INTEGRATION WITH RESPECT TO FUNCTIONS OF UNBOUNDED VARIATION

J.K. Brooks - D.Candeloro

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1 Introduction

In this paper we shall study the problem of constructing an integral $(C)\int_a^b f dg$, where f and g are real valued functions defined on an interval $[a,b]$ and g is a continuous function, not necessarily of bounded variation. Of special interest is the case when B is normalized Brownian motion on $[0,T]$ and g is the continuous Brownian sample path $B(\cdot,\omega)$. One of our main results is that if f is continuous, then $f(B(\cdot,\omega))$ is $B(\cdot,\omega)-$integrable (Definition 5.5) and $(C)\int_0^T f(B(t,\omega))dB(t,\omega)$ is equal to $((S)\int^T -0 f(B)dB)(\omega)$, the Stratonovitch stochastic integral, for each path ω. Thus for such integrands, a pathwise integration is possible for the Stratonovitch stochastic integral.

As a result, if $f\in C^1$, then by means of It$\hat{o}$'s formula, a pathwise integration can be given for $(I)\int_0^t f(B)dB$, the It$\hat{o}$ stochastic integral (section 2).

The (C) appearing before the above integrals is in honor of R. Caccioppoli, who studied in [1] an integration theory with respect to continuous functions of unbounded variation. Although Caccioppoli's paper has serious gaps, he formulated an ingenious approach involving an approximating sequence (g_n) of continuous functions of bounded variation which converges to g. In our paper we shall present a construction of a "generating sequence" of functions (g_n) which occur in [1]. Our construction is motivated by Caccioppoli's approach, but it is significantly different and it puts the desired integration theory on a firm foundation.

The analysis leading to $(C)\int_a^b f dg$ is rather delicate and intricate and will be presented in detail, but here is a rough sketch of the development. To define (g_n) we first construct g_1 as follows. By means of Zorn's lemma we obtain a maximal family of disjoint open "zero-intervals"-these are intervals having the property that g takes on the same values at the endpoints. In terms of these components we define a continuous monotonic function g_1 which agrees with g outside the open set formed by the union of these intervals. Of course it is impossible to visualize such a g_1 for functions g as complicated as a Brownain path, which is monotone in no subinterval (g_1 would be much more complicated than Lebesgue's singular function). Once we have g_1 in hand, we define g_2 by a certain splitting procedure; to obtain g_3 we perform a splitting procedure on g_2 and so on. With (g_n) defined, where each g_n is continuous and of bounded variation on $[a, b]$, we say f is $g-$integrable if the sequence of functions $F_n(x) = \int_a^x f dg_n$ are uniformly absolutely continuous with respect to (g_n). In this case, the sequence (F_n) will converge uniformly on $[a, b]$ to what we define to be $(C)\int_a^x f dg$, which in general will depend upon the choice of (g_n). In case g is of bounded variation, $(C)\int_a^b f dg$ coincides with the usual Riemann-Stieltjes integral.

One of our main theorems is the following (Theorem 7.2). Let g be as above and assume $U : R \to R$ is measurable and locally bounded. Then $u(g)$ is $g-$integrable and $(C)\int_a^x u(g)dg = U(g(x)) - U(g(a))$ for all x in $[a, b]$, where U is the indefinite integral of u with respect to Lebesgue measure. In addition, the above integral is independent of any generating sequence (g_n). With our integral established, we prove a Vitali convergence theorem (Theorem 6.2) and a characterization of a continuous function G on $[a, b]$, where G is absolutely continuous with respect to g, as an infinite integral of a function which is $g-$integrable (Theorem 6.1). In Section 7 we present a counter example to an assertion made in [1] concerning the integrability of $u(g)$ when u is locally integrable. Our purpose in presenting these theorems, which are found in [1], is not only to clear up the gaps in Caccioppoli's treatment of these results, but also to streamline the proofs. We hope that this paper will both render

Cacioppoli's creative work more popular than it seems to be at the present time and to be of use in future applications.

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2 Stochastic Integrals

Let B denote the standard Brownian Motion on a real interval $[0,T]$. B can be considered as a suitable family of real continuous functions $t \to B(t), t \in [0,T]$: these are the paths of B. It is well-known that, almost surely, these trajectories have unbounded variations on every sub-interval of $[0,T]$, and therefore, a Stieltjes integration is not available with respect to these functions: this is the main motivation for the complicated definition of Itô's Integral. If we restrict ourselves to integration of functions of B, i.e. to processes $Y = f(B)$, for suitable real functions f, then Itô's Integral

$$(\mathrm{I}) \int_0^T f(B)dB$$

can be defined as the limit, in probability, of the random variables

$$R(Y) = \sum_{i=1}^{n} f(B(t_{i-1}))(B(t_i) - B(t_{i-1})) \quad (2.1)$$

where $[t_0, \ldots, t_n]$ is any decomposition of $[0,T]$, and the limit is taken by letting the *mesh* if the decompositions tend to 0.

Without entering into details, we simply recall here Itô's Formula: assuming that f is a C^1 function on the real line, and denoting by F any primitive function of f, then

$$(\mathrm{I}) \int_0^T f(B)dB\text{exists} \text{ , and equals } F(B(T)) - F(B(0)) - \int_0^T f'(B(t))dt$$

where the last integral is in the classical sense, and the last integral is computed pathwise. Itô's Formula can also be expressed as follows:

(I.F.) $\quad dF(B) = F'(B)dB + F''(B)dt.$

Another way to obtain a stochastic integral is due to Stratonovitch: if we set $Y = f(B)$, the Stratonovitch Integral is defined in a similar way, but the sums $R(Y)$ are replaced by

$$S(Y) = \sum_{i=1}^{n} f((B(t_i) + B(t_{i-1})/2)(B(t_{i-1})). \quad (2.2)$$

In [7] it is shown that the Stratonovitch integral of Y, which will be denoted by $(S)\int f(B)dB$, exists as soon as f is continuous, and then it satisfies $(S)\int_0^T f(B)dB = F(B(T)) - F(B(0))$, i.e. the classical chain rule holds. Many other different types of stochastic integrals could be defined, by simply changing the first term in the sums $R(Y)$, for instance replacing the value of f at $B(t_{i-1})$ with its value at $B(t_i)$ ("Backward Integral"), but none of them is defined "pathwise", i.e. fixing ω and "integrating" each path of Y with respect to the corresponding path of B. The latter procedure is exactly what can be done by means of Caccioppoli's integral. We shall see later that, as soon as f is locally bounded, it is possible to define unambiguously an integral

$$\mathrm{(C)}\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f(B)dBforanysub - interval[a,b]} \subset \mathrm{[0,T]},$$

where now the particular path ω can be kept fixed, although arbitrary, and we have:

$$\mathrm{(C)}\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f(B)dB = F(B(b)) - F(B(a))}. \quad (2.3)$$

thus showing that, at least when f is continuous, Caccioppoli's integral gives the same result as the Stratonovitch integral, but in a stronger and more natural form. Now it is clear that if $f \in C^1$, by Itô's Formula, (I) $\int_a^b f(B)dB$ can be computed pathwise using (C) $\int_a^b f(B)dB$.

3 Generating Sequences

Caccioppoli's definition of integral is based upon the concept of "generating sequence": given any continuous function $g : [a,b] \to R$, we shall find the integral with respect to g as a limit of integrals with respect to CBV (continuous bounded variation) functions uniformly converging to g.

3.1. Definitions. Denote by $|u, v|$ an interval with endpoints u and v, which may or may not contain a u or v. Given any non-trivial sub-interval $|u, v|$ of $[a, b]$, we shall say that $|u, v|$ is a *zero-interval* if $g(v) = g(u)$. Unless g is strictly monotonic, there are zero-intervals in $[a, b]$, and there are maximal zero-intervals (with respect to inclusion) because g is continuous.

