ON THE CACCIOPPOLI INTEGRAL

J.K. Brooks - D.Candeloro

31/12/2002

ABSTRACT. In order to clarify the basic concepts involved in the definition of the Caccioppoli integral, the construction of a generating sequence is sketched, for functions nowhere differentiable; the definition of the Caccioppoli integral is then extended also to Banach-valued functions, together with a theorem of differentiation and a Vitali-type theorem. Moreover, it is shown that, in the scalar case, Riemann-Stieltjes integrability implies the existence of the Caccioppoli integral. Finally an announcement is given about some results concerning the space of all functions integrable in the sense of Caccioppoli.


KEY WORDS: Variation, Caccioppoli Integral, Riemann-Stieltjes Integral.

1 Introduction

The problem of constructing an integral \( \int_{a}^{b} f \, dg \), where \( f \) and \( g \) are real valued functions defined on an interval \([a, b]\) and \( g \) is a continuous function not necessarily of bounded variation, is a natural and intriguing one. It becomes apparent that it is a formidable task to obtain the desired integral when one considers the case when \( g \) is a Brownian Motion sample path \( B(\cdot, \omega) \) on a time interval \([a, b]\), which is of unbounded variation on every subinterval.

Caccioppoli in [1] formulated an ingenious approach in constructing the integral by using an approximating sequence \((g_n)\) (called a generating sequence) of continuous
functions with bounded variation, which converge uniformly to \( g \). Although this paper had serious gaps, the approach, using a generalized Cantor-like construction at each stage, is a marvelously creative \textit{tour de force}.

The authors, in [2], revisited this problem and motivated by Caccioppoli’s approach, significantly altered the construction and put the integration theory on a firm foundation. The analysis is rather delicate and intricate and the reader is referred to this paper for complete details.

In this paper we shall examine various aspects of this integration theory. In section 2, an expository section, we give a rough sketch of the construction, which we hope will serve as an interesting glimpse into the definition of \( \int f \, dg \), without the painstaking job of working through too many technical difficulties involved. The construction of the function \( g_1 \), the first and most laborious step, is discussed and a picture, to create the illusion of seeing what the first steps looks like, is provided. Keep in mind that the function \( g_1 \) is much more complicated than Lebesgue’s singular function, hence we feel that the word \textit{illusion} is justified. We also mention that in [2] we show that this integral provides a \textit{pathwise} integration for the Itô stochastic integral \( \int_0^t f(B) \, dB \), when \( f \in C^1 \).

In section 3 we list some of the theorems related to \((C) \int_a^b f \, dg\) (the \((C)\) is sometimes used in honor of Renato Caccioppoli). We further extend the theory by allowing \( f \) to take its values in a Banach space. A proof of the important Vitali convergence theorem is given in section 4, and this improves on the proof given in [2]. The relationship between our integral and the Riemann-Stieltjes integral is presented in section 5: if \( f \) is Riemann-Stieltjes integrable with respect to \( g \), where \( f \) is bounded and measurable, then \((C) \int f \, dg\) exists independently of the generating sequence for \( g \), and the two integrals coincide (the proof is more complicated than one might expect).

Finally, in section 6 we introduce a new space, \( C \), of functions integrable with respect to \( g \), relative to \((g_n)\). It turns out that a certain locally convex topology is the natural one for this space, and we present some structure theorems involving \( C \). This section serves as an announcement of results which will appear later.
2 Generating Sequences

In order to define Caccioppoli’s integral, we need the concept of “generating sequence” for a given continuous function $g : [a, b] \to \mathbb{R}$: this means a particular sequence of BV functions $g_n : [a, b] \to \mathbb{R}$, uniformly converging to $g$.

Definitions 2.1 Denote by $|u, v|$ an interval with endpoints $u$ and $v$, which may or may not contain $u$ or $v$. Given any non-trivial sub-interval $[u, v]$ of $[a, b]$, we shall say that $[u, v]$ is a zero-interval if $g(v) = g(u)$. Unless $g$ is monotonic, there are zero-intervals in $[a, b]$, and there are maximal zero-intervals (with respect to inclusion) because $g$ is continuous.

Given any open set $K \subset [a, b]$, $K$ is said to be admissible if each one of its components is a zero-interval.

A generating sequence is a sequence $(g_n)$ of continuous functions, defined on $[a, b]$, which satisfies:

i) $g_1$ is monotonic, and $g_n$ is BV for all $n$;
ii) for each $n$, there exists an admissible open set $K_n$, such that:
   ii.1) for any component $[u, v]$ of $K_n$, $g_n(x) = g(v)$, for all $x$ in $[u, v]$;
   ii.2) $g_n(x) = g(x)$ for all $x \not\in K_n$;
   ii.3) $K_n \supset K_{n+1}$ for all $n$;
   ii.4) $\lim \delta_n = 0$ where $\delta_n$ is the maximum length of the components of $K_n$;
iii) If $g$ is B.V. in some subinterval $[u, v]$, then $g_n$ eventually coincides with $g$ in $[u, v]$.

