Nonexistence of positive weak solutions of elliptic inequalities

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ABSTRACT

In this paper we give sufficient conditions for the nonexistence of positive entire weak solutions of coercive and anticoercive elliptic inequalities, both of the $p$-Laplacian and of the mean curvature type, depending also on $u$ and $x$ inside the divergence term, while a gradient factor is included on the right-hand side. In particular, to prove our theorems we use a technique developed by Mitidieri and Pohozaev in [E. Mitidieri, S.I. Pohozaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities, Proc. Steklov Inst. Math., 234 (2001) 1–362], which relies on the method of test functions without using comparison and maximum principles. Their approach is essentially based first on a priori estimates and on the derivation of an asymptotics for the a priori estimate. Finally nonexistence of a solution is proved by contradiction.

1. Introduction

The first part of this paper deals with the nonexistence of positive weak solutions of the inequalities

$$\text{div}(h(x)g(u)A(|Du|)Du) \geq f(x, u, Du),$$

$$\text{div}(g(u)A(|Du|)Du) - g'(u)A(|Du|) \geq f(x, u, Du),$$

in $\mathbb{R}^N$, $N \geq 2$, where $Du = (\partial_1 u, \ldots, \partial_N u)$ is understood in the sense of distributions. These inequalities, as in [1], concern the so-called coercive case. In the second part of the paper we treat the anticoercive case, namely

$$-\text{div}(h(x)g(u)A(|Du|)Du) \geq f(x, u, Du).$$

Throughout the paper we assume

$$A \in C(\mathbb{R}^+), \quad A > 0 \quad \text{in} \quad \mathbb{R}^+, \quad s \mapsto sA(s) \text{ is strictly increasing in} \quad \mathbb{R}^+;$$

$$\lim_{s \to 0^+} sA(s) = 0, \quad A(s) = \int_0^s \sigma A(\sigma) d\sigma, \quad s > 0; \quad (H)$$

$$g \in C^1(\mathbb{R}^+), \quad h \in C(\mathbb{R}^N \setminus \{0\}), \quad g > 0 \quad \text{in} \quad \mathbb{R}^+, \quad h > 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\};$$

$f$ is a nonnegative Caratheodory function in $\mathbb{R}^N \times \mathbb{R}^+_0 \times \mathbb{R}^N$ and there exist $q > 0$, $\theta \geq 0$ and a measurable function $a(x) \geq 0$ in $\mathbb{R}^N$ such that

$$f(x, z, \xi) \geq a(x)|z|^q|\xi|^{\theta} \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^+_0 \times \mathbb{R}^N,$$

where $\mathbb{R}^+_0 = [0, \infty)$ and $\mathbb{R}^+ = (0, \infty)$.
Important examples of operators $A$ are the $p$-Laplacian operator

$$A(s) = s^{p-2}, \quad s > 0, \: p > 1,$$

(1.4)

and the mean curvature operator

$$A(s) = \frac{1}{\sqrt{1 + s^2}}, \quad s \geq 0.$$  

(1.5)

The relevance of inequalities (1.1)-(1.3) has been widely recognized in recent years. Indeed there is a great number of papers dealing with nonexistence of entire solutions, namely solutions defined in all of $\mathbb{R}^N$, starting from the study of the semilinear case, see [2–9] and the references therein. Then for the quasilinear case we refer to [10–13], [14–20], and the references therein. For possible geometrical and physical models related to the semilinear case, see papers dealing with nonexistence of entire solutions, namely solutions defined in all of $\mathbb{R}^N$.

As in [1], we deal with the set

$$W^{1,\hat{p},0}_{\alpha,\text{loc}}(\mathbb{R}^N) = \{u : \mathbb{R}^N \to \mathbb{R} : a(x)|Du|^\hat{p}, u^\beta - u^\gamma \in L^1_{\text{loc}}(\mathbb{R}^N)\},$$

where $\alpha > 0$ is sufficiently large and $\beta \in \mathbb{R}$. Of course the standard solution space $C^1(\mathbb{R}^N)$ is trivially contained in $W^{1,\hat{p},0}_{\alpha,\text{loc}}(\mathbb{R}^N)$.

As a consequence of our theorems we have the following results.

**Corollary 1.1.** Let $\beta < 0$, $\gamma < p - \theta$ and

$$0 \leq \theta < p - 1, \quad \beta + p - 1 < q + \theta.$$ 

(1.7)

If $\theta > 0$, then problem

$$\text{div}(u^\beta|Du|^{p-2}Du) - \frac{\beta}{p} u^\beta - 1|Du|^p \geq a(x)u^\theta|Du|^\gamma \quad \text{in} \; \mathbb{R}^N,$$  

(1.8)

has no other positive solutions, except constants, in the class $W^{1,\hat{p},0}_{\alpha,\text{loc}}(\mathbb{R}^N)$.

While if $\theta = 0$ in (1.7), problem

$$\text{div}(u^\beta|Du|^{p-2}Du) - \frac{\beta}{p} u^\beta - 1|Du|^p \geq a(x)u^\theta \quad \text{in} \; \mathbb{R}^N,$$  

(1.9)

has no positive solutions in the class $W^{1,\hat{p},0}_{\alpha,\text{loc}}(\mathbb{R}^N)$. \hfill $\Box$

The case $\beta = 0$ of Corollary 1.1 is covered by Corollary 1.3 below when $\sigma = 0$. Moreover Corollary 1.1 extends Corollary 5 of [23] which gives the same nonexistence result but in the class of regular solutions and when $\theta = 0$. On the other hand, in Corollary 3 of [24] it is proved that inequality (1.9) admits entire regular solutions if and only if $q \leq \beta + p - 1$. 

From now on we assume

$$1 < p < N, \quad q > 0, \quad \theta \geq 0,$$

and that there exist constants $c_0, R_0 \in \mathbb{R}^+$ and an exponent $\gamma \in \mathbb{R}$ such that

$$a(x) \geq c_0|x|^{-\gamma} \quad \text{for all} \; x \text{ with} \; |x| \geq R_0.$$  

(1.6)
Concerning the mean curvature operator we have

**Corollary 1.2.** Let \( \beta < 0, \gamma < 2 - \theta \) and

\[
0 \leq \theta < 1, \quad \beta + 1 < q + \theta.
\]

If \( \theta > 0 \), then problem

\[
\text{div} \left( u^\beta \frac{Du}{\sqrt{1 + |Du|^2}} \right) - \frac{\beta}{p} u^{p-1} \left( \sqrt{1 + |Du|^2} - 1 \right) \geq a(x) u^\theta |Du|^\theta \quad \text{in } \mathbb{R}^N,
\]

has no other positive solutions, except constants, in the class \( W^{1,2(0)}_{a,loc}(\mathbb{R}^N) \).

While if \( \theta = 0 \) in (1.10), problem

\[
\text{div} \left( u^\beta \frac{Du}{\sqrt{1 + |Du|^2}} \right) - \frac{\beta}{p} u^{p-1} \left( \sqrt{1 + |Du|^2} - 1 \right) \geq a(x) u^\theta \quad \text{in } \mathbb{R}^N,
\]

has no positive solutions in the class \( W^{1,2(0)}_{a,loc}(\mathbb{R}^N) \). \( \square \)

Also here, the case \( \beta = 0 \) of Corollary 1.2 can be deduced by Corollary 1.4 below when \( \sigma = 0 \).

In the next results we include also a dependence on \( x \) inside the divergence.

**Corollary 1.3.** Let (1.6) and (1.7) hold. Suppose that \( \sigma < p - \theta - \gamma \). If \( \theta > 0 \), then problem

\[
\text{div}(\alpha^\sigma x^\alpha |Du|^{p-2}Du) \geq a(x) u^\theta |Du|^\theta \quad \text{in } \mathbb{R}^N,
\]

has no other positive solutions, except constants, in the class

\[
W^{1,(p,0)}_{a,\sigma,loc}(\mathbb{R}^N) = \{ u : \mathbb{R}^N \rightarrow \mathbb{R} : a(x)u^{\theta+a-1} |Du|^\theta, u^{\theta+a+\sigma} |Du|^{p-2} \in L^1_{loc}(\mathbb{R}^N) \}
\]

for \( \alpha > 0 \) sufficiently large.