Given any open set of $R, K \subset [a, b], K$ is said to be *admissible* if each one of its components is a zero-interval.

An interval $[u, v] \subset [a.b]$ is said to be a $BV-$*interval* if g is BV on it. An interval $|s, t| \subset [a, b]$ is said to be an $LBV-$*interval* if every closed subinterval of it is a $BV-$interval. Of course, if there are non-trivial $BV-$intervals, there are also maximal $LBV-$intervals (with respect to inclusion); it's possible that some maximal $LBV-$interval is also a BV interval, and thus closed.

If there are any, the maximal $LBV-$intervals are pairwise disjoint; they will play a useful role in the construction of generating sequences.

We now turn to the concept of a "generating sequence"; we begin with a lemma.

3.2. Lemma Let K be any admissible open set, and denote by $(]u_n, v_n[)$ the sequence of its components. If we define

$$g_K(x) = g_k(x), \qquad \mathrm{if x \notin K : g(u_n) = g(v_n), if x \in]u_n, v_n[, for some n;}$$

then g_K is continuous.

Proof. Fix $\varepsilon > 0$. By continuity of g, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|g(x) - g(y)| < \varepsilon$. Now, let us fix x, y in $[a, b], x < y - \delta$. If x and y are both in the complement of K, it is trivial that

$$|g_K(x) - g_K(y)| = |g(x) - g(y)| < \epsilon$$

If $x \in K, y \notin K$, we can set $x \in]u_n, v_n[$, and $v_n < y$, so that $g_K(x) = g(v_n),\ g_K(y) = g(y)$, and from $y - v_n < \delta$ it follows that

$$|g_K(x) - g_K(y)| = |g(y) - g(v_n)| < \epsilon.$$

The proof is similar in the other cases: $x \notin K; y \in K; x \in K, y \in K$.

3.3. Definition. Let g be as above. A *generating sequence* is a sequence (g_n) of continuous functions, defined on $[a, b]$, which satisfies:

i) g_1 is monotonic, and g_n is BV for all n;

ii) for each n, there exists an admissible open set K_n, such that;

ii.1) for any component $]u, v[\mathrm{afK_n, g_n(x) = g(v)}$, for all x in $[u, v]$;

ii.2) $g_n(x) = g(x)\mathrm{forallx} \notin \mathrm{k}$;

ii.3) $K_n \supset K_{n+1}\mathrm{foralln}$;

11.4) $\lim \delta_n = 0$ where δ_n is the maximum length of the components of K_n;

iii) if g is BV on some interval $[s, t] \subset [a, b]$, then there exists $m \in N$ such that g and g_n agree on $[s, t]$, for all $n > m$.

There are important differences between this concept and the one introduced by Caccioppoli: in condition i), Cacioppoli replaces BV by "piece-wise monotonic"; moreover, he does not impose iii). We shall postpone the construction of a generating sequence until after the next theorem.

3.4.Theorem. *If $g : [a, b] \to R$ is any continuous function, and (g_n) is a generating sequence for it, then (g_n) is uniformly convergent to g. Moreover, for any sub-interval $[u, v] \subset [a, b]$, we have*

(3.4.1) $\lim_n V(g_n; [u, v]) = V(g; [u, v])$.

Proof. First, we show the uniform convergence. Fix $\epsilon > 0$; then, by continuity, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|g(x) - g(y)| < \epsilon$. Now, by ii.4), we can find $k \in N$ such that $\delta_n < \delta$ for all $n > k$. Thus, if $n > k$, and x is any point in $[a, b]$, we have $|g_n(x) - g(x)| < \epsilon$; indeed, this is trivial if $x \notin K_n$, by ii.2); if $x \in K_n$, let us denote by $]u, v[$ the component of K_n including x. Then $|g_n(x) - g(x)| = |g(u) - g(x)| < \epsilon$ since $v - u < \delta$.

We now turn to the variations. Of course, (3.4.1) is a trivial consequence of (iii) in 3.3 if $[u, v]$ is a BV-interval. On the other hand, if this is not the case, by lower-semi continuity we find

$$+\infty = V(g; [u, v]) \leq \lim V(g_n; [u, v]) \leq +\infty,$$

and (3.4.1) holds.

4 The Construction

We now proceed to the construction of a generating sequence. First of all, we shall assume that g is of unbounded variation on $[a, b]$, otherwise we simply set $g_n = g$ for all $n > 1$, and $K_n = \emptyset$, no matter what g_1 is.

Next, we define g_1. To this end, we shall distinguish between two possibilities: $g(a) = g(b)\mathrm{and g(a) \neq g(b) :}$ in the first case, we set $K_1 =]a, b[\mathrm{and g_1(x) = g(a)}$ for all x.

So, the interesting case is the second one; without loss of generality, we shall assume that $g(a) < g(b)$.

We shall construct a sequence $(]u_n, v_n[)$ of a pairwise disjoint zero-intervals, with the following properties:

4.1) No zero-interval is disjoint from $K_1 := \bigcup]u_n, v_n[$.

4.2) $g(u_n) < (u_m)$ as soon as $u_n < u_m$.

4.3) If $x \notin K_1$, and $v_n < x\mathrm{or x < u_n}$ for suitable n, then $g(v_n) < g(x)\mathrm{or g(x) < g(u_n)}$.

Once this sequence is found, we shall define

$$g_1(x) = \{ \, g \ (u_n) = g(v_n)\mathrm{if u_n \leq x \leq v_n g(x)} \qquad \text{otherwise}$$

We first show that g_1 is increasing, provided the sequence $(]u_n, v_n[)_n$ satisfies 4.1, 4.2, and 4.3 above.

Fix x, y in $[a, b], x < y$. If x and y both belong to K_1, say $x \in [u_n, v_n], y \in [u_m, v_m], n \neq m$. Then $u_n < u_m$ and $g_1(y) = g(u_m) > g(u_n) = g_1(x)$. (Clearly, if $n = m$ then $g_1(x) = g_1(y)$.)

If $x \in [u_n, v_n]$ and $y \notin K_1$, then $y > v_n\mathrm{and g_1(y) = g(y) > g(v_n) = g_1(x)}$ by property 4.3.

We can proceed similarly, if $x \notin K_1, \mathrm{and y \in K_1}$.

Finally, let us assume that x and y both belong in the complement of K_1; in this case we may assume $[x, y] \cap K_1, = \emptyset$.

We shall show again that $g(x) > g(y)$ is impossible.

Indeed, assume that $g(x) > g(y)$, and consider those intervals $]u_n, v_n[$ such that $v_n < x$ (if there are any): since $g(y) > g(v_n)$ for all these intervals, from continuity of g we deduce the existence of some $t < x$ such that $g(t) = g(y)$ and $[t, y]$ is disjoint from K_1; this contradicts 4.1. Also, if all intervals $[u_n, v_n]$ are contained in $]y, b]$, a similar reasoning yields some point $w > y$ such that $g(w) = g(x)$ and $[x, w] \cap K_1, = \emptyset$: again, this is impossible. In conclusion, we must have $g(x) \leq g(y)$ when $x < y$.

Our next step is the construction of a sequence $(]u_n, v_n[)_n$ satisfying 4.1, 4.2, 4.3. To this end, let us denote by $\mathcal{K}$ the family of all (finite or denumerable) sets $\{]u_n, v_n[: n \in N. n \leq N \leq +\infty\}$ of pairwise disjoint zero-intervals, satisfying 4.2 and 4.3. We first observe that $\mathcal{K}$ is non-empty: indeed, as g is non-monotone, there are zero-intervals, and hence there are also maximal zero-intervals; if $[u, v]$ is one, it's easy to see that $g(x) < g(u)$ for all $x < u$ and $g(y) > g(v)$ for all $y > v$: so $\{]u, v[\}$ is an element of $\mathcal{K}$.

