F-Vectors of Pure Complexes and Pure Multicomplexes of Rank Three

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Abstract

Let \( \mathcal{C} \) be a set of multisets that is closed under taking submultisets (i.e. when \( X \in \mathcal{C} \) and \( Y \subseteq X \), we have \( Y \in \mathcal{C} \)); then \( \mathcal{C} \) is a \textit{(simplicial) multicomplex}. If every member of \( \mathcal{C} \) is a set, \( \mathcal{C} \) is a \textit{(simplicial) complex}. We denote by \( F_i \) the number of multisets of cardinality \( i \) in \( \mathcal{C} \). When the multiset of largest cardinality has \( r \) elements, the \textit{F-vector} of \( \mathcal{C} \) is \( (F_0, F_1, \ldots, F_r) \). A complex or multicomplex is \textit{pure} of \textit{rank} \( r \) if (1) the multiset of largest cardinality has \( r \) elements, and (2) every multiset in \( \mathcal{C} \) is contained in at least one multiset of \( \mathcal{C} \) that has \( r \) elements. In 1977, in examining questions in commutative algebra, Stanley asked when an integer vector arises as the F-vector of a pure complex. Despite many partial results, this remains open.

The main purpose of this talk is to explore a surprising connection with problems in combinatorial design theory. We focus on rank three. Fixing the numbers of sets of cardinality one and two, we determine the possible numbers of sets of cardinality three as follows. An upper bound is established using shifting arguments. Techniques from combinatorial design theory are used to establish a lower bound. Then it is shown that every number of sets of cardinality three between the lower and the upper bound can be realized. Hence we establish necessary and sufficient conditions for an integer vector to be the F-vector of some pure complex of rank three (and do the same for pure multicomplexes). This characterization is restated to determine the precise spectrum of possible numbers of sets of cardinality two for specified numbers of sets of cardinality one and three. Recently, Zanello asked whether these spectra form intervals. For pure complexes, these spectra are not always intervals, and the gaps are determined precisely.

This is joint work with Missy Keranen and Don Kreher at Michigan Technological University.