While if \( \theta = 0 \), then problem

\[
\text{div}(\alpha^\sigma x^\alpha |Du|^{p-2}Du) \geq a(x) u^\theta \quad \text{in } \mathbb{R}^N,
\]

has no positive solutions in the class \( W^{1,(p,0)}_{a,\sigma,loc}(\mathbb{R}^N) \) for \( \alpha > 0 \) sufficiently large. \( \square \)

The above corollary extends Theorem 14.1 of [1] in which the case \( \sigma = \beta = \theta = 0 \) is treated.

**Corollary 1.4.** Let (1.10) hold and \( \sigma < 2 - \theta - \gamma \). If \( \theta > 0 \), then problem

\[
\text{div} \left( |x|^\sigma x^\alpha \frac{Du}{\sqrt{1 + |Du|^2}} \right) \geq a(x) u^\theta |Du|^\theta \quad \text{in } \mathbb{R}^N,
\]

has no other positive solutions, except constants, in the class \( W^{1,2(0)}_{a,\sigma,loc}(\mathbb{R}^N) \) for \( \alpha > 0 \) sufficiently large.

While if \( \theta = 0 \), then problem

\[
\text{div} \left( |x|^\sigma x^\alpha \frac{Du}{\sqrt{1 + |Du|^2}} \right) \geq a(x) u^\theta \quad \text{in } \mathbb{R}^N,
\]

has no positive solutions in the class \( W^{1,2(0)}_{a,\sigma,loc}(\mathbb{R}^N) \) for \( \alpha > 0 \) sufficiently large. \( \square \)

The strict bound for \( \theta \) given in (1.7), that is \( \theta < p - 1 \), or equivalently for \( p = 2 \) in (1.7), appears in the literature in other nonexistence results of entire solutions, see for instance [18] in which nonexistence of nonnegative entire solutions is studied for elliptic inequalities of the \( p \)-Laplacian type with no dependence on \( u \) inside the divergence but having possibly singular weights and a general gradient term on the right-hand side which could be both a nondecreasing function and a nonincreasing function of the gradient. The technique used in [18] is based on a change of variables introduced in [25] and then a comparison theorem yields the result. Of course the problem of nonexistence of entire solutions is strictly connected with the problem of existence of large solutions on bounded domains, namely solutions which explodes on the boundary. For instance, in Theorem 4.4 of [13], existence of nonnegative large solutions, is proved for the problem \( \text{div}(|Du|^{p-2}Du) = f(u) |Du|^p \) in bounded convex domains satisfying a uniform internal sphere condition again under the condition \( \theta < p - 1 \). A necessary condition for the nonexistence of entire solutions of these type of problems, namely of coercive type, is the well-known Keller–Osserman condition, say (KO) condition, on the nonlinearity \( f \), introduced in the pioneering papers [2,3] for semilinear problems, and which is, roughly speaking, the superlinearity of \( f \). In particular, the generalization of (KO) to the case when a gradient term is included on the right-hand side of the inequality, has been first
introduced in [12] for the generalized mean curvature equation and in Section 4 of [13] for the nonweighted \( p \)-Laplacian equation. Finally in [18] is used to treat the \( p \)-Laplacian with weights. For inequalities (1.1) and (1.2), the (KO) condition is represented by \( q > p - 1 - \theta + \beta \), which can be found in [18] in the case \( \beta = 0 \).

In the case \( \theta = 0 \), as conjectured by Mitidieri and Pohozaev in Section 14 of Chapter 1 of [1], the nonexistence results given above remain true, at least for regular solutions, if the exponent \( \gamma \) in (1.6) is such that \( \gamma = p \) for the \( p \)-Laplacian or \( \gamma = 2 \) for the mean curvature operator. This has been done by Naito and Usami, in [11] and by Usami, in [10], respectively. In particular, in Theorem 1 of [11], in order to have nonexistence of positive regular solutions of (1.9) with \( \sigma = \beta = 0 \), the condition assumed on \( a \) is that \( \lim_{|x| \to \infty} |x|^p a(x) > 0 \). Thus, the case \( \gamma = p \) in Corollary 1.1 when \( \theta = 0 \), can occur. The same conclusion holds true for the mean curvature operator if \( \gamma = p = 2 \) in Corollary 1.2 when \( \theta = 0 \), as proved in the corollary of [10].

In the second part of the paper the anisotropic case is investigated by following the scheme of Sections 12, 15 and 16 of Chapter 1 in [1]. In particular, in this setting, we remove the parameter \( a \) in the definition space of the solution, indeed in a similar way as in [1], set for \( q \geq 0 \), \( \theta \geq 0 \), \( \beta, \sigma \in \mathbb{R} \),

\[
W_{\sigma, \beta, \text{loc}}(\mathbb{R}^N) = \{a(x)u^\sigma|Du|^p, |x|^\sigma u^\sigma \in L^1_{\text{loc}}(\mathbb{R}^N)\}.
\]

In the case \( a \equiv 1 \), \( \theta = \beta = \sigma = 0 \), the following corollary, and consequently the corresponding theorem in Section 2 below, has to be understood as a nonexistence result in the class \( W_{1, p}^{1, p}(\mathbb{R}^N) \) being \( q < p^* \), as proved in [1].

**Corollary 1.5.** Assume (1.6) with \( \gamma < N - \theta \). Let \( \beta > 1 - p \),

\[
0 \leq \theta < p - 1, \quad p - N < \sigma < p - \gamma - \theta,
\]

\[
p - 1 + \beta - \theta < q \leq \frac{(N - \gamma)(p - 1 + \beta)}{N - p + \sigma} - \theta \frac{N - 1 + \beta + \sigma}{N - p + \sigma}.
\]

Then, if \( \theta > 0 \), problem

\[
-\text{div}(|x|^\sigma u^\beta |Du|^{p-2}Du) \geq a(x)u^\sigma |Du|^p \quad \text{in} \ \mathbb{R}^N,
\]

has no other positive solutions, except constants, in the class \( W_{\sigma, \beta, \text{loc}}^{1, p}(\mathbb{R}^N) \).

If \( \theta = 0 \), then problem

\[
-\text{div}(|x|^\sigma u^\beta |Du|^{p-2}Du) \geq a(x)u^\sigma \quad \text{in} \ \mathbb{R}^N,
\]

has no positive solutions in the class \( W_{\sigma, \beta, \text{loc}}^{1, p}(\mathbb{R}^N) \). \( \square \)

The corollary above covers the so-called superlinear case. Now in the next result we include both the linear and the sublinear case in order to give a complete description of the range of nonexistence of solutions of (1.18), finally we shall give an example to show the sharpness of the results.

**Corollary 1.6.** Let (1.6) hold with \( \gamma < N - 1 - \theta \). Assume also

\[
0 \leq \theta < p - 1, \quad p - N < \sigma < p - 1 - \gamma - \theta, \quad 0 < q \leq p - 1 + \beta - \theta,
\]

then the same conclusions of Corollary 1.5 hold true.

Corollaries 1.5 and 1.6 extend Theorem 12.1 in [1] in which it is considered the case \( \theta = \sigma = \beta = 0 \) while the Case \( \theta > 0 \) of Corollary 1.5 extends also Theorem 15.1 in [1], where the case \( \sigma = \beta = 0 \) is investigated.