Since the elements of $\mathcal{K}$ are nothing but sequences of intervals, from now on we shall denote them with $(]u_n, v_n[)_n$.

Secondly, we remark that an element $(]u_n, v_n[)_n$ of $\mathcal{K}$ satisfies 4.1 as soon as it is maximal (according to inclusion: here, "elements" are the intervals): indeed, assuming that $(]u_n, v_n[)_n$ is an element of $\mathcal{K}$, and that $]u, v[$ can be chosen maximal with this property, and the new family $\{]u_n, v_n[,]u, v[, n \in I\!N\}$ is a larger set, enjoying both 4.2 (because $(]u_n v_n[)_n$ enjoys 4.2 and 4.3), and 4.3; to see this, all we must show is that $g(x) < g(u)$ as soon as x is not an element of $K* :=]u, v[\cup (\cup]u_n, v_n[)$, satisfying $x < u$ and $g(x) > g(v)$ as soon as $x \notin K*$ satisfies $x > v$. We study only the first case, since the other one is similar. Thus, fix $x \notin K$, with $x < u$: in case there exists some i such that $x < u_i < v_i < u$, then if follows from 4.2 and 4.3 applied to $(]u_n, v_n[)_n$ that $g(x) < g(u_i) < g(u)$. Now, if the interval $]x, u[$ does not contain any of the intervals $]u_n, v_n[$, let us consider those intervals $]u_i, v_i[$ such that $v_i < x$ (if there are any). As $g(v_i) < g(u)$ for these points, from $g(x) > g(u)$ we must deduce the existence of some point $t < x$ such that $g(t) = g(u)$, and such that $]t, v[$ is disjoint from all the $]u_n, v_n[$'s against maximality of $]u, v[$. So, the only possibility is that all the intervals $]u_n, v_n[$ satisfy $u < u_n$, and the same reasoning shows that $g(x) > g(u)$ would imply the existence of some point $\omega > v$, such that

$g(\omega) = g(x)$and$]$x$, \omega[\cap$K $= \emptyset$, contradicting the maximality of $]u, v[$. This shows that a sequence $(]u_n, v_n[)_n$ in $\mathcal{K}$ can be enlarged with an extra interval, if there exists some zero-interval $]u, v[$ disjoint from $\cup]u_n, v_n[$. This also shows that $(]u_n, v_n[)_n$ is maximal, whenever no zero-interval $]u, v[$ is disjoint from $\cup]u_n, v_n[$.

Now, it is easy to see that $\mathcal{K}$ is inductive, i.e. every increasing sub-family of elements of $\mathcal{K}$ has an upper bound.

This implies, by Zorn lemma, that $\mathcal{K}$ admits a maximal element, and this is the requested sequence $(]u_n, v_n[)_n$, satisfying 4.1, 4.2, and 4.3.

We now turn to the definition of the whole generating sequence (g_n), by introducing some special terminology.

P1. Given any closed sub-interval $J \subset [a, b]$, it is always possible to reproduce the same construction of g_1, in J, simply starting from g/J: this will be called *procedure* P.1 in J.

P2. If J is as above, and is a zero interval, we can split it into two parts by its midpoint. Call them J^1 and J^2, and apply procedure P.1 in both. This will be called *procedure* P.2 in J. We remark here that the sub-intervals may be zero-intervals as well: Procedure P.2 in such case will give rise to the same constant function g_1 both in J^1 and in J^2, and the admissible open set arising now is $J^{1^0} \cup J^{2^0}$ rather than J^0.

Now, to define g_2, and the subsequent functions g_n, we begin by setting

$\sigma_n := \frac{b-a}{2^n}$, for all n. Then we denote by (I_k) the sequence of all maximal $LBV-$intervals. For each k, there exists an increasing sequence $(I_k^n)_n$ of closed $BV-$intervals, such that

$$I_k = \bigcup_{n=1}^{\infty} I_k^n$$

We can rearrange the double sequence (I_k^n) in a single sequence (J_j), in such a way that for every m there exists a j such that every interval I_k^n, for which k and n are less than m, is one of the J_i's $i \leq j$.

In this way, if $[u, v]$ is any $BV-$interval, it is contained in some I_k, and then there exists an index i such that $[u, v] \subset \bigcup_{j \leq i} J_j$.

Before introducing the admissible open sets K_2, K_3, and so on, we must still introduce certain families of intervals.

Let us go back to K_1, and denote by $\mathcal{A}^{\infty}_{\setminus}$ the set of those components of K_1, that are contained in J_n, for each $n \geq 1$: we shall denote by A^1_n the union of all sets in $\mathcal{A}^{\infty}_{\setminus}$.

It is obvious that g is BV on A^1_n, for all n.

Next, we denote by $\mathcal{A}^{\in}_{\infty}$ the set of those components of K_1, that are not $BV-$intervals, and whose width exceeds σ_2 (thus $\mathcal{A}^{\in}_{\infty}$ is a finite set).

Finally, we denote by $\mathcal{A}^{\ni}_{\infty}$ the set of the components of K_1 that are not in $\mathcal{A}^{\infty}_{\infty} \cup \mathcal{A}^{\in}_{\infty}$: thus, an element of it can be either a $BV-$interval, not included in J_1, or an interval where g has infinite variation, those length is less than σ_2.

We are now in position to introduce K_2: we begin by saying that all the elements of $\mathcal{A}^{\ni}_{\infty}$ are among its components. The other components of K_2 will result from the following operation:

Take any element $]u, v[$ from $\mathcal{A}^{\in}_{\infty}$ and apply there procedure P.2; thus for each interval $]u, v[$ of that kind two new disjoint open sets arise, both contained in $]u, v[$: then, K_2 will be the union of the elements of $\mathcal{A}^{\ni}_{\infty}$ and of all the admissible open sets described above, as $]u, v[$ runs in $\mathcal{A}^{\in}_{\infty}$. Of course, we get $K_2 \subset K_1$, and we point out that K_2 is obtained from K_1 by removing some components (i.e., the elements of $\mathcal{A}^{\infty}_{\infty}$), by keeping fixed some other components (i.e. the elements of $\mathcal{A}^{\ni}_{\infty}$), and replacing each of the remaining components (which are a finite number) by two disjoint admissible open subsets.

Of course, (ii.1) and (ii.2) of 3.3 give then g_2 uniquely: furthermore, 3.2 ensures that g_2 is continuous. Now we shall show that g_2 is BV.

4.1.<u>Theorem</u>. *the function g_2 is of bounded variation.*

<u>Proof</u>. In each element $]u, v[$ of $\mathcal{A}^{\in}_{\infty}$ Procedure P.2 yields a function which is monotone on both the half intervals of $]u, v[$. Since $\mathcal{A}^{\in}_{\infty}$ is finite, one deduces that g_2 is BV in the union of the closures of such intervals $]u, v[$: we denote this union by T.

We shall now prove that g_2 is BV in J_1. If we choose two points x, y in J_1, there are four possibilities: $x \notin K_2$ and $y \notin K_2; x \in K_2$ and $y \notin K_2; x \in K_2$ and $y \in K_2$; $x \notin K_2$ and $y \in K_2$.

In the first case, $|g_2(x) - g_2(y) = |g(x) - g(y)|$.

In the second case, let $]s,t[$ denote the component of K_2 containing x: then we have $|g_2x) - g_2(y)| = |g(t) - g(y)|$ (of course, $t < y$. so $t \in J_1$).

