On Minimal Weighted Clones
(or On Maximal Valued Constraint Languages)

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2 Related Work

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4 Minimal weighted clones on a Boolean domain

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Valued Constraint Satisfaction Problems

**Definition**

A VCSP is a 4-tuple \( P = (V, D, \Omega, C) \)

- \( V \) is a finite set of *variables*
- \( D \) is a finite *domain*
- \( \Omega \) is a *valuation structure* (typically \( \mathbb{Q}^+ \cup \{\infty\} \))
- \( C \) is a finite set of *valued constraints*. Each *valued constraint* is a pair \((\sigma, \phi)\):
  - \( \sigma \) is the *scope*, a list of variables
  - \( \phi \) is the *cost function*, a mapping from \( D^{\left|\sigma\right|} \) to \( \Omega \) specifying the cost of each tuple of values
Valued Constraint Satisfaction Problems

Let $\mathcal{P} = (V, D, \Omega, C)$ be a VCSP. For any assignment $s : D^{|V|} \rightarrow D$, we define

\[
\text{Cost}_\mathcal{P}(s) = \sum_{\langle \sigma, \phi \rangle \in C} \phi(s(\sigma))
\]

An optimal solution to $\mathcal{P}$ is an assignment with minimal cost.
A *Valued Constraint Language* is a set of cost functions over a finite set $D$ and valuation structure $\Omega$.

For every valued constraint language $\Gamma$, we have the corresponding class of problems $\text{VCSP}(\Gamma)$: the VCSPs $\langle V, D, \Omega, C \rangle$ in which every $\langle \sigma, \phi \rangle \in C$ has $\phi \in \Gamma$.

The *expressive power* of a language $\Gamma$, denoted $\langle \Gamma \rangle$, is the set of all cost functions which can be expressed as elements of $\text{VCSP}(\Gamma)$.
Tractable languages

We say a Valued Constraint Language $\Gamma$ is *tractable* if there exists a polynomial time algorithm which can compute an optimal solution to every $P \in \text{VCSP}(\Gamma)$.

We say $\Gamma$ is *maximal* if, for any $\phi \notin \langle \Gamma \rangle$, $\langle \Gamma \cup \{\phi\} \rangle$ is equal to the set of all cost functions.

**Question:** Which maximal Valued Constraint Languages are tractable?
Our result

**Theorem**

There are precisely 9 maximal Valued Constraint Languages on a Boolean domain, 8 of which are tractable.

**Proof.**

Uses the Galois connection of (Cohen et al., ’06 and ’11) and a characterisation of all minimal weighted clones.

Previously proved using gadgets in (Cohen et al., ’03)
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Clones and Crisp Constraint Languages

Let $D$ be a finite set.

**Definition**

A *k-ary operation* on $D$ is a mapping $f : D^k \rightarrow D$.

A set of operations $F$ is called a *clone* if $F$ contains all projections and is closed under composition. For any set of operations $F$, we define $\text{Clone}(F)$ to be the smallest clone containing $F$.

**Definition**

An *r-ary relation* on $D$ is a set $R \subset D^r$.

For every set of relations (a.k.a. crisp constraint language) $\Phi$ we define $\langle \Phi \rangle$ to be the set of relations expressible as CSPs over $\Phi$.

Note that $\text{Clone}(\text{Clone}(F)) = \text{Clone}(F)$ and $\langle \langle \Phi \rangle \rangle = \langle \Phi \rangle$. 
The Galois connection $\text{Pol} – \text{Inv}$

Sets of relations

$R_D$

$\text{Inv}(\text{Pol}(\Phi))$

$\langle \Phi \rangle$

$\Phi$

$\emptyset$

Sets of operations

$O_D$

$\text{Inv}$

$\text{Pol}(

\Phi

)$
The Galois connection $\text{Pol} - \text{Inv}$

Sets of relations

$\text{Pol}(\text{Inv}(F))$

$\text{Inv}(F)$

$\emptyset$

Sets of operations

$\text{Clone}(F)$

$\emptyset$

$F$
The Galois connection $\text{Pol} - \text{Inv}$

Sets of relations

$\emptyset = \text{Inv}(\text{Pol}(\emptyset))$

Sets of operations

$F = \text{Pol}(\text{Inv}(F))$

$\emptyset = \text{Pol}(\text{Inv}(\emptyset))$

$\emptyset = \text{Inv}(\text{Pol}(\emptyset))$
Minimal clones and maximal constraint languages

The Galois connection gives a bijection between crisp constraint languages (closed under expressibility) and clones.

**Fact**

*Minimal clones correspond to maximal (crisp) constraint languages.*

Schaefer’s Dichotomy Theorem for Boolean CSP (Schaefer, ’78) can be inferred from Post’s classification of all clones on a Boolean domain (Post, ’41).

Many more applications of this Galois connection to identifying the tractable crisp constraint languages.
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Weighted operations

There is a Galois connection for the VCSP using *weighted operations* (Cohen et al., ’11).

Definition

Let $C$ be a clone of operations on some finite domain $D$. A $k$-ary weighted operation supported by $C$ is a mapping $\omega : C^{(k)} \to \mathbb{Q}$ satisfying:

1. $\sum_{f \in C} \omega(f) = 0$;
2. $\omega(f) < 0$ only if $f$ is a projection ($\exists i$ such that $f(x) = x_i$ for all $x \in D^k$).

