can be applied to (51). \( \square \)

**Remark.** In spite of the fact that we cannot immediately apply Proposition 1, because of the fact that we do not assume \( F(T,u_0) > 0 \) in Theorem 6 above, we obtain the same assertion of Proposition 1, namely that
\[
|u(r)| \geq U_0 \quad \text{for all} \quad r \in \mathbb{R}_0^+,
\tag{52}
\]
for some \( U_0 > 0 \), by virtue of Step 1 of the proof of Theorem 4.

We now study the asymptotic behavior of the global radial solutions of (44). In particular, we obtain the following result.

**Theorem 7.** Assume that all the hypotheses of Theorem 6 hold. Suppose in addition that there exists a constant \( a_1 > 1 \) such that
\[
s^2A(s) \geq a_1A(s) \quad \text{for all} \quad s > 0. \tag{53}
\]

Finally, suppose also that for every $U > 0$, there exists a positive constant $a_2$ such that
\begin{align}
\langle f(r,u), u \rangle & \geq a_2 \quad \text{for all } (r,u) \in \mathbb{R}_0^+ \times \mathbb{R}^N, \ |u| \geq U.
\end{align}
(54)
\begin{align}
a_1 g(u) + \langle \nabla_u g(u), u \rangle & \geq 0 \quad \text{for all } u \in \mathbb{R}^N, \ |u| \geq U.
\end{align}
(55)
Let $u$ be a radial global solution of (44), then
\begin{align}
\lim_{|x| \to \infty} |u(|x|)| = \infty.
\end{align}
(56)

**Proof.** Consider the auxiliary function $Z(r) = g(u) A(|u'|) (u', u)$, $r \in \mathbb{R}_0^+$. We claim that
\begin{align}
Z(r) > 0 \quad \text{for all } r > 0.
\end{align}
(57)
Indeed, by using (51) and (53), we have
\begin{align}
Z'(r) & \geq \left[ a_1 g(u) + \langle \nabla_u g(u), u \rangle \right] A(|u'|) + \langle f(r,u), u \rangle - \frac{n-1}{r} Z(r).
\end{align}
(58)
Now note that by (52) we can apply (54) and (55), obtaining from (58)
\begin{align}
Z'(r) + \frac{n-1}{r} Z(r) & \geq \left[ a_1 g(u) + \langle \nabla_u g(u), u \rangle \right] A(|u'|) + a_2 \geq a_2,
\end{align}
(59)
or equivalently $[r^{n-1} Z(r)]' \geq a_2 r^{n-1} > 0$ for all $r > 0$. Thus $r^{n-1} Z(r)$ is strictly increasing for all $r > 0$. Since $Z(0) = 0$, we have $Z(r) > 0$ if $r > 0$, and the claim (57) is proved.

Now, to prove (56), first we shall show that
\begin{align}
\limsup_{r \to \infty} |u(r)| = \infty.
\end{align}
(60)
Hence assume for contradiction that
\begin{align}
\limsup_{r \to \infty} |u(r)| < \infty.
\end{align}
(61)
Consequently, there exists $U_1 > 0$ such that $|u(r)| \leq U_1$ for all $r > 0$, thus, by (52), it follows that $|u(r)| \in [U_0, U_1]$ for all $r > 0$. Since $F(r,u) = \int_0^1 \langle f(r,tu), u \rangle \, dt$, and by (R1) and the Cauchy–Schwarz inequality, it follows that $|F(r,u)| \leq C |u|^p \leq C U_1^p$. Consequently, from the fact that
\begin{align}
E(0) & \geq E(r) = H(u,u') - F(r,u) \geq - F(r,u) \geq - C |u|^p \geq - C U_1^p,
\end{align}
we deduce that, since $F$ is bounded, then also $H$ must be bounded along the solutions. Hence by (R2), which holds in this setting thanks to (50) and (47), there exists $L_1 > 0$ such that $|u'(r)| \leq L_1$ for all $r \geq 0$. Consequently, the boundness of both $|u|$ and $|u'|$ implies, by the Cauchy–Schwarz inequality and by (C1), that $Z(r) \leq C_0$, $C_0 > 0$, for all $r > 0$. Thus $\lim_{r \to \infty} (n-1) Z(r)/r = 0$ and by (59) and (61), we get that $\liminf_{r \to \infty} Z'(r) \geq C_1$, namely $Z'(r) \geq C_2$ when $r \geq r_2$, for $r_2$ sufficiently large. Consequently, by integration and thanks to (57),
\begin{align}
Z(r) & \geq C_2 (r - r_2) + Z(r_2) > C_1 (r - r_2), \quad r \geq r_2,
\end{align}
which implies that $Z(r) \to \infty$ as $r \to \infty$. This contradicts the boundness of $Z$, thus (61) cannot occur and this proves (60).
Finally, by virtue of (C1), (C2), and (57), we deduce that \( d/dr(|u|^2)/2 = \langle u, u' \rangle > 0 \) for all \( r > 0 \). This implies that the function \( |u|^2 \) is increasing, thus \( |u| \) tends to a limit at infinity, namely \( \lim_{r \to \infty} |u(r)| = \ell \in (0, \infty] \). By (60), we immediately deduce that \( \ell = \infty \) and consequently (56) is proved. \( \square \)