We can work in a similar way in all cases, replacing $|g_2(x) - g_2(y)|$ with the jump of g in a suitable sub-interval of J_1, hence g_2 turns out to be BV in J_1.

Finally, we consider $(T \cup J_1)^c$, which is the union of a finite number of intervals: if we show that g_2 is BV in each, we are finished. So, let us denote by $]u,v[$ any such interval, fix x, y there, $x < y$, and consider the four possibilities as above in case $x \notin K_2$, then $x \in K_1$ (since $x \notin T \cup J_1$).

Then, by definition, $g_2(x) = g(x) = g_1(x)$. Thus, we can infer that $|g_2(x) - g_2(y)| = |g_1(x) - g_1(y)|$ as soon as both x and y are not in K_2.

In case $x \in K_2, y \notin K_2$, as usual we can replace x by a suitable point $t \in]x,y[\setminus K_2$ thus obtaining $|g_2(x) - g_2(y)| = g_1(t) - g_1(y)|$. In the remaining two cases, we proceed in the same fashion, replacing the variation of g_2 in $[u,v]$ with the one of g_1 there. This concludes the proof.

The definition of K_3, and therefore g_3, is quite similar: we shall consider $\mathcal{A}_{\in}^{\infty}$ instead of $\mathcal{A}_{\infty}^{\infty}$ and denote by $\mathcal{A}_{\in}^{\in}$ the family of those components of K_2 that are not $BV-$intervals, and whose width exceeds σ_3. Then $\mathcal{A}_{\in}^{\ni}$ will denote the set of the remaining components of K_2; finally K_3 will consist of the elements of $\mathcal{A}_{\in}^{\ni}$, and the new admissible open sets arising from application of Procedure P.2 to each element of $\mathcal{A}_{\in}^{\in}$. The corresponding function g_3 will be continuous in view of 3.2; to show that it is BV, one can proceed as in the proof of the above theorem.

Now, it is clear how to define the whole sequence (g_n) with the corresponding (K_n), in such a way that (i), and (ii.1) to (ii.3) of 3.3 are fulfilled. It only remains to prove (ii.4), and (iii). Let us check that (ii.4) is satisfied.

To this end, we first observe that (δ_n) is a decreasing sequence, hence it is enough to find any sub-sequence whose limit is 0.

Let's start from K_1, and consider those elements of $\mathcal{A}_{\infty}^{\ni}$, that are $BV-$intervals, and whose width exceeds σ_2. Since they are finite in number there exists $p_1 \in \mathbb{N}, p_1 > 2$, such that all of them are contain in $\bigcup_{j \leq p} J_j$, hence they are in all the families $\mathcal{A}_{\setminus}^{\infty}$, for $n \geq p_1$. Thus, if we consider K_{p_2}, none of its components is contained in any of the considered elements of $\mathcal{A}_{\infty}^{\ni}$: this means that any component of K_{p_1} is contained either in some element of $\mathcal{A}_\infty^\in$ or in some element of $\mathcal{A}_\infty^\ni$, it must have length less than $\frac{b-a}{2} = \sigma_1$: hence, $\delta_{p_1} \leq \sigma_1$.

Starting from K_{p_1}, and replacing σ_2 with σ_3 leads then to an integer $p_2 > p_1$ such that $\delta_{p_2} \leq \sigma_2$.

Iterating the process yields the required sub-sequence.

Finally, (iii) holds, because each $BV-$interval $[u, v]$ is contained in $\bigcup_{i\leq j} J_i$ for some j, and then g_n coincides with g in $[u, v]$for all $n > j$.

5 Absolute Continuity and Integration

Let $g : [a, b] \to I\!R$ be a continuous function, and (g_n) a generating sequence for g.

5.1. Definitions. Let $F[a, b] \to I\!R$ be a (continuous) function: we shall say that F is *absolutely continuous* with respect to $g(F << g)$ if, for any $\epsilon > 0$ there exists $\delta > 0$ such that

(5.1.0) $\sum_j F(y_j) - f(x_j)| < \epsilon$

(5.1.1) $\sum_j |g(y_j) - g(x_j)| < \delta$, and

(5.1.2)$\{\max |y_j - x_j| : j \in N\} < \delta$.

¿From now on, whenever $([x_j, y_j[)$ is a sequence of pariwise disjoint intervals, satisfying (5.1.2), we shall call it a $\delta-$*small sequence.* If the sequence also satisfies (5.1.1), then we shall say that it is a $(g, \delta)-$*small sequence.*

Let $(F_n), (g_n)$ be sequences of continuous functions on $[a, b]$. We shall say that (F_n) is *uniformly absolutely continuous* with respect to (g_n) if, for any $\epsilon > 0$ there exists $\delta >$ such that

(5.1.3) $\sum |F_n(y_j) - F_n(x_j)| < \epsilon$

holds whenever $(|x_j, y_j[)$ is a (g_n, δ)-small sequence, for each index n.

We point out that condition (5.1.2) is needed in order to avoid severe restrictions: for instance, if we drop it in the first definition of absolute continuity, then $F << g$ would imply that $g(x) = g(y)$ entails $F(x) = F(y)$, which in turn forces F to be functionally dependent on g.

The following proposition ties these conditions of absolute continuity with the classical ones, for those cases in which a comparison makes sense. We shall omit the proof, because of its rather technical and tedious nature.

5.2.Proposition. Let $g : [a, b] \to R$ *have bounded variation. If* $F : [a, b] \to R$ *is absolutely continuous with respect to* g*, then* F *is B.V. too, and its variation measure is absolutely continuous with respect to the variation of* g*.*

Conversely, if F *is of bounded variation, and its variation measure is absolutely continuous with respect to the variation of* g *then* $F << g$*.*

5.3. Remark. The uniform absolute continuity doesn't behave as well; one can prove that the direct part of 5.2 holds for sequences too: i.e., if (F_n) is uniformly absolutely continuous with respect to (g_n), and if each g_n is $B.V.$ than all F_n's are $B.V.$ and their variation measures are uniformly absolutely continuous with respect to the variation measures of the g_n's (in the sense of definition 5.1). However, the converse statement doesn't hold in general as we shall see in example 5.9(c).

5.4. Theorem. *Assume that* $F_n << g_n$ *uniformly, and that* $\lim F_n = F$ *pointwise in* $[a, b]$*. then, if* (g_n) *is pointwise convergent to some continuous function* g*, we have* $F << g$*.*

Proof. Fix $\epsilon > 0$, and let $\delta > 0$ be the corresponding number given by the uniform absolute continuity. Now choose any $(g, \delta/2)-$small sequence $(]u_n, v_n[)$ of pairwise disjoint intervals in $[a, b]$. Choose also any integer N : since $g_n \to g$, it is possible to find k' such that

$$\Big|\sum_{n=1}^{N} |g(u_n) - g(v_n)| - \sum_{n=1}^{N} |g_k(u_n) - g_k(v_n)|\Big| \text{forallk} > \text{k}'.$$

Then we find $\sum_{n=1}^{N} |g_k(v_n) - g_k(u_n)| < \epsilon$ for all $k > k'$ hence

$$\sum_{n=1}^{N} |F_k(v_n) - F_k(u_n)| < \epsilon \text{forallk} > \text{k}'.$$

Since $F_k \to F$, we have $\sum_{n=1}^{N} |F(v_n) - F(u_n)| \le \epsilon$. Since N is arbitrary, we have

$$\sum_{n=1}^{\infty} |F(v_n) - F(u_n)| \le \epsilon,$$

and this proves the theorem.

Now we turn to the definition of the integral. First of all, we fix the function g, together with some generating sequence (g_n): in general, Caccioppoli's integral will depend on (g_n).