The following result is known (see [1, 2]).

Theorem 2.2 If $g : [a, b] \to \mathbb{R}$ is any continuous function, and $(g_n)$ is a generating sequence for it, then $(g_n)$ is uniformly convergent to $g$.

The construction of a generating sequence follows several steps: the complete description can be found in [2], so we simply give some intuitive ideas. Just to face directly the interesting cases, we shall assume that our function $g$ is nowhere differentiable, like the typical trajectory of Brownian Motion.
Therefore, condition (iii) in the previous definition is now meaningless.

First, the monotonic function $g_1$ is defined, by finding a sequence $(]u_n, v_n[)$ of pairwise disjoint zero-intervals, with the following properties:

i) No zero-interval is disjoint from $K_1 := \bigcup_{n=1}^{+\infty} ]u_n, v_n[.$

ii) $g(u_n) < g(u_m)$ as soon as $u_n < u_m.$

iii) If $x \not\in K_1$, and $v_n < x$ (resp. $x < u_n$) for suitable $n$, then $g(v_n) < g(x)$ (resp. $g(x) < g(u_n)$).

(Zorn’s lemma is used in the construction of these intervals).

Once this sequence is found, one defines

$$g_1(x) = \begin{cases} g(u_n) = g(v_n) & \text{if } u_n \leq x \leq v_n \\ g(x) & \text{otherwise.} \end{cases}$$

In [2] all details concerning the existence of the sequence $(]u_n, v_n[)_n$ and properties of $g_1$ are presented.

We now turn to the definition of $g_2$ and, by induction, of the whole sequence $(g_n)$.

In view of 2.1, all we must do is to construct $K_2$ and then the sequence $(K_n)$ of admissible open sets, but here we limit ourselves to sketch only the open set $K_2$. This is done as follows: for any component $]\alpha, \beta[$ of $K_1$, let us denote by $\gamma$ its midpoint, and assume, for the moment, that $g(\gamma) \neq g(\alpha)$ (hence $g(\gamma) \neq g(\beta)$ because $]\alpha, \beta[$ is a zero-interval). Then we may repeat in both the half-intervals $[\alpha, \gamma]$ and $[\gamma, \beta]$ the same construction needed to define $g_1$: this gives two disjoint sequences of zero-intervals, one for each half-interval; the union of these two families is an admissible open set, included in $]\alpha, \beta[$. The set $K_2$ is then the union of all these admissible open sets, obtained by varying $]\alpha, \beta[$ among the components of $K_1$. In the previous construction, we did not consider the case $g(\gamma) = g(\alpha)$: when this happens, one simply replaces $]\alpha, \beta[$ by $]\alpha, \gamma[ \cup ]\gamma, \beta[$ and passes to a different component of $K_1$. (See the enclosed picture.)

It is now clear how to define also $K_3$, and the other admissible open sets, and to obtain the generating sequence $(g_n)$. Details can be found also in [1]; in [2] a more general situation is considered, taking into account the possibility that $g$ is BV in
Integration

Trajectory of B. Motion

Construction of function $g_1$

Function $g_1$ is the first element of the generating sequence

Construction of function $g_2$

The function $g_3$
some intervals.

3 Absolute Continuity and Integration

In order to define the integral of a function \( f \) with respect to a continuous function \( g \), the concept of absolute continuity is needed. We shall assume that \( f \) takes its values in a Banach space \( X \). In the sequel we shall require that \( X \) enjoys the so-called Radon-Nikodým Property (RNP). All topological properties involved with \( X \)-valued functions are to be intended in the strong sense.

Definitions 3.1 Let \( F : [a, b] \to X \) be any continuous function: we shall say that \( F \) is absolutely continuous with respect to \( g \) (\( F << g \)) if for any \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
\sum_j ||F(y_j) - F(x_j)|| < \varepsilon
\]

holds, whenever \([x_j, y_j]_j \) is a sequence of pairwise disjoint intervals, satisfying both

\[
\sum_j |g(y_j) - g(x_j)| < \delta, \quad \text{and}
\]

\[
\{\max |y_j - x_j| : j \in \mathbb{N}\} < \delta.
\]

From now on, whenever \([x_j, y_j]_j \) is a sequence of pairwise disjoint intervals, satisfying (3.1.2), we shall call it a \( \delta \)-small sequence. If the sequence also satisfies (3.1.1), then we shall say that it is a \((g, \delta)\)-small sequence.

For every integer \( n \), let \( F_n \) and \( g_n \) be continuous functions, with \( F_n : [a, b] \to X \) and \( g_n : [a, b] \to \mathbb{R} \). We shall say that the sequence \((F_n)\) is uniformly absolutely continuous with respect to \((g_n)\) if, for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
\sum ||F_n(y_j) - F_n(x_j)|| < \varepsilon
\]

holds whenever \([x_j, y_j]_j \) is a \((g_n, \delta)\)-small sequence, for each index \( n \).