The range of nonexistence of solutions of (1.18) when \( \sigma = \beta = \gamma = \theta = 0 \) and \( a(x) \equiv 1 \) is sharp, as emphasized by Mitidieri and Pohozaev at the end of the proof of Theorem 12.1 of [1], where they exhibit an appropriate example. The same phenomenon happens for inequalities (1.18), indeed if \( q \) does not belong to the range given by Corollaries 1.5 and 1.6, namely

\[
q > \frac{(N - \gamma)(p - 1 + \beta)}{N - p + \sigma} - \theta \frac{N - 1 + \beta + \sigma}{N - p + \sigma},
\]

with \( p - N < \sigma < p - \gamma - \theta \) and \( 0 \leq \theta < p - 1 \), then the function

\[
u(x) := |x|^{(p - \theta - \gamma - \sigma)/(p - 1 - \theta) - (q - \theta - p + 1 - \beta)/(q + \theta - p + 1 - \beta)}
\]

is a solution of (1.18) with \( a(x) = |x|^\gamma \), provided that

\[
0 < \varepsilon \leq \left(\frac{q(N - p + \sigma) + \theta(N - 1 + \beta + \sigma) - (N - \gamma)(\beta + p - 1)}{q + \theta - p + 1 - \beta}ight)^{1/(q + \theta - p + 1 - \beta)} \times \left(\frac{p - \theta - \gamma - \sigma}{q + \theta - p + 1 - \beta}\right)^{(p - 1 - \theta)/(q + \theta - p + 1 - \beta)}.
\]

Finally, to complete the picture, we give a somewhat Bernstein type result to cover the case when \( N \leq p + \sigma \), for details see Section 6 of Chapter 1 in [1] and Section 5 of [9]. In particular the next corollary was proved in Theorem 16.2 of [1] in the case \( \beta = 0 \).
Corollary 1.7. Let $N \geq 1$ and $\beta > 1 - p$. Let $u$ be a positive solution of
\[ -\text{div} (|x|^\alpha u^\beta |Dv|^{p-2}Du) \geq 0 \quad \text{in } \mathbb{R}^N. \tag{1.20} \]
If $N \leq p + \sigma$, then $u \equiv \text{const. a.e.}$ in $\mathbb{R}^N$.

2. Nonexistence for coercive problems

Theorem 2.1. Let $1 < p < N$, $0 \leq \theta < p - 1$, $\gamma < p - \theta$, $(1.6)$ and $(H)$ hold. Assume that there exist constants $c_1, c_2, c_3 > 0$ and exponents $\kappa > 0$ and $\beta \in \mathbb{R}$ such that
\[ c_1t^{p-1} \leq tA(t) \leq c_2t^{p-1} \quad \text{in } \mathbb{R}_0^+ \tag{2.1} \]
$u^\gamma g(u)$ is nondecreasing in $\mathbb{R}_0^+$,
\[ g(u) \leq c_3u^\beta \quad \text{in } \mathbb{R}_0^+. \tag{2.3} \]
If $\beta + p - 1 < q + \theta$, then problem
\[ \text{div} (g(u)A(|Du|)Du) - g'(u)A(|Du|) \geq f(x, u, Du) \quad \text{in } \mathbb{R}^N, \tag{2.4} \]
has no other positive solutions in the class $W^{1,(p,0)}_{a,\text{loc}}(\mathbb{R}^N)$ for $\alpha$ sufficiently large, except constants, whenever $\theta > 0$. While if $\theta = 0$, then problem $(2.4)$ has no positive solutions in $W^{1,(p,0)}_{a,\text{loc}}(\mathbb{R}^N)$ for $\alpha$ sufficiently large.

Proof. Let $\alpha > 0$ to be chosen later and $u$ be a positive solution of $(2.4)$. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be a standard nonnegative cut-off function that will be specified later. Now multiply $(2.4)$ by $u^\gamma \varphi$. Integrating and using $(H)$ and the fact that $A(t) \leq t^2A(t)$ we get
\[ \int_{\mathbb{R}^N} a(x)u^{\alpha + \gamma} |Du|^\gamma \varphi \, dx + \int_{\mathbb{R}^N} \left[ a(x)u^{\alpha + \gamma} |Du|^\gamma \varphi \right] \leq \int_{\mathbb{R}^N} a(x)u^\alpha |Du|A(|Du|) \varphi \, dx + \int_{\mathbb{R}^N} g(u)u^\alpha A(|Du|) |Du|^\gamma \varphi \, dx \]
where in the last inequality we have used $(2.1)$. Set
\[ v = \frac{p\alpha + \alpha(p - \theta) - \theta \beta + \theta}{q(p - 1) + \alpha(p - \theta) - \theta \beta} \quad \text{and} \quad v' = \frac{p\alpha + \alpha(p - \theta) - \theta \beta + \theta}{q + \alpha + \theta}. \tag{2.6} \]
Since $\theta < p - 1$, then $v, v'$ are well defined for $\alpha$ sufficiently large. Of course, whenever $\theta = 0$ then $v = p'$ and $v' = p$. Now applying Young inequality with exponents $v > 1$, $v'$ and $\epsilon > 0$, it results that
\[ \int_{\mathbb{R}^N} g(u)u^\alpha |Du|^{p-1} |Du|^\gamma \varphi \, dx \]
Consequently, from $(2.5)$, $(H)$ and inequality $\nu^p \leq pA(t)/c_1$, $t \geq 0$, which follows from $(2.1)$, we deduce
\[ \int_{\mathbb{R}^N} \left[ a(x)u^{\alpha + \gamma} |Du|^\gamma \varphi \right] \leq \int_{\mathbb{R}^N} a(x)u^{\alpha + \gamma} |Du|^\gamma \varphi \, dx \]
by also $(2.3)$. Now, for $\alpha$ sufficiently large $\alpha - c_2\rho\nu/(c_1\nu) \geq \kappa$, so that
\[ |\alpha - c_2\rho\nu/(c_1\nu)| g(u) + u^\gamma g(u) \geq g(u) \left[ \log (u^\rho g(u)) \right] \geq 0 \]
by $(H)$ and $(2.2)$. In turn, inequality $(2.7)$ yields
\[ \int_{\mathbb{R}^N} a(x)u^{\alpha + \gamma} |Du|^\gamma \varphi \, dx \leq \int_{\mathbb{R}^N} a(x)u^{\alpha + \gamma} |Du|^\gamma \varphi \, dx \]
(2.7)

\[ \int_{\mathbb{R}^N} a(x)u^{\alpha + \gamma} |Du|^\gamma \varphi \, dx \]
(2.8)
Now, by Hölder inequality with exponents $\zeta$ and $\zeta'$, we get

$$\int_{\mathbb{R}^N} u^{\beta + \alpha + 1/(v-1)} |Du|^{p-\nu} \frac{|D\psi|}{\varphi^{v-1}} \, dx \leq \left( \int_{\mathbb{R}^N} u^{\theta + \alpha + \zeta/(v-1)} |Du|^{(p-v')\varphi} a(x) \, dx \right)^{1/\zeta} \left( \int_{\mathbb{R}^N} \frac{|D\psi|^{\zeta'}}{\varphi^{v-1}} \, dx \right)^{1/\zeta'} .$$

If we now choose

$$(\beta + \alpha) \zeta + \frac{\zeta}{v-1} = q + \alpha \quad \text{and} \quad \zeta(p-v') = \theta,$$

which reduces only to the first equality when $\theta > 0$, we get from (2.8)

$$\int_{\mathbb{R}^N} a(x) u^{\theta + \alpha} |Du|^\nu \varphi \, dx \leq c_e \int_{\mathbb{R}^N} \frac{|D\psi|^{\nu'}}{\varphi^{\nu'}} a(x)^{-\zeta/(\zeta-1)} \, dx \leq c_e \int_{\mathbb{R}^N} \frac{|D\psi|^{\nu'}}{\varphi^{\nu'}} \frac{\zeta}{\zeta-1} \, dx,$$

where $c_e = (c_2 c_3 (e^{v'} v'))^{1/(\zeta-1)}$, with $1/(\zeta-1) = \zeta'/\zeta = (\beta + \alpha + p - 1)/(q + \theta - \beta - p + 1) \to \infty$ as $\alpha \to \infty$. Now, we specialize the choice of the cut-off function $\varphi$. Let $\xi \in C_0^\infty(\mathbb{R}^N)$ and

$$\xi(x) = \xi_0 \left( \frac{|x|}{R} \right), \quad R > 0, \ 0 \leq \xi_0 \leq 1,$$

where $\xi_0 \in C^\infty(\mathbb{R}^n)$ and

$$\xi_0(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & 2 \leq t. \end{cases}$$