Next, we choose any bounded measurable map, $f : [a, b] \to R$, as our integrand. (Actually, the definition will be applicable to f, even if it is unbounded: all we need is that f is Stieltjes-integrable with respect to dg_n, for every n). Then we set

$(5.a) F_n(x) = \int_a^x f dg_n, \mathrm{foralln} \in \mathrm{Nandx} \in \mathrm{[a, b]}.$

It is clear that F_n is continuous, and that $F_n << g_n$, for each n.

5.5. Definition. We shall say that f is *integrable* with respect to g (with respect to the generating sequence (g_n)) if the functions F_n defines in (5.a) are uniformly absolutely continuous with respect to (g_n).

Now we need some results in order to define the integral of f.

One of the basic tools we shall use is a result by Goffman and Serrin, ([2]).

5.6.Lemma. *Let u, v be weakly differentiable functions on an open set $T \subset R$, and let their derivative measures be denoted by du and dv respectively. Assume that, for every point t of some Borel set $E \subset T$, we have $u(t + 0) = v(t + 0)$, and also $u(t - 0) = v(t - 0)$. Then $du(E) = dv(E)$.*

In particular, if u and v are continuous, BV functions on $[a, b]$, and if they agree on some Borel set E, then du and dv agree for all Borel subsets of R, and the same is true for their total variations.

5.7Theorem.*Let $f : [a, b] \to R$ be integrable with respect to g (relative to (g_n)). Then the functions F_n defined in (5.a) above are uniformly convergent to some continuous function F, such that $F << g$.*

Proof. Fix n, m in $N, n < m$, and evaluate

$$F_m(x) - F_n(x) = \int_a^x f dg_m - \int_a^x f dg_n, \quad x \in [a, b].$$

In the open set $K_m \subset K_n$ both g_m and g_n have zero variations, hence

$$F_m(x) - F_n(x) = \int_{[a,x] \cap K_m^c} f d(g_m - g_n).$$

Moreover, if we consider the Borel set K_n^c on it g_m and g_n agree (with g) hence, by

5.6, dg_m and dg_n agree on every subset of K_n^c. Thus

$$F_m(x) - F_n(x) = \int_{(K_n \setminus K_m)\cap[a,x]} f dg_m - \int_{(K_n \setminus K_m)\cap[a,x]} f dg_n.$$

Since dg_n is zero in K_n, the last integral vanishes. Then

$$F_m(x) - F_n(x) = \int_{K_n\cap[a,x]} f dg_m, \text{since} \int_{\mathrm{K_m\cap[a,x]}} \mathrm{fdg_m} = 0$$

Now, let us set $K_n \cap [a,x] = (\cup]s_j,t_j[) \cup]s,x]$, where $]s_j,t_j[$ are those components of K_n that are contained in $[a,x]$, and s is the left endpoint of that component (if it exists) containing x: in case $x \notin K_n$, we put $s = x$, so that the last interval is empty. By $\sigma-$additivity, we have

$F_m(x) - F_n(x) = \sum_j (F_m(t_j) - F_m(s_j)) + F_m(x) - F_m(s)$.

Now fix $\epsilon > 0$, and let δ' be the corresponding number by the uniform absolute continuity of (F_n) with respect to (g_n). Since g is continuous, we can find a number $\delta > 0$ such that $|s - t| < \delta$ implies $|g(s) - g(t)| < \delta'$.

Let n be large enough so that $\delta_n < \delta$. We have for $m > n$:

$\sum_j |g_m(t_j) - g_m(s_j)| + |g_m(x) - g_m(s)| = |g_m(x) - g_m(s)|$

since each $]s_j,t_j[$ is a zero-interval for g_m. In addition, $|g_m(x) - g_m(s)|$ is different from 0 only if $x \in K_N$, in which case we find $|g_m(x) - g_m(s)| = |g(x) - g(s)| < \delta'$. Hence by absolute continuity: $\sum_j |F_m(t_j) - F_m(s_j)| + F_m(x) - F_m(s)| < \epsilon$, which means $|F_m(x) - F_n(x)| < \epsilon$.

Since the size of n does not depend on x, this is a uniform Cauchy property for (F_n), and therefore the sequence is uniformly convergent. The limit function F is trivially continuous, and $F << g$ by 5.4.

5.8.Definition. Let $f : [a,b] \to R$ be integrable with respect to g (relative to (g_n)). Then the function F, arising as uniform limit of (F_n), is called the *indefinite integral* of f with respect to g. We also write

$$(C)\int_a^x f dg \equiv \int_a^x f dg = F(x), x \in [a,b].$$

5.9.Remarks.

a) Once g and (g_n) have been fixed, it is easy to see that, for each integrable function f, $f1_{[u,v]}$ is integrable too, for every sub-interval $[u,v]$. We shall also write:

$$\int f1_{[u,v]} dg = \int_u^v f dg$$

and one can easily prove that this integral equals $F(v) - F(u)$, where F is the indefinite integral of f.

It is also immediate that, in case g is $B.V.$ in $[a, b]$, every bounded measurable f is integrable (relative to *any* generating sequence), and the indefinite integral is the usual one.

b) In general, given any generating sequence, there are very simple functions that aren't integrable: in fact, unless g is $B.V.$ there are Borel sets $A \subset [a, b]$ such that 1_A is not integrable. For, assume that 1_A is integrable for every A: then it is clear that $\int_a^b 1_A dg_n$ equals $dg_n(A)$ for all n, and integrability yields setwise convergence of the measures dg_n, on the Borel $\sigma-$field: the Nikodym Boundedness Theorem would then imply that these measures are uniformly bounded, i.e.

$$\sup V(g_n) : n \in N < +\infty.$$

In view of (3.4.1) we then find $V(g) < +\infty$.

c) The last remark gives also a counterexample for the analogous statement of the reverse implication in 5.2. for sequences (this was mentioned in 5.3): indeed, if g is not $B.V.$, there is some Borel set $A \subset [a, b]$ such that 1_A is not integrable: however $|dF_n| << |dg_n|$ uniformly, where

$$F_n(x) = \int_a^x 1_A dg_n,$$

since $dF_n(B) = dg_n(B \cap A)$ for every Borel set B, and therefore

$$|dF_n| \leq |dg_n| \text{foralln}.$$

Hence the functions F_n give us the desired example, since it is false that $F_n << g_n$ uniformly.

d) In a similar fashion, we can see that, unless g is $B.V.$, there are *continuous* functions f that are not integrable: otherwise, integration with respect to the generating sequence (g_n) would yield a converging sequence of bounded functionals on $C([a, b])$; so, by the Uniform Boundedness Principle such a sequence should be a bounded subset of the dual of $C([a, b])$; this implies that the functions g_n have uniformly bounded variations, and therefore (3.4.1) again shows that g is $B.V.$

6.Two Important Theorems

One of the mist important results in [1] is the characterization of those functions F such that $F << g$. We have already seen that if f is integrable, then its indefinite integral is absolutely continuous with respect to g. We mention that in this setting f need not be bounded: it is sufficient that it be measurable and $|dg_n|-$integrable for all n.

Cacciopoli proved the converse of this result; we provide here a modernized proof of his result. Later we present a Vitali convergence theorem which essentially follows his proof.

6.1.Theorem *Let $F : [a,b] \to R$ be any continuous function, $F << g$. then there exists an integrable function f, such that F is the indefinite integral of f. (We fix some generating sequence (g_n)).*

Proof. For all $n \in N$, let us define

$$F_n(x) = \{\, F\ (x) \qquad \mathrm{if x \in K_n^c F(s_j^n)} \qquad \mathrm{if x \in]s_j^n, t_j^n[for some j},$$

where $(]s_j^n, t_j^n[)$ are the components of K_n.