Now we turn to the definition of the integral. First of all, we fix the function \( g \), together with some generating sequence \((g_n)\): in general, Caccioppoli’s integral will depend on \((g_n)\).

Next, we choose any bounded strongly measurable map, \( f : [a, b] \to \mathbb{R} \), as our integrand. Then we set

\[
F_n(x) = \int_a^x f \, dg_n, \quad \text{for all } n \in \mathbb{N} \text{ and } x \in [a, b],
\]

(3.a)
where the integral is in the Bochner sense.
It is clear that $F_n$ is continuous, and that $F_n << g_n$, for each $n$.

**Definition 3.2**

We shall say that $f$ is integrable with respect to $g$ (with respect to the generating sequence $(g_n)$) if the functions $F_n$ defined in (3.a) are uniformly absolutely continuous with respect to $(g_n)$.

The next result allows us to define the integral. The proof is given in [2] for the case $X = \mathbb{R}$, and it can be repeated word by word, but replacing absolute value with $|| \cdot ||$.

**Theorem 3.3** Let $f : [a, b] \to \mathbb{R}$ be integrable with respect to $g$ (relatively to $(g_n)$). Then the functions $F_n$ defined in (3.a) above are uniformly convergent to some continuous function $F$, such that $F << g$.

Now, we are in position to define the integral.

**Definition 3.4**. Let $f : [a, b] \to \mathbb{R}$ be integrable with respect to $g$ (relatively to $(g_n)$). Then the function $F$, obtained as the uniform limit of $(F_n)$, is called the indefinite integral of $f$ with respect to $g$. We also write

$$ (C) \int_a^x f dg \equiv \int_a^x f dg = F(x), \quad x \in [a, b]. $$

### 4 A Vitali Theorem and differentiation

One of the main results in any integration theory is a convergence theorem: for Caccioppoli’s integral a kind of Vitali convergence theorem is available. Though in [2] a proof is provided, here a new proof is presented, which from one hand is somewhat simpler and on the other hand it takes into account that $f$ is Banach-valued.

**Theorem 4.1** (Vitali) Let $g : [a, b] \to R$ and suppose $(g_n)$ is a generating sequence for $g$. Assume that $(f^j)$ is a sequence of $X$-valued integrable functions, such
that their integrals $F^j$ are uniformly absolutely continuous with respect to $g$. If the sequence $(f^j)$ is pointwise convergent to some function $f$, then $f$ is integrable, and its integral function $F$ is the uniform limit of $(F^j)$.

**Proof:** We first introduce an auxiliary notation: for each point $x \in [a, b]$ and every $n \in \mathbb{N}$, whenever $x \in K_n$ we denote by $[s^n(x), t^n(x)]$ the component of $K_n$ containing $x$ (here $K_n$ is the admissible open set which defines $g_n$); in case $x \notin K_n$ we simply set $s^n(x) := t^n(x) := x$.

The generic component of $K_n$ will be denoted by $[s^n_k, t^n_k], k \in \mathbb{N}$.

Let us denote by $F^j_n$ the integral function of $f^j$ with respect to $g_n$.

Now, if we keep $j$ and $n$ fixed and choose $u$ and $v$ in $K_n^c, u < v$, then for every $m \geq n$ we obtain

$$
(F^j_n(v) - F^j_n(u)) - (F^j_m(v) - F^j_m(u)) = \sum_{u < s^n_k < t^n_k < v} (F^j_m(t^n_k) - F^j_m(s^n_k))
$$

hence

$$
(F^j_n(v) - F^j_n(u)) - (F^j(v) - F^j(u)) = \lim_{m \to \infty} \sum_{u < s^n_k < t^n_k < v} (F^j_m(t^n_k) - F^j_m(s^n_k))
$$

$$
= \sum_{u < s^n_k < t^n_k < v} (F^j(t^n_k) - F^j(s^n_k)).
$$

Thus, we obtain

$$
||F^j_n(v) - F^j_n(u)|| \leq ||F^j(v) - F^j(u)|| + \sum_{u < s^n_k < t^n_k < v} ||F^j(t^n_k) - F^j(s^n_k)|| \tag{1}
$$

Choose now $x \in [a, b]$ and assume $u = a$. For every $m \geq n$ we have

$$
F^j_n(x) - F^j(s^n(x)) = (F^j_n(s^n(x)) - F^j_n(a)) - (F^j(s^n(x)) - F^j(a)) =
$$

$$
= \lim_{m \to \infty} \sum_{s^n_k < t^n_k < s^n(x)} (F^j_m(t^n_k) - F^j_m(s^n_k)) = \sum_{s^n_k < t^n_k < s^n(x)} (F^j(t^n_k) - F^j(s^n_k)).
$$

Hence

$$
||F^j_n(x) - F^j(x)|| \leq ||F^j(x) - F^j(s^n(x))|| + \sum_{s^n_k < t^n_k < s^n(x)} ||F^j(t^n_k) - F^j(s^n_k)|| \tag{2}
$$
We shall now prove that $F^j_n << g_n$, uniformly both in $n$ and $j$.