We define $\varphi(x) = \xi^{\nu'}(x)$ with $\lambda$ sufficiently large so that $\lambda > v' \zeta'$, thus

$$\frac{|D\varphi|^{\nu'}}{|\varphi^{\zeta'-1}|} \leq \left( \frac{\lambda}{R} \right)^{\nu'} \xi^{\zeta'-v' \zeta'} \leq \left( \frac{\lambda}{R} \right)^{\nu'} \text{ in } B_{2R} \setminus B_R,$$

for some positive constant $C$ and being $0 \leq \xi \leq 1$. Consequently

$$\int_{\mathbb{R}^N} \frac{|D\varphi|^{\nu'}}{|\varphi^{\zeta'-1}|} |x|^{\gamma/(\zeta-1)} \, dx = \int_{B_{2R} \setminus B_R} |D\varphi|^{\nu'/\zeta'} |x|^{\gamma/(\zeta-1)} \, dx \leq \left( \frac{\lambda}{R} \right)^{\nu'} \int_{B_{2R} \setminus B_R} |x|^{\gamma/(\zeta-1)} \, dx \leq (\log 2)^{\nu'} |\varphi|^{\nu'} \begin{cases} (\log 2)^{R^{-\nu'}} & \text{if } N - 1 + \gamma/(\zeta - 1) = -1, \\ (\zeta - 1)(2^{N-1} N^{1+1/(\zeta-1)} - 1) R^{-\nu' \zeta' + \gamma/(\zeta-1)} & \text{if } N - 1 + \gamma/(\zeta - 1) \neq -1. \end{cases}$$

Thus, by (2.10) and the choice of $\varphi$, we deduce

$$\int_{B_R} a(x) u^{\theta + \alpha} |Du|^\nu \varphi \, dx \leq \int_{B_{2R} \setminus B_R} a(x) u^{\theta + \alpha} |Du|^\nu \varphi \, dx = \int_{\mathbb{R}^N} a(x) u^{\theta + \alpha} |Du|^\nu \varphi \, dx \leq c_e \int_{\mathbb{R}^N} \frac{|D\varphi|^{\nu'}}{|\varphi^{\zeta'-1}|} |x|^{\gamma/(\zeta-1)} \, dx \leq c_e \int_{B_{2R} \setminus B_R} \frac{|D\varphi|^{\nu'}}{|\varphi^{\zeta'-1}|} |x|^{\gamma/(\zeta-1)} \, dx$$

which yields,

$$\int_{\mathbb{R}^N} a(x) u^{\theta + \alpha} |Du|^\nu \varphi \, dx \leq \tilde{C} R^{\mu},$$

for some $\tilde{C} > 0$ and where

$$\mu = \begin{cases} -\nu' \zeta' & \text{if } N - 1 + \gamma/(\zeta - 1) = -1, \\ N - \nu' \zeta' + \gamma/(\zeta - 1) & \text{if } N - 1 + \gamma/(\zeta - 1) \neq -1. \end{cases}$$

Of course, for $\alpha$ sufficiently large only the latter case can occur and since

$$\zeta = \frac{q + \alpha + \theta}{\beta + \alpha + p - 1}, \quad \zeta' = \frac{q + \alpha + \theta}{q + \theta - \beta - p + 1},$$

(2.13)
we obtain
\[
\mu = \frac{(N - p)q + \theta (N - 1 + \beta) - (N - \gamma)(\beta + p - 1) - \alpha (p - \gamma - \theta)}{q + \theta - \beta - p + 1},
\]
so that, whenever \(\alpha\) is sufficiently large, the conclusion \(\mu < 0\) holds since \(\gamma < p - \theta\) and \(\beta + p - 1 < q + \theta\). By letting \(R \to \infty\) in (2.12) we obtain
\[
\int_{\mathbb{R}^N} a(x) u^{\theta + \alpha} |Du|^\theta \varphi \, dx = 0,
\]
which contradicts our assumptions on \(a, u\), concluding the proof of the theorem. \(\Box\)

**Remark.** From the requirement that \(\kappa > 0\) in assumption (2.2), we deduce that the case when \(g\) is nondecreasing is not covered by Theorem 2.2. Indeed, as noted in the Introduction, for this type of monotonicity for \(g\) the nonexistence result given in Theorem 2.2 would be included in the subcase \(\sigma = 0\) of the next result.

**Proof of Corollary 1.1.** It is enough to apply Theorem 2.1 with \(A(t) = t^{p-2}, \kappa = -\beta, g(u) = u^\theta, \beta > 0\). \(\Box\)

**Theorem 2.2.** Let \(1 < p < N\) and \(0 < \theta < p - 1\). Let (H), (1.6), (2.1) and (2.3) hold.
Suppose that there exist a function \(h \in C(\mathbb{R}^N \setminus \{0\}; \mathbb{R}^+), \) constants \(a_0, R_1 \in \mathbb{R}^+\) and an exponent \(\sigma < p - \theta - \gamma\) such that
\[
h(x) \leq a_0 |x|^\sigma \quad \text{for all } |x| \geq R_1,
\]
(2.14)
If \(\beta + p - 1 < q + \theta\), then problem
\[
\text{div} (h(x) g(u) A(|Du|) Du) \geq f(x, u, Du) \quad \text{in } \mathbb{R}^N,
\]
(2.15)
has no other positive solutions in the class \(W^{1,p}_{a,\sigma,loc} \) for \(\alpha\) sufficiently large, except constants, whenever \(\theta > 0\). While if \(\theta = 0\), then (2.15) has no positive solutions in the class \(W^{1,p}_{a,\sigma,loc}\) for \(\alpha\) sufficiently large.

**Proof.** We proceed as in the proof of Theorem 2.1 by multiplying inequality (2.15) by \(u^\nu \varphi\) and integrating so that, using also (H),
\[
c_4 \int_{\mathbb{R}^N} a(x) u^{\theta + \nu} |Du|^\theta \varphi \, dx + \int_{\mathbb{R}^N} \left[\alpha c_1 - c_2 \varphi^{\nu/v} g(u) u^{a - 1} h(x) |Du|^p \right] \varphi \, dx
\]
\[
\leq \frac{c_2 c_3}{\nu / \varphi^{\nu/v}} \int_{\mathbb{R}^N} u^{\beta + \alpha + 1/(\nu - 1)} h(x) |Du|^{p - \nu} \frac{|D \varphi|^{\nu'}}{\varphi^{\nu'/v - 1}} \, dx,
\]
(2.16)
holds thanks to (2.1), (2.3) and Young inequality with exponents \(\nu > 1\) and \(\nu'.\) Consequently, for \(\alpha\) sufficiently large, using Hölder inequality with \(\zeta\) and \(\zeta'\) given in the proof of Theorem 2.1, from (2.9) and (2.16), we deduce
\[
\int_{\mathbb{R}^N} a(x) u^{\theta + \alpha} |Du|^\theta \varphi \, dx \leq c \int_{\mathbb{R}^N} \frac{|D \varphi|^{\nu'/\zeta'}}{\varphi^{\nu'/\zeta - 1}} h(x) \zeta' a(x)^{-\zeta'/\zeta} \, dx
\]
\[
\leq C \int_{\mathbb{R}^N} \frac{|D \varphi|^{\nu'/\zeta'}}{\varphi^{\nu'/\zeta - 1}} |x|^{\sigma \zeta' + \gamma/(\zeta - 1)} \, dx,
\]
(2.17)
where \(C = c_4 c_3\) with \(c_3\) is given in (2.10) and by (2.14). Now, using the same cut-off function as in the proof of Theorem 2.1, by (2.11) we obtain
\[
\int_{\mathbb{R}^N} a(x) u^{\theta + \alpha} |Du|^\theta \varphi \, dx \leq CR^\mu,
\]
with \(\mu = N - \nu \zeta' + \gamma/(\zeta - 1) + \sigma \zeta',\) namely
\[
\mu = \frac{q(N - p + \sigma) + \theta(N - 1 + \beta + \sigma) - (N - \gamma)(\beta + p - 1) - \alpha (p - \gamma - \theta - \sigma) }{q + \theta - \beta - p + 1},
\]
so that, whenever \(\alpha\) is sufficiently large, the conclusion \(\mu < 0\) holds being \(\gamma - p + \theta + \sigma < 0\) and \(\beta + p - 1 < q + \theta\). By letting \(R \to \infty\) we obtain the required contradiction concluding the proof of the theorem. \(\Box\)

**Remark.** By the proof of the theorem above, we must have that
\[
\gamma < p_a = \frac{\alpha (p - \theta - \sigma) + N(\beta + p - 1 - q - \theta) + pq - \theta \beta + \theta - \sigma (q + \theta)}{\alpha + \beta + p - 1}.
\]
In particular \( p_a < p - \theta - \sigma \), being \( N - p + \sigma > 0 \) and \( q + \theta - \beta - p + 1 > 0 \). Thus in correspondence of the given \( \gamma \), we can take \( \tilde{\alpha} = \tilde{\alpha}(\gamma) \) given by

\[
\tilde{\alpha} = \frac{\gamma(\beta + p - 1) + N(q + \theta - \beta - p + 1) - pq + \theta \beta - \theta + \sigma(q + \theta)}{p - \theta - \sigma - \gamma},
\]

such that problem (1.8) has no solutions in the class \( W^{1,p}_{a,\sigma,\text{loc}}(\mathbb{R}_+^N) \) for \( \alpha \geq \tilde{\alpha}(\gamma) \). But, since \( p_a \not\geq p - \sigma - \sigma \) as \( \alpha \to \infty \), then we get the assertion for all \( \gamma < p - \theta - \sigma \) whenever \( \alpha \geq \tilde{\alpha} \).