Now we apply the results above to two model elliptic cases. Precisely we obtain the following corollary.

**Corollary 6.** Assume that \( m > 1, \ p > 1, \) and \( \gamma \geq p - m \). Then the elliptic system

\[
\text{div}(|u|^\gamma |\nabla u|^{m-2}\nabla u) - \frac{\gamma}{m} |u|^{p-2}u|\nabla u|^m = |u|^{p-2}u
\]

admits a one parameter family of nontrivial entire radial solutions \( u : \mathbb{R}^+ \to \mathbb{R}^N \setminus \{0\} \) such that \( \lim_{|x| \to \infty} |u(|x|)| = \infty \), namely (62) admits entire radial solutions.

**Proof.** It is enough to apply Theorems 6 and 7 with \( A(s) = s^{m-2} \) and \( g(u) = |u|^\gamma \). Indeed, (46)–(47) are trivially satisfied with \( b_1 = b_2 = 1 \), while \( a_1 = m \) in (53) of Theorem 7 so that (55) holds since \( \gamma + m \geq p > 1 \). \( \square \)

**Corollary 7.** Assume that \( 1 < m \leq 2, \ p > 1, \) and \( \gamma \geq p - m \). Then the elliptic system

\[
\text{div}(|u|^\gamma (1 + |\nabla u|^2)^{(m-2)/2} - \frac{\gamma}{m} |u|^{p-2}u(1 + |\nabla u|^2)^{(m-2)/2} - 1) = |u|^{p-2}u
\]

admits a one parameter family of nontrivial entire radial solutions \( u : \mathbb{R}^+ \to \mathbb{R}^N \setminus \{0\} \) such that \( \lim_{|x| \to \infty} |u(|x|)| = \infty \), namely (63) admits entire radial solutions.

**Proof.** As in the corollary above, we apply Theorems 6 and 7 with \( A(s) = (1 + s^2)^{m/2-1} \) and \( g(u) = |u|^\gamma \). Consequently \( b_1 = b_2 = 1 \) in (46) and (47), respectively. Moreover, (53) of Theorem 7 holds with \( a_1 = m \). Indeed, by direct calculation (see [1, Corollary 7]),

\[
mA(s) = (1 + s^2)^{m/2} - 1 \leq (1 + s^2)^{m/2-1} - 1 = A(s)s^2,
\]

and so multiply by \((1 + s^2)^{-m/2+1}\) and use the facts that \( s \geq 0 \) and \(-m/2 + 1 \geq 0\).

Finally, (55) holds since \( \gamma + m \geq p > 1 \). \( \square \)

If we now combine Corollary 6 with [1, Corollary 12], which continue to hold for every \( \gamma < p - m \), without further restrictions on \( p \) and \( m \), we obtain the following result.

**Corollary 8.** The elliptic inequality

\[
\text{div}(|u|^\gamma |\nabla u|^{m-2}\nabla u) - \frac{\gamma}{m} |u|^{p-2}u|\nabla u|^m \geq |u|^{p-2}u, \quad p, m > 1,
\]

admits entire solutions if and only if \( \gamma \geq p - m \).