Fix $\varepsilon > 0$ and let $\delta$ be the number corresponding to $\varepsilon$ in the condition $F^j << g$ uniformly (which is true by assumption); choose any integer $n$, and any $(g_n, \delta)$—small sequence $\{x_i, y_i\}$. Then we have:

$$\sum_i ||F^j_n(y_i) - F^j_n(x_i)|| = \sum_i ||F^j_n(s^n(y_i)) - F^j_n(t^n(x_i))||.$$

As

$$\sum_i |g(s^n(y_i)) - g(t^n(x_i))| = \sum_i |g_n(s^n(y_i)) - g_n(t^n(x_i))| = \sum_i |g_n(y_i) - g_n(x_i)| \leq \delta$$

we obtain $\sum_i ||F^j(s^n(y_i)) - F^j(t^n(x_i))|| \leq \varepsilon$.

By virtue of (1), we see that

$$\sum_i ||F^j_n(y_i) - F^j_n(x_i)|| = \sum_i ||F^j_n(s^n(y_i)) - F^j_n(t^n(x_i))|| \leq \varepsilon$$

$$\leq \sum_i ||F^j(s^n(y_i)) - F^j(t^n(x_i))|| + \sum_i \sum_{s^n(x_i) < t^n_k < t^n_l} ||F^j(t^n_k) - F^j(t^n_l)|| \leq 2\varepsilon$$

since the intervals $[s^n_k, t^n_l]$ are a $\delta$—small family and satisfy $\sum_k |g(t^n_k) - g(s^n_k)| = 0$.

This shows that $F^j_n << g_n$ uniformly in $n$ and $j$.

Thus, by the classical Vitali Theorem, we deduce that $f = \lim f^j$ is $g_n$—integrable for all $n$, and

$$\lim_{j \to \infty} \int_a^x f^j dg_n = \int_a^x f dg_n \text{ for all } n \text{ and all } x.$$

Uniformity in $n$ implies that $f$ is integrable, that is $F_n << g_n$ uniformly; to see this, it is enough to let $j$ tend to $\infty$ in the last chain of inequalities above.

The final step is to prove that $F = \lim F^j$: but from (2) it follows easily that $\lim_{n \to \infty} F^j_n(x) = F^j(x)$, uniformly both in $x$ and $j$. Therefore an exchange of limits in the sequence $(F^j_n)$ gives the conclusion. $\square$

Another important result concerning Caccioppoli’s integral is the possibility to obtain a kind of derivative for every absolutely continuous function $F$. This can be done also in the present framework, but assuming the space $X$ has the Radon-Nikodým property. We shall only sketch the proof, because it is quite similar to the one given in [2].
Theorem 4.2 Let $F : [a, b] \to X$ be any continuous function, $F << g$, and assume $X$ has the Radon-Nikodým property. Then there exists an integrable function $f$, such that $F$ is the indefinite integral of $f$. (We fix some generating sequence $(g_n)$).

Sketch of the Proof: For all $n \in \mathbb{N}$, and $x \in [a, b]$, let us define
\[ F_n(x) = F(s^n(x)), \]
where $s^n(x)$ has the same meaning as in the previous proof.

Then it turns out that:

1) $F_n$ is right-continuous and $B.V.$ for all $n$: this implies that for every $n$ there exists an $X$-valued countably additive measure $\mu_n$ on the Borel $\sigma$-field in $[a, b]$, such that $F_n(x) = \mu_n([a, x])$ for all $x \in [a, b]$. As in the real valued case, the measures $\mu_n$ can be decomposed in the sense of Lebesgue with respect to the scalar measures $dg_n$; let us denote by $\mu^*_n$ the absolutely continuous part of $\mu_n$, and define $F^*_n(x) = \mu^*_n([a, x])$ for all $n$ and $x \in [a, b]$;

2) the functions $F^*_n$ are uniformly absolutely continuous with respect to $g_n$;

3) the Radon-Nikodym derivatives $dF^*_n/dg_n$ have increasing supports, and so they can be “pasted” together in such a way to yield a measurable function $f$, which is the required function. \( \Box \)

5 Independence on the generating sequence

In some particular cases, integrability (and the integral) do not depend on the generating sequence: of course, these are the most important cases, and give rise to useful applications.

There are (at least) two main situations in which a function $f$ is integrable with respect to $g$ via any generating sequence, and with the same integral: one occurs when $f$ is Riemann-Stieltjes integrable with respect to $g$, (in which case, Caccioppoli’s integral coincides with the classical one), and the second is when $f$ is functionally dependent on $g$, i.e. $f = u(g)$ for some suitable function $u$.

We are able to prove the former result only in the scalar case; the latter was proved in [2], under the hypothesis that $u$ is locally bounded (also, an example
was given showing that local boundedness cannot be dropped). Here, we give a corresponding result for the case of Banach-valued $f$.