**Proof of Corollary 1.3.** It is enough to apply Theorem 2.2 with \( A(t) = t^{p-2}, g(u) = u^\beta, h(x) = |x|^\gamma \). □

The theorems above cannot be applied to the mean curvature operator because assumption (2.1) fails being \( tA(t) \) bounded as \( t \to \infty \). Thus to get the nonexistence result we need to modify slightly the proof, as in [1] where, instead of (2.1), it is assumed the boundedness of \( A \). Precisely we can prove the following result.

**Theorem 2.3.** Let \( 1 < p < N, 0 \leq \theta < 1 \). Assume \((H),(1.6)\) with \( \gamma < 2 - \theta \), (2.2) and (2.3). Suppose that there exist a constant \( C > 0 \) such that

\[
A(t) \leq C \quad \text{in } \mathbb{R}^N_+.
\]

If \( \beta + 1 < q + \theta \), then the same conclusions of Theorem 2.1 hold true in the class \( W^{1,(2,\theta)}_{a,\sigma,\text{loc}}(\mathbb{R}_+^N) \) for \( \alpha \) sufficiently large.

**Proof.** Arguing as in the proof of Theorem 2.1, multiply (2.4) by \( u^\sigma \varphi \), integrate in \( \mathbb{R}^N \) and apply Young inequality with exponents \( \nu \) and \( \nu \) given in (2.6) where now \( p = 2 \) and \( \varepsilon \in (0, \nu \alpha)^{1/\nu} \), by (H), (2.3) and (2.18), we arrive to

\[
\int_{\mathbb{R}^N} \left[ |\alpha - \varepsilon^\nu/\nu| g(u) + u^\beta \right] u^{\alpha - 1} A(\|Du\|) \varphi \, dx
\]

\[
+ \int_{\mathbb{R}^N} a(x) u^{\beta + \alpha} |Du|^\alpha \, dx \leq C_{\nu} \left( \frac{1}{\nu^\nu} \right) \int_{\mathbb{R}^N} u^{\beta + \alpha + 1/(1-\nu)} |Du|^2 |\varphi|^\nu \, dx.
\]

(2.19)

Now, for \( \alpha \) sufficiently large \( \alpha - \varepsilon^\nu/\nu \geq k \), so that by (2.2) and (H) we deduce (2.8) with \( p = 2 \) and \( c_2 = C \). From now on it is enough to repeat word by word the proof of Theorem 2.1 with \( p = 2 \) to conclude the proof. □

**Proof of Corollary 1.2.** It is enough to apply Theorem 2.3 with \( A(t) = t^{1/2} + t^{\beta}, g(u) = u^\beta, \beta > 0 \). □

**Theorem 2.4.** Let \( 1 < p < N, 0 \leq \theta < 1 \). Assume \((H),(2.18),(1.6),(2.3)\) and (2.14) with \( \sigma < 2 - \theta - \gamma \). If \( \beta + 1 < q + \theta \), then the same conclusions of Theorem 2.2 hold true in the class \( W^{1,(2,\theta)}_{a,\sigma,\text{loc}}(\mathbb{R}_+^N) \) for \( \alpha \) sufficiently large.

**Proof.** Arguing as in the proof of Theorem 2.2, multiply (2.4) by \( u^\sigma \varphi \), integrate in \( \mathbb{R}^N \) and apply Young inequality with exponents \( \nu \) and \( \nu \) given in (2.6) where now \( p = 2 \), by (2.18) and (2.3), we arrive to

\[
\int_{\mathbb{R}^N} h(x) [\alpha - \varepsilon^\nu/\nu] g(u) u^{\alpha - 1} A(\|Du\|) \varphi \, dx
\]

\[
+ \int_{\mathbb{R}^N} a(x) u^{\beta + \alpha} |Du|^\alpha \, dx \leq C_{\nu} \left( \frac{1}{\nu^\nu} \right) \int_{\mathbb{R}^N} h(x) u^{\beta + \alpha + 1/(1-\nu)} |Du|^2 |\varphi|^\nu \, dx.
\]

(2.20)

Now, for \( \alpha \) sufficiently large we deduce (2.17) where \( c_2 = C \) in the expression of \( c_\nu \). From now on it is enough to repeat word by word the proof of Theorem 2.2 with \( p = 2 \) to conclude the proof. □

**Proof of Corollary 1.4.** It is enough to apply Theorem 2.4 with \( A(t) = t^{1/2} + t^{\beta}, g(u) = u^\beta, h(x) = |x|^\gamma \). □

3. Nonexistence for anticoercive problems

In this section we consider anticoercive problems having a diffusion term depending on the solution inside the divergence and the right-hand side depending also on the gradient of the solution. In particular the case without diffusion has been investigated by Mitidieri and Pohozaev in [1], for similar results see [26]. As in [1], we consider for the next result the space of solutions \( S^2_\sigma \) defined in the Introduction. In particular, the following nonexistence result has been proved in the case \( \beta = \gamma = \sigma = 0 \) in Theorem 15.1 of [1].

**Theorem 3.1.** Let \( \theta < p - 1 \) and \( \beta > 1 - p \). Assume \((H),(2.1),(1.6)\) and (2.14) with \( \sigma \) such that \( p - N < \sigma < p - \gamma - \theta \). If

\[
p - 1 + \beta - \theta < q \leq \frac{(N - \gamma)(p - 1 + \beta)}{N - p + \sigma} - \frac{1}{N - p + \sigma},
\]

(3.1)
then, the problem

\[- \text{div} ( h(x)g(u)A(|Du|)Du) \geq f(x, u, Du) \quad \text{in} \quad \mathbb{R}^N, \]  

has no other positive solutions, except constants, in the class

\[ W_{\text{loc}}^{1, (p, \theta)} (\mathbb{R}^N) = \{ (u(x))u^p|Du|^\theta, h(x)g(u)|Du|^\theta, h(x)g(u) \in L_{\text{loc}}^1 (\mathbb{R}^N) \}, \]

whenever \( \theta > 0 \). While if \( \theta = 0 \), then problem (3.2) has no positive solutions in \( W_{\text{loc}}^{1, (p, 0)} (\mathbb{R}^N) \).

**Proof.** The proof of the theorem follows the same patterns of Theorem 12.1, Step 1 and 2, and of Theorem 15.1 of [1], even if adapted to the new case with a dependence both on \( x \) and on the solution \( u \) inside the divergence and the dependence on \( x \) on the right-hand side. Let \( \varphi \in C_0^1 (\mathbb{R}^N) \) be a standard nonnegative cut-off function and \( \alpha < 0 \) be a parameter to be chosen later. Assume by contradiction that \( u \) is a positive solution of (3.2). Now we divide the proof into two parts.