For the special case where $\omega(f) = -1$ whenever $f$ is a projection, we will use the shorthand $\omega = \{(\omega(f), f) : \omega(f) > 0\}$. 
Weighted clones

Definition

A set of weighted operations \( W \) (supported by some clone \( C \)) is a weighted clone if it contains the 0-weight operation of every arity and is closed under:

1. Addition
2. Translation (similar to composition)
3. Scaling by positive constants

For any set of weighted operations \( W \), we write \( \text{wClone}(W) \) for the smallest weighted clone containing \( W \).

Note that \( \text{wClone}(\text{wClone}(W)) = \text{wClone}(W) \).
Weighted Polymorphisms

Definition

Let $\omega$ be a weighted operation, supported by some clone $C$, and $\phi$ a cost function. We say $\omega$ is weighted polymorphism of $\phi$ (or $\phi$ is improved by $\omega$) if for all $x_1, x_2, \ldots, x_k \in D^r$

$$\sum_{f \in C} \omega(f)\phi(f(x_1[1], \ldots, x_k[1]), \ldots, f(x_1[r], \ldots, x_k[r])) \leq 0$$

We say $\omega \in \text{wPol}(\phi)$ and $\phi \in \text{Imp}(\omega)$. 
Weighted Polymorphisms: example

Suppose $\omega$ is a binary weighted operation with:

$$\omega(e_1^{(2)}) = \omega(e_2^{(2)}) = -1 \quad \omega(\text{min}) = \omega(\text{max}) = 1$$

Let $\phi$ be an $r$-ary cost function.

Then $\omega$ is a weighted polymorphism of $\phi$ if $\forall t_1, t_2 \in D^r$:

$$-\phi(t_1) - \phi(t_2) + \phi(\text{min}(t_1, t_2)) + \phi(\text{max}(t_1, t_2)) \leq 0$$

That is, if and only if $\phi$ is submodular:

$$\phi(\text{min}(t_1, t_2)) + \phi(\text{max}(t_1, t_2)) \leq \phi(t_1) + \phi(t_2)$$
Weighted Polymorphisms: example

Suppose $\omega$ is a binary weighted operation with:

$$\omega(e_1^{(2)}) = \omega(e_2^{(2)}) = -1 \quad \omega(\text{min}) = \omega(\text{max}) = 1$$

Suppose $\phi : \{0, 1\}^2 \rightarrow \mathbb{Q}$ has values

$$\phi(0, 0) = 1; \quad \phi(0, 1) = 1; \quad \phi(1, 0) = 1; \quad \phi(1, 1) = 0.$$

Applying $\omega$ to the pair of tuples $t_1 = \langle 0, 1 \rangle$ and $t_2 = \langle 1, 0 \rangle$ gives

$$-1 \cdot \phi(0, 1) - 1 \cdot \phi(1, 0) + 1 \cdot \phi(0, 0) + 1 \cdot \phi(1, 1) = -2 + 1 = -1 \leq 0.$$

So, $\phi$ is submodular, i.e. $\phi \in \text{Imp}(\omega)$. 
The Galois connection $wPol$ – $Imp$

$\mathcal{F}_D$  $\mathcal{W}_D$

Sets of cost functions

$\text{Imp}(wPol(\Gamma))$  $\text{Imp}(\langle \Gamma \rangle)$

$\langle \Gamma \rangle$  $wPol(\Gamma)$

Sets of weighted operations
The Galois connection $\text{wPol} - \text{Imp}$

- Sets of cost functions
- $\text{Sets of weighted operations}$

- $\text{wPol}(\text{Imp}(W))$
- $\text{wClone}(W)$
The Galois connection $\text{wPol} - \text{Imp}$

Sets of cost functions

$\Gamma = \text{Imp}(\text{wPol}(\Gamma))$

Sets of weighted operations

$W = \text{wPol}(\text{Imp}(W))$
Minimal weighted clones

The Galois connection $wPol - Imp$ gives a bijection between valued constraint languages and weighted clones.

**Fact**

*Maximal valued constraint languages correspond to minimal weighted clones.*
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Minimal weighted clones on a Boolean domain

Theorem

There are precisely 9 minimal weighted clones on a Boolean domain, generated by the following weighted operations:

1. \{ (1, f_0) \}
2. \{ (1, f_1) \}
3. \{ (1, 1 - x) \}
4. \{ (2, \text{min}) \}
5. \{ (2, \text{max}) \}
6. \{ (1, \text{min}), (1, \text{max}) \}
7. \{ (3, \text{Minority}) \}
8. \{ (3, \text{Majority}) \}
9. \{ (1, \text{Minority}), (2, \text{Majority}) \}
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5. \(\{(2, \text{max})\}\)
6. \(\{(1, \text{min}), (1, \text{max})\}\)
7. \(\{(3, \text{Minority})\}\)
8. \(\{(3, \text{Majority})\}\)
9. \(\{(1, \text{Minority}), (2, \text{Majority})\}\)
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Conclusions and Open Problems

- Valued constraint languages are in bijection with weighted clones
- Maximal languages correspond to minimal weighted clones
- We have obtained a weighted version of Rosenberg’s Classification Theorem, which gives conditions minimal weighted clones must satisfy (in the paper)
- We have identified the minimal weighted clones on a Boolean domain
- What are the minimal weighted clones on larger domains?
- What are the other weighted clones on a Boolean domain?
- Are integer weights sufficient?