We start by establishing a connection between the Riemann-Stieltjes integral and the $(C)$-integral, only for scalar functions $f$. While the Lebesgue-Stieltjes integral requires that the function $g$ is BV, the Riemann-Stieltjes one does not need this condition. So, while Lebesgue-Stieltjes integrability of $f$ with respect to $g$ does not use the Caccioppoli integral, we are going to prove that Riemann-Stieltjes integrability implies that $f$ is $C$-integrable with respect to $g$ even when neither $f$, nor $g$, are BV.

**Definition 5.1** Let $f$ and $g$ be two real-valued functions, defined on $[a, b]$. We say that $f$ is *Riemann-Stieltjes integrable with respect to $g$*, if there exists a real number $I$, satisfying the following condition:

for all $\varepsilon > 0$ there exists a $\delta > 0$ such that, for every decomposition $D = \{a = t_0 < t_1 < ... < t_n = b\}$, for which $\max\{t_{i+1} - t_i : i = 0, ...n - 1\} < \delta$, one has

$$\left| \sum_{i=0}^{n-1} f(\tau_i)(g(t_{i+1}) - g(t_i)) - I \right| < \varepsilon,$$

for any choice of the points $\tau_i$ in the intervals $[t_i, t_{i+1}]$. (The quantity $\max\{t_{i+1} - t_i : i = 0, ...n - 1\}$ is often called the mesh of $D$, and denoted by $m(D)$. Moreover, $D$ is also identified with the set of the intervals $[t_i, t_{i+1}] , i = 0, ..., n - 1$.)

Usually, the number $I$ is called the *Riemann–Stieltjes integral* of $f$ with respect to $g$, and is denoted by: $I = (R - S) \int_a^b f dg$.

The first result we shall prove concerns absolute continuity of the integral function: $F(x) = \int_a^x f dg$, whenever $(R - S) \int f dg$ exists. (We recall that R-S integrability in $[a, b]$ implies integrability in every sub-interval). A technical lemma is useful, which can be found in [5].

**Lemma 5.2** Assume that $f$ is Riemann-Stieltjes integrable with respect to $g$. Then

$$\lim_{m(D) \to 0} \sum_{J \in D} \varpi(f, g ; J) = 0, \quad (3)$$
where \( \varpi(f, g; J) = \sup_{D_1, D_2} |\sum_{I \in D_1} f(\tau_I) \Delta g(I) - \sum_{I' \in D_2} f(\tau_{I'}) \Delta g(I')| \), and \( D_1, D_2 \) run along all decompositions of \( J \) \( (J \in D) \), \( \tau_I, \tau_{I'} \) are arbitrarily chosen in \( I \) and \( I' \) respectively, and \( \Delta g([u, v]) \) means \( (g(v) - g(u)) \) for every interval \( [u, v] \).

Conversely, condition (3) is also sufficient for R-S-integrability of \( f \).

**Corollary 5.3** Assume that \( f \) is Riemann-Stieltjes integrable with respect to \( g \). Then:

\[
\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } \sum_{I \in D} |\int_I f \, dg - f(\tau_I) \Delta g(I)| < \varepsilon,
\]

for all decompositions \( D \), \( m(D) < \delta \), and every choice of \( \tau_I \in I \).

**Proof:** Fix \( \varepsilon > 0 \). Then there exists \( \delta > 0 \), such that \( \sum_{J \in D} \varpi(f, g; J) < \varepsilon \) whenever \( m(D) < \delta \).

Now, pick any decomposition \( D \), such that \( m(D) < \delta \), and choose any decomposition \( D' \), finer than \( D \). Then we have

\[
\sum_{I \in D} |f(\tau_I) \Delta g(I) - \sum_{J \subset I, J' \in D'} f(\tau_{I'}) \Delta g(J')| \leq \sum_{I \in D} \varpi(f, g; I) < \varepsilon.
\]

Letting \( m(D') \) tend to 0, we find \( \sum_{I \in D} |\int_I f \, dg - f(\tau_I) \Delta g(I)| < \varepsilon \), and this concludes the proof. \( \Box \)

**Theorem 5.4** Assume that \( f \) is bounded, and \( (R - S) \)-integrable with respect to \( g \).

**Setting:**

\[
F(x) = \int_a^x f \, dg, \quad \text{for all } x \in [a, b],
\]

we get \( F \ll g \).

**Proof:** Fix \( \varepsilon > 0 \), and let \( M > 0 \) be any upper bound for \( |f| \). Corresponding with \( \varepsilon \), there exists \( \delta > 0 \) such that \( \delta M < \varepsilon / 2 \) and \( \sum_{I \in D} |\int_I f \, dg - f(\tau_I) \Delta g(I)| < \varepsilon / 2 \), whenever \( D \) is a decomposition with \( m(D) < \delta \), and for every choice of \( \tau_I \in I \).

Now, let \( ([u_i, v_i]) \) be any finite \((g, \delta)\)--small family of disjoint intervals in \([a, b]\).