**Case 1.** First we prove the theorem when the strict inequality holds in (3.1). Now multiply (3.2) by \( u^\alpha \varphi \) and integrate. Of course, without loss of generality we can assume that \( u^\alpha \in L_1^1 (\mathbb{R}^N) \) when \( \alpha < 0 \), since otherwise we can consider \( u_\varepsilon = u + \varepsilon \) and then let \( \varepsilon \searrow 0 \). By using (2.1), we get

\[
\int_{\mathbb{R}^N} f(x, u, Du)u^\alpha \varphi dx - \alpha \int_{\mathbb{R}^N} g(u)u^{\alpha-1}h(x)A(|Du|)|Du|^2 \varphi dx \\
\leq c_2 \int_{\mathbb{R}^N} g(u)u^\alpha h(x)|Du|^{p-1} |D\varphi| dx.
\]

Set \( \nu \) and \( \nu' \) as in (2.6). Of course for \( \alpha < 0 \) sufficiently small they are well defined since, by assumptions on \( \beta \) and \( \theta \), we have \( q > \beta \) so that \( q(p - 1) > \beta (p - 1) > \beta \theta \) whenever \( \beta > 0 \), otherwise is trivial. In particular, it is enough to choose \( \alpha \) negative with \( \alpha > -(q(p - 1) - \theta \beta)/(q - \beta) \) to obtain

\[
pq + \alpha(p - \theta) - \theta \beta + \theta = q(p - 1) + \alpha(p - \theta - 1) - \theta \beta + q + \alpha + \theta \\
> q(p - 1) - \theta \beta + q + \alpha + \theta > 0
\]

since by the choice of \( \alpha \) we have

\[
q(p - 1) - \theta \beta + q + \alpha + \theta > 0 \quad \text{and} \quad q + \theta > \frac{q(p - 1) - \theta \beta}{q - \beta} > -\alpha.
\]

Apply now Young inequality on the right-hand side of (3.3) with those exponents \( \nu \) and \( \nu' \) and a parameter \( \varepsilon > 0 \), by (H), (2.1) and (2.2), we deduce, as in (2.16),

\[
\int_{\mathbb{R}^N} a(x)|Du|^\theta \varphi dx + \int_{\mathbb{R}^N} |[\alpha c_1 - c_2 \varepsilon^\nu]/\nu| g(u)u^{\alpha-1} h(x)|Du|^{p-1} |D\varphi| \varphi^{\nu'/\nu} dx \\
\leq \frac{C_2\nu}{\varepsilon^\nu} \int_{\mathbb{R}^N} u^{\theta + \alpha + 1/(\nu - 1)} h(x)|Du|^\nu \varphi^{\nu'/\nu} dx.
\]

Now, choosing \( \varepsilon = \varepsilon (\alpha) \) so that \( 0 < \varepsilon < \min (|\alpha c_1 - c_2 \varepsilon^\nu|/\nu) \), let \( \tau_\varepsilon = |\alpha c_1 - c_2 \varepsilon^\nu|/\nu > 0 \), and using again Young inequality on the right-hand side of (3.4), with exponents \( \zeta \) and \( \zeta' \) given in (2.13), we get

\[
\int_{\mathbb{R}^N} a(x)|Du|^\theta \varphi dx + \tau_\varepsilon \int_{\mathbb{R}^N} g(u)u^{\alpha-1} h(x)|Du|^{p-1} \varphi dx \\
\leq \tilde{\tau}_\varepsilon \int_{\mathbb{R}^N} u^{\theta + \alpha + 1/(\nu - 1)}|Du|^{\nu}/\nu |D\varphi|^{\nu}/\nu \varphi^{\nu'/\nu} dx,
\]

where \( \tilde{\tau}_\varepsilon = c_2 c_3 \varepsilon^\nu/(\nu' \varepsilon^\nu) \) and \( \tilde{\tau}_\varepsilon = c_2 c_3/(\nu' \varepsilon^\nu + \varepsilon^\nu) \). Here to have \( \zeta \) and \( \zeta' \) well defined we need a further restriction for the range of \( \alpha \), namely \( \alpha \) negative and \( \alpha > \bar{\alpha} \), where \( \bar{\alpha} = \max \{ -(q(p - 1) - \theta \beta)/(q - \beta), 1 - p - \beta \} \). Consequently, by the choice of \( \zeta \) and \( \zeta' \), the above inequality yields

\[
c_\varepsilon \int_{\mathbb{R}^N} a(x)|Du|^\theta \varphi dx + \tau_\varepsilon \int_{\mathbb{R}^N} g(u)u^{\alpha-1} h(x)|Du|^{p-1} \varphi dx \\
\leq \tilde{\tau}_\varepsilon \int_{\mathbb{R}^N} h(x)|D\varphi|^{\nu'/\nu} dx,
\]

with \( c_\varepsilon = 1 - \tilde{\tau}_\varepsilon > 0 \). Now to estimate the first integral in (3.5), multiply (3.2) by \( \varphi \), integrate and use Hölder inequality with exponents \( \mu \) and \( \mu' \) so that
we arrive to

\[ \int_{\mathbb{R}^N} a(x)u^\alpha |Du|^p \varphi \, dx \leq c_2 \int_{\mathbb{R}^N} g(u)h(x)|Du|^{p-1} |D\varphi| \, dx \]

\[ \leq c_2 c_3^{1/\mu'} \left( \int_{\mathbb{R}^N} g(u)u^{\alpha-1}h(x)|Du|^p \varphi \, dx \right)^{1/\mu} \left( \int_{\mathbb{R}^N} h(x)|Du|^{p-\mu}u^{\mu-(\alpha-1)/(\mu-1)} |D\varphi|^{\mu'/\mu} \right)^{1/\mu'} \]

(3.6)

Apply again Hölder inequality, with exponents \( \chi \) and \( \chi' \) given by

\[ \left( \beta - \frac{\alpha - 1}{\mu - 1} \right) \chi = q + \alpha \quad \text{and} \quad (p - \mu') \chi = \theta, \]

which yields

\[ \mu = \frac{qp + \theta - \beta + \alpha(p - \theta)}{q(p - 1) - \beta + \alpha(p - 1)}, \quad \mu' = \frac{qp + \theta - \beta + \alpha(p - \theta)}{q + \theta + \alpha(1 - \theta)}, \]

\[ \chi = \frac{q + \theta + \alpha(1 - \theta)}{p + \beta - 1 - \alpha(p - \theta)}, \quad \chi' = \frac{q + \theta + \alpha(1 - \theta)}{q + \theta - p + 1 - \beta + \alpha(p - \theta)}, \]

we obtain

\[ \int_{\mathbb{R}^N} a(x)u^\alpha |Du|^p \varphi \, dx \leq c_2 c_3^{1/\mu'} \left( \int_{\mathbb{R}^N} g(u)u^{\alpha-1}h(x)|Du|^p \varphi \, dx \right)^{1/\mu} \]

\[ \times \left( \int_{\mathbb{R}^N} [h(x)]^{\chi'/\chi} [a(x)]^{-\chi'/\chi} |Du|^{\chi'/\chi} |D\varphi|^{\mu'/\mu} \right)^{1/\chi' \mu'}. \]

Thus by (3.5), (1.6), (2.14) and choosing the same cut-off function as in the proof of Theorem 2.1 we arrive to

\[ \int_{\mathbb{R}^N} a(x)u^\alpha |Du|^p \varphi \, dx \leq \tilde{c}_e \left( \int_{\mathbb{R}^N} [h(x)]^{\chi'/\chi} [a(x)]^{-\chi'/\chi} |D\varphi|^{\mu'/\mu} \right)^{1/\chi' \mu'} \]

\[ \times \left( \int_{\mathbb{R}^N} [h(x)]^{\chi'/\chi} [a(x)]^{-\chi'/\chi} |D\varphi|^{\mu'/\mu} \right)^{1/\chi' \mu'} \leq c_4 \]

\[ \leq c_4 \left( \int_{\mathbb{R}^N} [x]^\sigma \varphi \, dx \right)^{1/\chi' \mu'} \leq c_4 \left( \int_{\mathbb{R}^N} [x]^\sigma \varphi \, dx \right)^{1/\chi' \mu'}. \]

where \( \tilde{c}_e = c_2 c_3^{1/\mu'} (\tilde{\tau}_e/\tau_e)^{1/\chi' \mu'}/c_4 \) and \( c_c = \tilde{c}_e (\tilde{\tau}_e/c_c) \). By using (2.11), we obtain

\[ \int_{\mathbb{R}^N} a(x)u^\alpha |Du|^p \varphi \, dx \leq \tilde{c}_e \left( \frac{\lambda c}{R} \right)^{\nu \chi' [1/\chi' (1/\mu' + 1)] + 1} \]

\[ \times \left( \int_{B_{2\tilde{\tau}_e} \setminus B_{\tilde{\tau}_e}} [x]^\sigma \varphi \, dx \right)^{1/\chi' \mu'} \leq c_5 \left( \int_{B_{2\tilde{\tau}_e} \setminus B_{\tilde{\tau}_e}} [x]^\sigma \varphi \, dx \right)^{1/\chi' \mu'}. \]