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Appendix A

Here we shall see in detail what announced in the Introduction, namely the fact that the problem of existence and nonexistence of solutions of (4) is strictly related to the competition between the terms \(|u|m^{2-2} u|\) and \(|u|p-2 u|\). Consider the simple special case of (4) when \(N = n = 1\), that is the initial value problem

\[
[u|m^{2-2} u'] = [u|p-2 u, \quad r > 0, \quad u(0) = u_0 > 0, \quad u'(0) = 0. \tag{A.1}
\]

Let \(J_R = [0, R), 0 < R \leq \infty\), denote the maximal interval of existence a solution \(u\) of (A.1). We claim that

\[
u(r) > u_0, \quad u'(r) > 0, \quad r \in (0, R), \tag{A.2}
\]

Indeed, since \(u(0) = u_0 > 0\), there exists \(r_0 \in (0, R)\) such that \(u(r) > 0\) for all \(r \in [0, r_0)\). In turn \([u|m^{2-2} u'] = u^{p-1} > 0\) for all \(r \in (0, r_0)\). Consequently, the function \([u|m^{2-2} u']\) is strictly increasing and since \(u'(0) = 0\) then \([u|m^{2-2} u'] > 0\) for all \(r \in (0, r_0)\). Hence \(u'(r) > 0\) for all \(r \in (0, r_0)\). Now let \(r_1 \in (0, R)\) be the first \(r > r_0\) such \(u'(r) > 0\) in \((0, r_1)\) and \(u'(r_1) = 0\). Integrating (A.1) from 0 to \(r_1\), we get a contradiction, namely 0 = \([u'(r_1)]m^{2-1} - [u'(0)]m^{2-1}\int_{0}^{r_1} u^{p-1} ds > 0\). Hence (A.2) is proved.

Consequently, by direct integration of (A.1) from 0 to \(r > 0\), one has

\[
m - 1 \frac{1}{m}(u')^m = \frac{u^p}{p} - \frac{u_0^p}{p}, \tag{A.3}
\]

which can be written as

\[
u^{- \frac{p}{m}} u' = c \left[1 - \left(\frac{u_0}{u}\right)^p\right]^{1/m}, \quad r \geq 0, \quad c = \left(\frac{m}{p(m-1)}\right)^{1/m}. \tag{A.4}
\]

Since \(u\) is strictly increasing for \(r \geq r_2 > 0\), we have \(u_0/u(r) \leq 1 - \varepsilon, \varepsilon \in (0, 1)\), and so \(u^{- \frac{p}{m}} u' \geq c \varepsilon^{1/m}\) for all \(r \geq r_2\). By integration and using (A.2), we get

\[
m \frac{m}{m-p} [u(r)]^{(m-p)/m} - m \frac{m}{m-p} [u(r_2)]^{(m-p)/m} \geq c \varepsilon^{1/m}(r - r_2), \quad r \geq r_2,
\]

and if \(p > m\), we deduce

\[
m [u(r_2)]^{(m-p)/m} \geq c(m-p) r^{1/m} (r - r_2), \quad r \geq r_2.
\]

Thus \(r\) cannot be \(\infty\) if \(p > m\). More precisely, we obtain that any solution of (A.1) cannot be continued to the entire \(\mathbb{R}^+_0\), indeed both \(u\) and \(u'\) approach \(\infty\) at some finite point \(r^* > 0\), see (A.3).

If \(1 < p \leq m\) from (A.4) and from the fact that \(u' > 0\) we deduce that \(u^{- \frac{p}{m}} u' \leq c\). Thus, if \(p < m\), by integration, we get

\[
u(r) \geq u_0^{(m-p)/m} + c \left(1 - \frac{p}{m}\right) r, \quad r > 0,
\]

while if \(p = m\) we get \(u(r) \leq u_0 e^{cr}, r > 0\). Hence in both cases we obtain that \(u\) and \(u'\) are bounded in every bounded set, by using also (A.3). This implies that \(u\) can be continued to the entire \(\mathbb{R}^+_0\) if \(1 < p \leq m\).

Thus we have proved that, global solutions of (A.1) exist if and only if \(p \leq m\).
References