We can add points, in order to get a decomposition \( D \), with \( m(D) < \delta \), such that \([u_i, v_i] \) is an interval of \( D \), for all \( i \). So we have:

\[
\sum_i |F(v_i) - F(u_i)| \leq \sum_i |\int_{u_i}^{v_i} f \, dg - f(\tau_i) \Delta g([u_i, v_i])| + \sum_i |f(\tau_i) \Delta g([u_i, v_i])| \leq \sum_{I \in D} |\int_I f \, dg - f(\tau_I) \Delta g(I)| + \sum_i |f(\tau_i) \Delta g([u_i, v_i])| \leq \varepsilon / 2 + \delta M < \varepsilon. \quad \Box
\]

Now, we deduce the announced result, concerning Caccioppoli integrability of \( f \) with respect to \( g \).
**Theorem 5.5** Let \( g, f \) be two real-valued functions on \([a, b]\). Assume that \( g \) is continuous, and \( f \) is bounded and measurable. If there exists \((R - S) - \int f \, dg\), then there exists \((C)\int f \, dg\), independently of the generating sequence for \( g \), and the two integrals coincide.

**Proof:** Assuming that \((R - S)\int f \, dg\) exists, for the previous results we can deduce that, for all \( \varepsilon > 0 \) there exists \( \sigma > 0 \) such that
\[
\sum_i |(R - S)\int_{u_i}^{v_i} f \, dg - \sum_{J \in D_i} f(\tau_J)\Delta g(J)| < \varepsilon/2,
\]
whenever \([u_i, v_i]\) is any finite \((g, \delta)-small\) sequence of disjoint intervals, and \(D_i\) is any decomposition of \([u_i, v_i]\), and for all \( i \). As a consequence, we have
\[
\sum_i \left| \sum_{J \in D_i} f(\tau_J)\Delta g(J) \right| < \varepsilon
\]
as soon as \([u_i, v_i]\) is \((g, \delta)-small\), and \( D_i \) is any decomposition of \([u_i, v_i]\), for all \( i \).

Now, let \((g_n)\) be any generating sequence for \( g \), and choose \( m \) in such a way that \( \delta(K_m) < \sigma \). Moreover, let \( \rho > 0 \), \( \rho < \sigma \), be so small that \( \sum_i |\int_{u_i}^{v_i} f \, dg| < \varepsilon \), whenever \([u_i, v_i]\) is any finite \((g_k, \rho)-small\) sequence, for \( k = 1, \ldots, m \).

(We are implicitly using the fact that \( f \) is also \((R-S)-integrable\) with respect to \( g_k \), for all \( k \). This is true, because the modulus of continuity of \( g_k \) is less than the modulus of continuity of \( g \), and in [5] it is proved that this is sufficient).

Now, pick \( n > m \), and let \([u_i, v_i]\) be any finite \((g_n, \delta)-small\) sequence. For every \( i \), there exists an interval \([u'_i, v'_i]\subset[u_i, v_i]\), such that \( g_n \) is constant in \([u_i, u'_i]\) and in \([v'_i, v_i]\), and moreover \( u'_i, v'_i \in K_n^c \). Thus we have
\[
\sum_i |\int_{u'_i}^{v'_i} f \, dg_n| = \sum_i |\int_{u'_i}^{v'_i} f \, dg_n|.
\]
Moreover, \([u'_i, v'_i]\) is \((g, \delta)-small\).

We observe that \( \int f \, dg_n \) exists both in the Riemann-Stieltjes and in the Lebesgue-Stieltjes sense, and the two integrals coincide. Hence for every \( i \) there exists a decomposition \( D_i \) of \([u'_i, v'_i]\), and a choice \( \tau_I \in I \) for every \( I \in D_i \), in such a way that
\[
\sum_i \left| \int_{u'_i}^{v'_i} f \, dg_n - \sum_{J \in D_i} f(\tau_J)\Delta g_n(J) \right| < \varepsilon. \tag{5}
\]

Moreover, as \(|dg_n|(K_n) = 0\), the endpoints of the intervals \( J \) can be taken in \( K_n^c \).
This device allows us to replace \( \Delta g_n \) by \( \Delta g \) in (5), thus obtaining:
\[
\sum_i |\int_{u_i'}^{v_i'} fdg_n| \leq \sum_i \left| \sum_{J \in D_i} f(\tau_J) \Delta g(J) \right| + \varepsilon \leq 2\varepsilon
\]
in view of (4).

This shows that \(\sum_i |F(v_i) - F(u_i)| \leq 2\varepsilon\), whenever \((u_i, v_i)\) is any finite \((g_n, \delta)\)-small sequence, for whatsoever index \(n\), and therefore \(f\) is integrable with respect to \(g\).

We shall now show that \((R-S) \int fdg\) equals \((C) \int fdg\), which will also imply the independence of the latter of the generating sequence \((g_n)\).

Fix \(\varepsilon > 0\), and pick \(\sigma > 0\) such that

\[
|(R - S) \int fdg - \sum_{J \in D} f(\tau_J) \Delta g(J)| < \varepsilon
\]
whenever \(m(D) < \sigma\) and for every choice of \(\tau_J \in J, J \in D\).