In turn, we deduce

\[ \int_{\mathbb{R}^N} a(x)u^\alpha |Du|^p \varphi \, dx \leq \int_{B_{\tilde{\tau}_e}} a(x)u^\alpha |Du|^p \varphi \, dx \leq KR^r, \quad K > 0, \]  

(3.7)

with \( \tau = (N + \sigma \chi' + \gamma \chi'/\chi - v \chi')/(1/\mu + 1/\chi \mu') + (N + \sigma \chi' + \gamma \chi'/\chi - v \chi')/\mu \chi' \). Consequently by the value of the parameters we have

\[ \tau = \frac{q(N - p + \sigma) + \theta(N - 1 + \beta + \sigma) - (N - \gamma)(\beta + p - 1)}{q + \theta - p + 1 - \beta}. \]

Thus, since (3.1) holds with the strict inequality, we have \( \tau < 0 \) so that, by letting \( R \to \infty \) in (3.7) we obtain the required contradiction, concluding the proof of the theorem in this case.
Case 2. Now, consider the case when
\[
q = \frac{(N - \gamma)(p - 1 + \beta)}{N - p + \sigma} - \theta^\prime \frac{N - 1 + \beta + \sigma}{N - p + \sigma},
\]
so that \( \tau = 0 \) in Case 1. For \( \varphi \) chosen above, if we denote by \( S(D\varphi) = \text{supp}(D\varphi) \), by multiplying (3.2) by \( \varphi \), and using (2.1) and (2.3) and Hölder inequality twice, we obtain
\[
\int_{B_R} a(x)u^\theta |Du|^\theta \varphi \, dx \leq c_2 \int_{\mathbb{R}^N} h(x)g(u)|Du|^{p-1}|D\varphi| \, dx
\]
\[
\leq c_2 c_3^{1/\lambda'} \left( \int_{S(D\varphi)} a(x)u^{(\beta-\lambda-1)\sigma} |Du|^{p-1} \varphi \, dx \right)^{1/\lambda'}
\]
\[
\times \left( \int_{S(D\varphi)} a(x)u^{(\beta-\lambda-1)m} |Du|^{m(p-\lambda')} \varphi \, dx \right)^{1/(m\lambda')}
\]
\[
\times \left( \int_{S(D\varphi)} [h(x)]^{m'} |a(x)|^{-m'/m} \frac{|Du|^{m\lambda'}}{\varphi^{m\lambda'-1}} \, dx \right)^{1/(m\lambda')}.
\]
for \( \alpha < 0 \) sufficiently small. By choosing
\[
\left( \beta - \frac{\alpha - 1}{\lambda - 1} \right) m = q \quad \text{and} \quad (p - \lambda')m = \theta,
\]
then
\[
\lambda = \frac{qp + \theta - \theta \beta - \alpha \theta}{q(p - 1) - \theta \beta}, \quad \lambda' = \frac{qp + \theta - \theta \beta - \alpha \theta}{q + \theta - \alpha \theta},
\]
\[
m = \frac{q + \theta - \alpha \theta}{p + \beta - 1 - \alpha (p - 1)}, \quad m' = \frac{q + \theta - \alpha \theta}{q + \theta - p + 1 - \beta + \alpha (p - 1)},
\]
where \( \lambda \) is well defined because \( q(p - 1) > \theta \beta \) follows from the fact that \( p > \sigma + \gamma + \theta \). Hence, we deduce by (3.5), (1.6) and (2.14)
\[
\int_{B_R} a(x)u^\theta |Du|^\theta \varphi \, dx \leq K \left( \int_{S(D\varphi)} \left| x \right|^\sigma |\phi|^{\gamma' \zeta'/\zeta} \frac{|Du|^{\gamma' \zeta'/\zeta}}{\varphi^{\gamma' \zeta'/\zeta - 1}} \, dx \right)^{1/\lambda'}
\]
\[
\times \left( \int_{B_R \setminus \bar{B}_R} a(x)u^\theta |Du|^\theta \varphi \, dx \right)^{1/(m\lambda')},
\]
\[
\left( \int_{S(D\varphi)} \left| x \right|^\sigma m' + \gamma m'/\sigma \frac{|Du|^{m\lambda'}}{\varphi^{m\lambda'-1}} \, dx \right)^{1/(m\lambda')}.
\]
In turn, since \( \varphi = 1 \) in \( B_R \),
\[
\int_{B_R} a(x)u^\theta |Du|^\theta \, dx \leq K_0 R^\kappa \left( \int_{B_{2R} \setminus \bar{B}_R} a(x)u^\theta |Du|^\theta \, dx \right)^{1/(m\lambda')},
\]
where
\[
\kappa = \left( N + \sigma \xi' + \frac{\gamma \zeta'}{\xi} - \nu \zeta' \right) \frac{1}{\lambda'} + \left( N + \sigma m' + \frac{\gamma m'}{m} - \lambda' m' \right) \frac{1}{\lambda m'} = 0,
\]
by the choice of \( \lambda \) and \( m \) and thanks to (3.8). On the other hand, from (3.7) with \( \tau = 0 \), it follows that \( a(x)u^\theta |Du|^\theta \in L^1(\mathbb{R}^N) \), thus from (3.9) we deduce that there exists a sequence \( (R_k)_k, R_k \to \infty \) as \( k \to \infty \), such that
\[
\lim_{k \to \infty} \int_{B_{R_k}} a(x)u^\theta |Du|^\theta \, dx = 0,
\]
which concludes the proof of the theorem. \( \square \)

**Proof of Corollary 1.5.** It is enough to apply Theorem 3.1 with \( A(t) = t^{p-2}, g(u) = u^\theta, h(x) = |x|^\gamma \). \( \square \)

In the next result we cover the sublinear and the linear case, namely we have the following.
**Theorem 3.2.** Let $\theta < p - 1$, $\gamma < N - 1 - \theta$ and $p - N < \sigma < p - 1 - \gamma - \theta$. Assume $(H)$, (1.6), (2.1) and (2.14). If
\[ 0 < q \leq p - 1 + \beta - \theta \] (3.10)
then if $\theta > 0$, problem (3.2) has no other positive solutions, except constants, in the class $W^{1,(p,\theta)}_{\text{loc}}(\mathbb{R}^N)$ with $a(x)|Du|^p \in L^{1}_{\text{loc}}(\mathbb{R}^N)$. While if $\theta = 0$, then (3.2) has no positive solutions in $W^{1,(p,0)}_{\text{loc}}(\mathbb{R}^N)$ with $a(x) \in L^{1}_{\text{loc}}(\mathbb{R}^N)$.

**Proof.** The proof of the theorem follows the same patterns of the last part of Theorem 12.1 but adapted to the new situation. Let $\varphi \in C_{0}^{\infty}(\mathbb{R}^N)$ be a standard nonnegative cut-off function and $\alpha < 0$ be a parameter to be chosen later. Assume by contradiction that $u$ is a positive solution of (3.2). Now we divide the proof into two parts.