We shall prove that

1) F_n is $B.V.$ for all n;

2) the functions f_n^*, arising as the absolute continuous part of F_n with respect to g_n, are uniformly absolutely continuous with respect to g_n;

3) the Radon-Nikodym derivatives dF_n^*/dg_n have increasing supports, and so they can be "pasted" together in such a way to yield a measurable function f, which is the required function.

We first prove 1). Since F is continuous, it is obvious that F_n is right-continuous at every point, and that the set of its singularities is contained in $\{t_j^n : j \in N\}$. Moreover, we have

$$|F_n(t_j^n) - F_n(t_j^n - 0)| = |F(t_j^n) - F(s_j^n)| \mathrm{for all j and all n}.$$

Let us set $D_n = \sum_j |F(t_j^n) - F(s_j^n)|, n \in N$. By (ii.4) of 3.3 we deduce

(6.1.0) $\lim\limits_{n \to +\infty} D_n = 0,$

since $F << g$, and $g(t_i^n) = g(s_j^n)$ for all j, n. By the same reason, it is also easy to deduce that F_n converges to F uniformly.

In view of (6.1.0) we see that $D_n < +\infty$, at least for n sufficiently large: let us set

$$N_0 = \{n \in N : D_n < +\infty\}.$$

Thus N_0 is the complement of a finite set (actually, it coincides with N, as we shall see later).

For every $n \in N_0$ we set

$$F_n^*(x) = F_n(x) - \sum_{t_j^n \leq x} (F(t_j^n) - F(s_j^n)),$$

where the last summation runs over those components $]s_j^n, t_j^n[$ that are contained in $[a, x]$.

Now we want to show that $F_n^* << g_n$ uniformly in N_0.

Fix $\epsilon > 0$; then there exists a $\delta > 0$ such that $\sum_j |F(v_j) - F(u_j)| < \epsilon/8$ whenever $(]u_j, v_j[)$ is a $(g\delta)-$ small sequence.

Now, fix $n \in N_o$, and choose any $(g_n, \delta)-$small sequence $(]u_j, v_j[)$. We shall compute $\sum_j |F_n^*(v_j) - F_n^*(u_j)|$ by splitting it into 4 summands, $\sum^1, \sum^2, \sum^3, \sum^4$, according to the following rules:

(A) $\sum^1$ runs over those indexes j, such that $u_j \notin K_n, v_j \notin K_n$.

(B) $\sum^2$ runs along those indexes j, for which $u_j \notin K_n, v_j \in K_n$.

(C) $\sum^3$ involves those indexes j, such that $u_j \in K_n, v_j \notin K_n$.

(D) $\sum^4$ runs along those indexes j for which $u_j \in K_n, v_j \in K_n$.

We have

(A) $\sum^1 |\mathrm{F_n^*(v_j) - F_n^*(u_j)}| \leq \sum^1 |\mathrm{F(v_j) - F(u_j)}| + \sum^1 \sum_\mathrm{j} |\mathrm{F(t_i^j) - F(s_i^j)}|$,

where $\sum_i$ runs over those components $]s_i^j, t_i^j[\mathrm{of K_n}$, that lie between $u_j \mathrm{and v_j}$. Since $(]u_j, v_j[)$ is $(g, \delta)-$small, and

$$\delta > \sum_j |g_n(v_j) - g_n(u_j)| = \sum_j |g(v_j) - g(u_j)|, \text{wehave}$$

$\sum^1 |F(v_j) - F(u_j)| < \epsilon/8.$

Since $g(s_i^j) = g(t_i^j)$ for all i, j, we also have $\sum^1 \sum_i |F(t_i^j) - F(s_i^j)| < \epsilon/8$ since $(]s_i^j, t_i^j[)_{i,j}$ is $\delta-$small. In conclusion, we find

$\sum^1 |F_n^*(v_j) - F_n^*(u_j)| < \epsilon/4.$

(B) In a similar way one can prove that

$\sum^2 |F_n^*(v_j) - F_n^*(u_j)| < \epsilon/4.$

(C) $\sum^3 |F_n^*(v_j) - F_n^*(u_j)| = \sum^3 |F_n(v_j) - F_n(u_j) - \sum_i (F(t_i^j) - F(s_i^j))| =$

$= \sum^3 |F(v_j) - F(s_j) - \sum_i (F(t_i^j) - F(s_i^j))|$, where $\sum_i$ runs along the components $]s_i^i, t_i^i[$ that lie between s_j and v_j (of which the first is $]s_j, t_j[$).

Hence we can simplify, and write:

$\sum^3 |F_n^*(v_j) - F_n^*(u_j)| = \sum^3 |F(v_j) - F(v_j) - F(t_j) - \sum_i' -F(s_i^j)|$

$\leq \sum^3 |F(v_j) - F(t_j)| + \sum^3 \sum_i' |F(t_i^j) - F(s_i^j)|$, where $\sum_i'$ runs over the components $]s_i^j, t_i^j[$ contained in $]t_j, v_j[$. Again, we see that $\sum^3 \sum' |F(t_i^j) - F(s_i^j)| < \epsilon/8$, by the same argument as above. Moreover, since $]t_j, v_j[\subset]u_j, v_j[$ for all j, and $\delta > \sum^3 |g_n(v_j) - g_n(u_j)| = \sum^3 |g(v_j) - g(t_j)|$, we see that $(]t_j, v_j[)$ is $(g, \delta)-$small, hence $\sum^3 |F(v_j) - F(t_j)| < \epsilon/8$. In conclusion, we find

$\sum^3 |F_n^*(v_j) - F_n^*(u_j)| < \epsilon/4$

(D) The same conclusion holds for $\sum^4$, by an analogous argument.

Thus, summing $\sum^1$ up to $\sum^4$ yields $\sum_j |F_n^*(v_j) - F_n^*(u_j)| < \epsilon$, and therefore $F_n^* << g_n$ uniformly in N_0. Our next step is to prove that $N_0 = N$.

Since $V(F_n) \leq V(F_n^*) + \sum_j |F(t_j^n) - F(s_j^n)|$ for all n, we see that F_n is $B.V.$ for all $n \in N_0$. Moreover, we observe that

$$F_n(x) = \{\ F_{\ n+1}\,(x). \qquad \mathrm{if x \notin K_n F_{n+1}(s_n^j) if x \in]s_n^j, t_n^j[.}$$

This follows since $K_n^c \subset K_{n+1}^c$ for all n, and (of course) $s_n^j \notin K_n$. Hence we can see that $V(F_n) \leq V(F_{n+1})$ for all n; indeed, if $\{x_0, x_1 \dots x_m\}$ is any decomposition of $[a, b]$, we have

$\sum_j |F_n(x_{j+1}) - F_n(x_j)| = \sum_j |F_{n+1}(s_{j+1}) - F_{n+1}(s_j)|,$

where s_j is a suitable point in $[a, x]$, and the intervals $]s_j, s_{j+1}[$ are disjoint. Thus

$$|F_n(x_{j+1}) - F_n(x_j)| \leq V(F_{n+1}) \mathrm{and so V(F_n) \leq V(F_{n+1}).}$$

Since $V(F_n) < +\infty$ for infinitely many integers n, we deduce that $V(F_n)$ is finite for all n in N. This also implies that $D_n < +\infty$ for all $n \in N$.