Furthermore, there exists \(\overline{n}\) such that

\[
|(C) \int fdg - \int fdg_n| < \varepsilon
\]
for all \(n > \overline{n}\).

Now, choose \(n > \overline{n}\), such that \(\delta(K_n) < \sigma\). Then, there exists a decomposition \(D\), with \(m(D) < \sigma\), made up by points of \(K_n^c\), and satisfying

\[
\left| \int fdg_n - \sum_{I \in D} f(\tau_I) \Delta g_n(I) \right| < \varepsilon
\]
for every choice of the points \(\tau_I \in I, I \in D\).

As the endpoints of \(I\) are in \(K_n^c\) for all \(I \in D\), we see that

\[
\sum_{I \in D} f(\tau_I) \Delta g_n(I) = \sum_{I \in D} f(\tau_I) \Delta g(I),
\]
so

\[
|(C) \int fdg_n - (R - S) \int fdg| < 2\varepsilon, \text{ by combining (6) and (8)}.
\]

Finally, as \(n > \overline{n}\), from (7) we find

\[
|(C) \int fdg - (R - S) \int fdg| < 3\varepsilon.
\]

Since \(\varepsilon\) is arbitrary, we get our result. \(\square\)
We shall now turn to the second case, i.e. we assume that \( f \) is a function of \( g \). More precisely, we assume that \( f = u \circ g \), where \( u : \mathbb{R} \to X \) is any strongly measurable, locally bounded map.

Under these assumption, we can define

\[
U(x) = \int_0^x u(t) dt
\]

in the Bochner sense, and we shall prove that \( f \) is integrable in the sense of Caccioppoli with respect to \( g \), via any generating sequence, and the integral \( F(x) := (C) \int_a^x u(g) dg \) satisfies

\[
F(x) = U(g(x)) - U(g(a))
\]

for all \( x \in [a, b] \).

We begin with the following proposition, which for real valued \( f \) has been proved in [2].

**Proposition 5.6** Let \( g \) be a continuous B.V. function on \([a, b]\) and \( u \) be any locally bounded, \( X \)-valued measurable map on the real line. Denote by \( U \) the indefinite integral of \( u \), with respect to Lebesgue measure, that is

\[
U(x) = \int_0^x u(t) dt.
\]

Then \( f := u \circ g \) is bounded and Stieltjes-integrable with respect to \( g \) and

\[
\int_a^x (u \circ g) dg = U(g(x)) - U(g(a)), \text{ for all } x.
\] (9)

**Proof:** We notice that the result holds, when \( X = \mathbb{R} \), as proved in [2].

In our setting, the integrability of \( f = u \circ g \) with respect to \( dg \) follows from bounded variation of \( g \) and local boundedness of \( u \). Formula (9) can be proved now in the weak sense: for each element \( x^* \in X^* \) the function \( \langle x^*, u \circ g \rangle \) satisfies the same hypotheses, but is real-valued, hence

\[
\int_a^x \langle x^*, u \circ g \rangle dg = \langle x^*, U(g(x)) - U(g(a)) \rangle.
\]

The conclusion now follows by the Hahn-Banach theorem. \( \Box \)

Now we can present an important theorem.
Theorem 5.7 Let $g : [a, b] \to \mathbb{R}$ be any continuous function, and $u : \mathbb{R} \to X$ be locally bounded and strongly measurable. Then $u \circ g$ is integrable with respect to $g$ for any generating sequence, and moreover

$$\int_a^x (u \circ g) \, dg = U(g(x)) - U(g(a)),$$

for every $x \in [a, b]$, where $U$ is the integral function of $u$ with respect to Lebesgue measure.

Proof: Fix any generating sequence $(g_n)$. We put

$$F_n(x) = \int_a^x (u \circ g) \, dg_n,$$

for all $x$, and observe that

$$F_n(x) = \int_{[a,x] \cap K_n} (u \circ g) \, dg_n = \int_{[a,x] \cap K_n} (u \circ g_n) \, dg_n = \int_a^x (u \circ g_n) \, dg_n = U(g_n(x)) - U(g_n(a)) \text{ by 5.6}.$$

Now, it is clear that $\lim_n F_n(x) = U(g(x)) - U(g(a))$ uniformly in $x$. However, our definition of integrability also requires that $F_n << g_n$ uniformly; which we now establish.

Since the sequence $(g_n)$ is uniformly convergent, there exists a constant $A > 0$ such that $|g_n(x)| \leq A$ for all $n$ and all $x$.