**Case 1.** First consider the case when (3.10) holds strictly. Choose
\[
\alpha = q(N - p + \sigma) + \theta(N - 1 + \beta + \sigma) - (N - \gamma)(\beta + p - 1).
\]
By (3.10), it follows that $\alpha < 0$ since $\alpha < -(p - 1 + \beta)(p - \gamma - \theta - \sigma)$. Furthermore we have $\alpha + p - 1 + \beta < -(p - 1 + \beta)(p - 1 - \gamma - \theta - \sigma) < 0$ thus $\alpha < 1 - p - \beta < -q - \theta \leq p - 1$ by (3.10) and $pq - \theta \beta + \theta - \alpha(p - \theta) < 0$, so that $\nu$ and $\nu'$, given in (2.6), as well as $\zeta$ and $\zeta'$, given in (2.13), are well defined and all strictly greater than 1. Hence we can argue as in Case 1 obtaining (3.4), then applying Hölder inequality with exponents $\zeta$ and $\zeta'$ to right-hand side of (3.4), we arrive to
\[
\int_{\mathbb{R}^N} a(x)|u|^p |Du|^q \varphi \, dx \leq K \int_{\mathbb{R}^N} |x|^{\zeta \zeta'} |\frac{|Du|^{q \zeta'} \varphi}{x^{\zeta' - 1}}| \, dx \leq K_0 R^p, \quad K > 0,
\]
\[\tau = N + \sigma \zeta' + \gamma \zeta' / \zeta - \nu \zeta' = \alpha + \gamma - p + 1 + \theta) / (q + \theta - \beta - p + 1) < 0, \text{ by assumptions. By letting } R \to \infty, \text{ the proof of the theorem is so completed.}
\]

**Case 2.** Now let $q = p - 1 + \beta - \theta$. We set, as in [1], $\alpha = 1 - p - \theta + \theta < 0$ so that (3.4), applied now with
\[
\nu = \frac{p - \theta}{p - 1} \quad \text{and} \quad \nu' = p - \theta,
\]
yields
\[
\int_{B_R} a(x)|Du|^p \, dx \leq \int_{\mathbb{R}^N} a(x)|Du|^p \varphi \, dx = \int_{\mathbb{R}^N} a(x)|u|^{p + \alpha} |Du|^q \varphi \, dx
\]
\[
\leq \int_{\mathbb{R}^N} a(x)|u|^{p + \alpha} |Du|^q \varphi \, dx + \kappa \int_{\mathbb{R}^N} g(u)|u|^{p - 1} h(x)|Du|^p \varphi \, dx
\]
\[
\leq \kappa \int_{\mathbb{R}^N} h(x)|Du|^p \varphi \, dx 
\]
\[
\int_{B_{2\delta}} h(x)|Du|^p \varphi \, dx \leq \frac{K_0}{R^{p - \theta - \gamma - \sigma}} \int_{B_{2\delta} \setminus B_{\delta}} a(x)|Du|^p \, dx.
\]
\[
(3.11)
\]

By virtue of (1.6), (2.14) and (2.11), this latter with $\nu' \zeta'$ replaced by $p - \theta$, we obtain
\[
\int_{\mathbb{R}^N} a(x)|Du|^q \varphi \, dx \leq \kappa \int_{\mathbb{R}^N} h(x)|Du|^q \varphi \, dx \leq \frac{K_0}{R^{p - \theta - \gamma - \sigma}} \int_{B_{2\delta} \setminus B_{\delta}} a(x)|Du|^p \varphi \, dx.
\]
Thus, being $p - \theta - \gamma - \sigma > 1$ and since $a(x)|Du|^p \in L^{1}_{\text{loc}}(\mathbb{R}^N)$, by letting $R \to \infty$ the required contradiction follows. The proof of the theorem is now completed. \[\square\]

**Remark.** The sharpness of the results given by Theorems 3.1 and 3.2 is discussed in the Introduction where an example of entire solution of (3.2) is given whenever $h(x) = |x|^\sigma$, $g(u) = u^\theta$, $f(x, u, Du) = |x|^\gamma u^\theta |Du|^p$, $\theta < p - 1$, $p - N < \sigma < p - \gamma - \theta$ but (3.1) and (3.10) fail.

**Proof of Corollary 3.2.** It is enough to apply Theorem 3.2 with $A(t) = t^{p - 2}$, $g(u) = u^\theta$, $h(x) = |x|^\sigma$. \[\square\]

Finally we give a Bernstein type theorem, which covers the case when $N \leq p + \sigma$. In particular, in Section 16 of Chapter 1 in [1] similar results are proved for operators which generates a $p$-Laplacian type operator, namely for those operators satisfying a property which implies the weak Harnack inequality. Of course the pure $p$-Laplacian belongs to this class. The next theorem was proved by Mitidieri and Pohozaev in the case $g(u) \equiv 1$ (cfr. Theorem 16.2 and Corollary 16.3 in [1]).
Theorem 3.3. Let \( N \geq 1 \) and \( \beta > 1 - p \). Assume (H) and (2.14) with \( N \leq p + \sigma \).

If \( u \) is a positive solution of

\[
- \text{div} (h(x)g(u)A(|Du|)Du) \geq 0 \quad \text{in} \quad \mathbb{R}^N,
\]

then \( u \equiv \text{const. a.e. in} \ \mathbb{R}^N. \)

Proof. As in Theorem 16.2 of [1], multiply (3.12) by \( u^\alpha \varphi \), where \( \varphi \in C^1_0(\mathbb{R}^N) \) is, as before, a standard nonnegative cut-off function and \( \alpha < 0 \) be a parameter to be chosen later. We divide the proof into two cases.

Case 1. We first consider the case when \( N < p + \sigma \). By (2.1)

\[
|\alpha|c_1 \int_{\mathbb{R}^N} h(x)g(u)u^{\alpha-1}|Du|^p \varphi \, dx \leq c_2 \int_{\mathbb{R}^N} h(x)g(u)u^{\alpha-1}|Du|^p \, dx + \frac{c_2c_3}{c_1|\alpha|p_\sigma} \int_{\mathbb{R}^N} h(x)u^{\beta+\alpha+p-1}|D\varphi|^2 \, dx.
\]

By Young inequality with exponents \( p \) and \( p' \) we have

\[
\int_{\mathbb{R}^N} h(x)g(u)u^{\alpha-1}|Du|^p \varphi \, dx \leq \frac{c_2c_3}{c_1|\alpha|p_\sigma} \int_{\mathbb{R}^N} h(x)|D\varphi|^p \, dx \leq K \int_{\mathbb{R}^N} |x|^p \frac{|D\varphi|^p}{\varphi^{p'/p}} \, dx \leq K_0 \int_{\mathbb{R}^N} |x|^p \, dx,
\]

where \( \tau_\sigma = 1 - c_2c_3/|\alpha|p_\sigma \) and \( K = c_2c_3a_0/|\alpha|p_\sigma \tau_\sigma \). In turn, since \( \varphi \equiv 1 \) in \( B_R \),

\[
\int_{B_R} h(x)g(u)u^{\alpha-1}|Du|^p \, dx \leq C \cdot R^{N-p+\sigma},
\]

which immediately implies the assertion being \( N < p + \sigma < 0 \).

Case 2. Let now \( N = p + \sigma \). In this case instead of Young inequality we use Hölder inequality on the right-hand side of (3.13) so that

\[
\int_{B_R} h(x)g(u)u^{\alpha-1}|Du|^p \, dx \leq \int_{\mathbb{R}^N} h(x)g(u)u^{\alpha-1}|Du|^p \varphi \, dx
\]

\[
\leq K \left( \int_{\mathbb{R}^N} h(x)g(u)u^{\alpha-1}|Du|^p \, dx \right)^{1/p'} \left( \int_{\mathbb{R}^N} h(x)u^{\beta+\alpha+p-1}|D\varphi|^{p'/p} \, dx \right)^{1/p} \int_{B_R} |x|^p \frac{|D\varphi|^p}{\varphi^{p'/p}} \, dx,
\]

where \( K, K_1 > 0 \). Consequently, by (3.14) with \( N = p + \sigma \) we deduce

\[
\int_{B_R} h(x)g(u)u^{\alpha-1}|Du|^p \, dx \leq C \left( \int_{B_R} h(x)g(u)u^{\alpha-1}|Du|^p \varphi \, dx \right)^{1/p'} \, , \quad C > 0,
\]

and also that \( h(x)g(u)u^{\alpha-1}|Du|^p \in L^1(\mathbb{R}^N) \). Thus by letting \( R \to \infty \) we obtain a contradiction if \( u \) is not constant. \( \square \)

Proof of Corollary 1.7. It is enough to apply Theorem 3.3 with \( A(t) = t^{p-2} \), \( g(u) = u^\sigma \), \( h(x) = |x|^\sigma \). \( \square \)

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References