So far we have proved that $F_n^* << g_n$ uniformly in N, hence $dF_n^* << dg_n$ for all n. Let's set

$$f_n = dF_n^*/dg_n \text{for all n}.$$

Without loss of generality, we assume that f_n vanishes in $K - N$ for all n. Moreover, if $t \in K_N^c$, we have $F_n^*(t) = F_k^*(t)$ for each $k > n$. By 5.6, this implies that $dF_n^*(a) = dF_k^*(A)$ for all $k > n$, and for every Borel set $A \subset K_n^c$: hence we can assume that $f_n(t) = f_k(t)$ for all $t \in K_n^c$, and for $k > n$. Now we define

$$f(x) = \{ f_n(x) \quad \text{if x} \in \mathrm{K_n^c} \text{for some n}; 0, \quad \text{otherwise}.$$

We shall prove that f is integrable, and that F is its indefinite integral. Indeed, if we fix n, f is integrable with respect to g_n, since g_n has its support in K_n^c, and there f agrees with f_n, which is integrable with respect to g_n by construction: hence

$$\int f dg_n = \int f_n dg_n \text{for all n}.$$

This also shows that $F_n^*(x) = \int_a^x f dg_n$, and therefore f is $g-$integrable. We have already shown that $F_n \to F$ uniformly; since $|F_n^*(x) - F_n(x)| < D_n$ for all n, from $D_n \to 0$ we deduce that $F_n^* \to F$ uniformly, hence F is the indefinite integral of f. This concludes the proof.

We now turn to the Vitali Theorem.

6.2. Theorem, *Let $g : [a, b] \to R$ and suppose (g_n) is a generating sequence for g. Assume that (f^j) is a sequence of integrable functions, such that their integrals F^j are uniformly absolutely continuous with respect to g. If the sequence (f^j) is pointwise convergent to some function f, then f is integrable, and its integral function F is the uniform limit if (F^j).*

Proof. Let us denote by F_n^j the integral function of f^j with respect to g_n. In 5.7 we proved that, whenever $m > n$, we have

$$|F_m^j(x) - F_n^j(x)| = |\int_{[a,x] \cap K_n} f dg| \leq \sum_x |F_m^j(t_k^n) - F_m^j(s_k^n)|,$$

where $\sum_x$ runs over those components $]s_k^n, t_k^n[$ of K_n, such that $t_k^n \leq x$, including possibly a last interval $]s_k^n, x]$ in case $x \in K_n$.

In a similar fashion, given $x, y, x < y$, and $n, m, n < m$, and $j \in N$, we see that

$$|(F_m^j(y) - F_m^j(x)) - F_n^j(y) - F_n^j(x))| = |\int_{[x,y]\cap K_n} f^j dg_m| \leq$$

$$\leq \sum_{(x,y)} |F_m^j(t_k^n) - F_m^j(s_k^n)|,$$

where $\sum_{(x,y)}$ runs over those components which are contained in $]x, y[$, including possible a first interval $[x, t[$ (in case $x \in K_n$), and last interval $]s, y]$ (in case $y \in K_n$).

Now fix $\epsilon > 0$. Since the modulus of continuity of g_n is less than the modulus of continuity of g, we can find an index N such that

$$\sum_{(x,y)} |F_m^j(t_k^n) - F_m^j(s_k^n)| \leq \sum_{k=1}^{n} |F_m^j(t_k^n) - F_m^j(s_k^n)| + \epsilon \mathrm{forallm} > \mathrm{n.Thus}$$

$$||F^j(y) - F^j(x)| - |F_n^j(y) - F_n^j(x)|| = \lim_{m\to\infty} ||F_m^j(y) - F_m^j(x)| - |F_n^j(y) - F_n^j(x)|| \leq$$

$$\leq \lim_{m\to\infty} |(F_m^j(y) - F_m^j(x)) - (F_n^j(y) - F_n^j(x))| \leq \lim_{m\to\infty} \sum_{(x,y)} |F_m^j(t_k^n) - F_m^j(s_k^n)| \leq$$

$$\leq \lim_{m\to\infty} \sum_{|\leq k\leq N} |F_m^j(T_k^n) - F_m^j(s_k^n)| + \epsilon = \sum_{|\leq k\leq N} |F^j(t_k^n) - F^j(s_k^n)| + \epsilon \leq$$

$\leq \sum_{(x,y)} |F^j)(t_k^n) - F^j(s_k^n)| + \epsilon$. Since ϵ is arbitrary, we obtain

$$(6.2.1) ||F^j(y) - F^j(x)| - |F_n^j(y) - F_n^j(x)|| \leq \sum_{(x,y)} |F^j(t_k^n) - F^j(s_k^n)|.$$

We shall now prove that $F_n^j << g_n$, uniformly both in n and j. In other words, we shall prove that, for each $\epsilon > 0$ there exists a $\delta > 0$ such that, for every n, whenever $(]x_j, y_j[)$ is a $(g_n, \delta)-$small sequence. We have

$\sum_i |F_n^j(y_i) - F_n^j(x_i)| < \epsilon \mathrm{forallj}$.

Indeed, let δ be the number corresponding to ϵ in the condition $F^j << g$ uniformly (which is true by assumption); choose any integer n, and any $(g_n, \delta)-$small sequence $(]x_j, y_i[)$.

Then we can associate with every x_i an element s_j, and with every y_i an element t_i in such a way that

a) $0 \leq t_j - s_j \leq y_i - x_i$ for all i.

b) s_i and t_i are endpoints of (possibly distinct) components of K_n;

c) g_n is constant in the interval from s_i to x_i, and from t_i to y_i, for all i.

Thus we have

$\sum_i |F_n^j(y_i) - F_n^j(x_i)| = \sum_i |F_n^j(t_i) - F_n^j(s^i)|$.Inviewofa)above

$\sum_i |g(t_i) - g(s_i)| \leq \delta$, and therefore $\sum_i |F^j(t_i) - F^j(s_i)| \leq \epsilon$.

By virtue of (6.2.1), we see that $\sum_i |F_n^j(y_i) - F_n^j(x_i)| =$

$\sum_i |F_n^j(t_i) - F_n^j(s_i)| \leq \sum_i |F^j(t_i) - F^j(s_i)| + \sum_i \sum_{(s_i,t_i)} |F^j(t_k^n) - F^j(s_k^n)| \leq 2\epsilon$

since the intervals $]s_k^n, t_k^n[$ are a $\delta-$small family and satisfy $\sum_k |g(t_k^n) - g(s_k^n)| = 0$. This shows that $F_n^j << g_n$ uniformly in n and j.

Thus, by the classical Vitali Theorem, we deduce that $f = \lim f^j$ is $g-$ integrable for all n, and

(6.2.2) $\lim\limits_{j\to\infty} \int_\alpha^x f^j dg_n = \int_\alpha^x f dg$forallnandallx.

Uniformity, in n implies that f is integrable, that is , $F_n << g_n$ uniformly; to see this, it is enough to let j tend to ∞ in the last chain of inequalities above.

Our final step is to prove that $F = \lim F^j$, which follows from the convergence $F_n^j \to F^j$, uniform both in x and j; indeed, if $\epsilon > 0$ is fixed, for $m > n$ we have

$|F_m^j(x) - F_n^j(x)| \leq \sum_x |F_m^j(t_k^n) - F_m^j(s_k^n)|$, and this is less than ϵ as soon as n is large enough, independently of x and j, because of the uniform absolute continuity of F_m^j with respect to g_m, uniform in j and m.

Now the conclusion follows by interchanging limits of F_n^j.

7.1. <u>Proposition</u>. *Let g be a continuous B.V function on $[a,b]$ and f be any locally bounded, measurable map on the real line. Denote by F the indefinite integral of f, with respect to Lebesgue measure, that is*

$$F(x) = \int_0^x f(t)dt.$$

Then $f \circ g$ is bounded and Stieltjes-intergrable with respect to g and

(7.1.0) $\int_a^x f \circ g dg = F(g(x)) - F(g(a)), \text{forallx}.$

Proof. We notice that (7.1.0) holds, when f is continuous and g is in class C^1.