Now, $u$ is bounded in $[-A, A]$, say $||u(x)|| \leq K$ for every $x \in [-A, A]$. If $x, y$ are in $[a, b], x < y$, then

$$F_n(y) - F_n(x) = U(g_n(y)) - U(g_n(x)) = \int_{g_n(x)}^{g_n(y)} u(t) \, dt \text{ and therefore}$$

$$||F_n(y) - F_n(x)|| \leq K |g_n(y) - g_n(x)|,$$

which shows that $F_n << g_n$ uniformly. \qed

6 The Space of Integrable Functions

We shall keep the same notations, namely $g$ is a continuous function defined on the interval $[a, b]$ and $(g_n)$ is a fixed generating sequence for $g$. Let $\theta_n$ denote the variation measure of $dg_n$, and define the finite measure $\lambda$ by

$$\lambda(\cdot) = \sum_n \frac{1}{2^n} \frac{\theta_n(\cdot)}{1 + \theta_n([a, b])}.$$
We shall also set
\[ \Theta_n = \sum_{i=1}^{n} \theta_i. \]

**Definitions 6.1** Let \( \mathcal{C} = \mathcal{C}(g, (g_n)) \) denote the space of all Borel measurable functions \( f : [a, b] \to \mathbb{R} \) which are \( g \)-integrable with respect to the generating sequence \( (g_n) \). Let us set \( L_n := L^1([a, b], \Theta_n) \). For \( f \in \mathcal{C} \), let \( ||f_n|| = ||f||_{L_n} \) and endow the vector space \( \mathcal{C} \) with the locally convex topology induced by the family of seminorms \( \{ || \cdot ||_n : n \in \mathbb{N} \} \).

What is the completion \( \hat{\mathcal{C}} \) of \( \mathcal{C} \)? This appears to be a very delicate problem, due to the seemingly necessary preservation of uniform absolute continuity of the sequences involved, however we have the encouraging result that \( \hat{\mathcal{C}} \) lies in the space of measurable functions.

**Theorem 6.2** \( \hat{\mathcal{C}} \) is a subspace of the space of Borel measurable functions on \([a, b]\). If \((f_n)\) is a Cauchy sequence of functions in \( \mathcal{C} \), there is a measurable function \( f \) and a subsequence \((f_{n_i})\) such that \( f_{n_i} \to f \lambda\)-a.e. in \([a, b] \), and \( f_n \to f \) in \( \hat{\mathcal{C}} \).

Now that we know \( \hat{\mathcal{C}} \) stays in the space of functions, the question is when does the function \( f \) in theorem 6.2 have the property that it is integrable with respect to \( g \)? The following is a partial answer.

**Theorem 6.3** Suppose \((f^j)\) is a Cauchy sequence in \( \mathcal{C} \), and \((F^j)\) is the sequence of indefinite integrals of the functions \( f^j \). Assume that \((F^j)\) is uniformly absolutely continuous with respect to \( g \) and that \((f^j)\) converges pointwise to \( f \) on \([a, b] \). Then \( f \in \mathcal{C} \) and \( f_n \to f \) in \( \mathcal{C} \).

Now we shall obtain a representation of \( \mathcal{C}' \), the dual space of \( \mathcal{C} \). Let \( T \in \mathcal{C}' \). Note that there exists an integer \( n \) such that \( T \) is continuous with respect to the seminorm \( || \cdot ||_n \), and hence it is continuous with respect to \( || \cdot ||_m \), where \( m \geq n \).

Let \( \mathcal{B} \) denote the space of all bounded Borel measurable functions. We define an equivalence relation \( \sim \) on \( \mathcal{B} \times \mathbb{N} \) as follows. We write \((k_1, n_1) \sim (k_2, n_2)\) if
\[ \int_a^b k_1 \ f \ d\Theta_{n_1} = \int_a^b k_2 \ f \ d\Theta_{n_2} \]
for all $f \in C$. Let $Z$ denote the set of equivalence classes $[(k, n)]$ defined by $\sim$. Scalar multiplication in $Z$ is defined by $\alpha[(k, n)] = [(\alpha k, n)]$.

Define $[k_1, n_1] + [k_2, n_2] = [(k_3, n_1 + n_2)]$, where $k_3$ is an element of $B$ such that

$$
\int_a^b f k_1 \, d\Theta_{n_1} + \int_a^b f k_2 \, d\Theta_{n_2} = \int_a^b f k_3 \, d\Theta_{n_1+n_2},
$$

for all $f \in C$. These operations make $Z$ a vector space. For each $[(k, n)]$ we associate the element $T \in C'$ defined by $T(f) = \int_a^b f k d\Theta_n$, for $f \in C$.

**Theorem 6.4** $C'$ is isomorphic to $Z$.

Using Theorem 6.4, we can characterize weak convergence as follows.

**Theorem 6.5** A sequence $(f_n)$ converges weakly to $f$ in $C$ if and only if $(f_n)$ converges weakly to $f$ in $L_m$, for each integer $m$.

**References**


J.K. Brooks                       D. Candeloro
Department of Mathematics         Dipartimento di Matematica
P.O. Box 118105                    e Informatica
University of Florida             Via Vanvitelli 1
Gainesville, Florida 32611-8105 (USA) 06123 Perugia (I)
brooks@math.ufl.edu               candelor@dipmat.unipg.it