First step: we assume that f is continuous and g is continuous and $B.V.$ Without loss of generality, we extend g out of $[a, b]$, by imposing that it is constant in the half lines $]-\infty, a], [b, +\infty[$. Then we pick any approximate identity on R, say (u_h), and set g_h =$G * u_h$($*$ =convolution). Since g_h converges to g uniformly, and in the sense of distributions, one can deduce that

(7.1.1) $\int \tilde{f} dg_h \to \int \tilde{f} dg \text{foranycontinuous} \tilde{f}$

Now let's turn to (7.1.0); we have $\int_a^x f \circ g_h dg_h = F(g_h(x)) - F(g_h(a))$, since each g_h is in class C^1. Moreover it is clear that $F(g_h(x)) - F(g_h(a))$ converges to $F(g(x)) - F(g(a))$. It remains to prove that $\int f \circ g_h dg_h$ converges to $\int f \circ g dg$. This follows by setting $= \tilde{f} \circ g$ in (7.1.1), and observing that $|\int f(g) dg_h - \int f(g_h) dg_h|$ tends to 0 with h, since the variations of g_h are uniformly bounded.

Second step: same hypotheses for g, but f is just locally bounded and measurable.

Let R be the range of g; denote by λ lebesgue measure, and set

$$\mu(A) = |dg|(g^{-1}(A)),$$

for all Borel sets $A \subset R$: it is clear that μ is a measure. Now pick any sequence (u_n) of continuous functions, uniformly bounded on R, and converging in R to $f(\lambda + u)$-almost everywhere; this implies that $u_n \circ g$ converges to $f \circ g dg - a.e.$in[a.b]. Since the convergence is dominated

$$\int_a^x u_n \circ g dg \to \int_a^x f \circ g dg \text{forallx}.$$

We already know, fromt he first set, that $\int u_n \circ g dg = \int_{g(a)}^{g(x)} f(t) dt$, again by dominated convergence. Since $\int_{g(a)}^{g(x)} f(t) dt = F(g(x)) - F(g(a))$, the theorem follows.

Now we can now present an important theorem.

7.2. Theorem. *Let $g : [a, b] \to R$ be any continuous function, and $u : R \to R$ be locally bounded and measurable. Then $u \circ g$ is integrable with respect to g for any* *generating sequence, and moreover*

$$\int_a^x u \circ g dg - U(g(x)) - U(g(a)),$$

for every $x \in [a,b]$, *where* U *is the integral function of* u *with respect to Lebesgue measure.*

Proof. Fix any generating sequence (g_n). We put

$$F_n(x) = \int_a^x u \circ g dg_n, \text{forallx},$$

and observe that $F_n(x) = \int_{[a,x]\cap K_n^c} u \circ g dg_n = \int_{[a,x]\cap K_n^c} u \circ g_n dg_n =$

$$= \int_a^x u \circ g_n dg_n = U(g_n(x)) - U(g_n(a)) \text{by7.1.}$$

Now, it is clear that $\lim_n F_n(x) = U(g(x)) - U(g(a))$ uniformly in x. However, our definition of integrability also requires that $F_n << g_n$ uniformly; which we now establish.

Since the sequence (g_n) is uniformly convergent, there exists a constant $A > 0$ such that $|g_n(x)| \leq A$ for all n and all x.

Now, u is bounded in $[-A, A]$, say $|u(x)| \leq K$ for every $x \in [-A, A]$. If x, y are in $[a,b], x < y$, then

$$F_n(y) - F_n(x) = U(g_n(y)) - U(g_n(x)) = \int_{g_n(x)}^{g_n(y)} u(t)dt \text{andtherefore}$$

$$|F_n(y) - F_n(x)| \leq K|g_n(y) - g_n(x)|,$$

which shows that $F_n << g_n$ uniformly.

7.3. Remark. In theorem 7.2 the condition of local boundedness cannot be replaced by the local summability of u, contrary to an assertion made by Cacioppoli, as the following example shows.

Define $g(x) = \{ x \sin(1/x), \quad x \in]0, \pi]$
0, $\quad x = 0,$

and define u as follows: $u(t) = \{ 1 /(2\sqrt{|t|}), \quad t \neq 0$
0, $\quad t = 0$

Note that u is locally summable, but not locally bounded, and

$$U(x) = \int_0^x u(t)dt = \sqrt{|x|} \text{forallx} \in \mathrm{R}.$$

Assume that $u \circ g$ is integrable, for some generating sequence (g_n). then $u \circ g_n$ must be integrable with respect to g_n for all n, as shown in the proof of 7.2., and we have $F_n(x) = \int_0^x u(g_n dg_n = U(g_n)(x) - U(g_n(0))$, for all n, and all $x \in [0, \pi]$. Therefore, $F(x) = \lim_{n\to\infty} F_n(x) = U(g(x)) = \sqrt{|g(x)|}$ must be absolutely continuous with respect to g; we shall now see that this is false.

Assume $F << g$; then, if we fix $\varepsilon < \frac{1}{2}$ there exists $\delta = \delta(\varepsilon) > 0$ in the definition of absolute continuity. Now choose $m \in I\!N$ so that

a) $n(n-1)\pi/2 > 1/\delta$,

b) $2[\sqrt{n}]/\pi n < \delta$,

c) $[\sqrt{n}]\sqrt{(2/\pi} \geq \sqrt{(n + [\sqrt{n}])}/2$

hold simultaneously for all $n > m$. Fix $n > m$, and choose intervals $J_j =]x_j, y_j[, j = n+1 \ldots n + [\sqrt{n}]$, by putting $x_j = 2/(j\pi), y = 2/((j-1)\pi)$. these intervals are pairwise disjoint, and are contained in $]1/(n\pi), 1/\pi[$. It's clear that $|J_j| = 2/(j^2\pi - j\pi) < \delta$ by a) above, for all j.

Moreover, if j is odd, we have $|g(y_j) - g(x_j)| = x_j$,

whereas, if j is even, we see that $|g(y_j) - g(x_j)| = y_j$.

In any case, we have $|g(y_j) - g(x_j)| \leq 2/(j\pi - \pi)$. In the same fashion we find $|F(y_j) - F(x_j)| \geq \sqrt{(2/\pi)}/\sqrt{j}$. Then $\sum_{j=n+1}^{n+[\sqrt{n}]} |g(y_j) - g(x_j)| \leq \sum_{j=n+1}^{n+[\sqrt{n}]} 2/(j\pi - \pi) < 2[\sqrt{n}]/\pi n < \delta$ by b), but $\sum_{j=n+1}^{n+[\sqrt{n}]} |F(y_j) - F(x_j)| \geq \sum_{j=n+1}^{n+[\sqrt{n}]} (2/\pi j)^{\frac{1}{2}} \geq [\sqrt{n}]\sqrt{(2/\pi}/\sqrt{(n + [\sqrt{n}])} \geq \frac{1}{2}$.

This contradicts the condition $F << g$, hence $u \circ g$ is not integrable.

<u>Remark</u>. As a result of Theorem 7.2, as seen in the discussion of the stochastic integrals in section 2, we see how pathwise versions of the Stratonovitch and Itô stochastic integrals in the restrictive setting earlier mentioned can be achieved. Our view is that the Caccioppoli integral, although delicate and applicable only in certain settings is a link between the sophisticated machinery of stochastic integration theory and the classical Stieltjes-type of integrals on the real line, which may find

use in applications.

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J.K. Brooks
Department of Mathematics
P.O. Box 118105
University of Florida
Gainesville, Florida 32611-8105

D. Candeloro
Dipartmento di Matematica
e Informatica
Via Vanvitelli 1
06123 Perugia (I)