Integration by parts with respect to the Henstock-Stieltjes integral in Riesz spaces

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ABSTRACT. Some versions of the formula of integration by parts with respect to the Henstock-Kurzweil and the Henstock-Stieltjes integrals in Riesz spaces are given.

1 Introduction.

The Henstock-Kurzweil integral for Riesz-space-valued maps was introduced by B. Riečan in [17], and in [19] and [20] some convergence theorems with respect to this integral were proved. Moreover, in [10], the Henstock-Stieltjes integral in Riesz spaces is introduced, and some theorems of existence are proved. For a recent story of the Henstock-Kurzweil and the Henstock-Stieltjes integral in vector lattices see [18], [1] and their bibliographies. Moreover, for related topics, see also [6], where some applications to various types of stochastic integral are given.

In this paper we deal with the formula of integration by parts for the Henstock-Stieltjes integral in Riesz spaces and we give some applications to Fourier analysis. (For analogous formulas relatively to the Riemann-Stieltjes integral, see [9] and [6]). In the real case, there are many versions of it, which historically are given firstly relatively to Perron-Stieltjes integral, and afterwards with respect to Henstock-Stieltjes integral, although these integrals are equivalent (see also [11]). For a survey about it, see [3].

Our thanks to Prof. M. Duchtina and B. Riečan for suggesting this research and to Prof. D. Candeloro for his helpful suggestions.

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A.M.S. CLASSIFICATION: 28A15, 28B05.
KEY WORDS: Riesz spaces, Henstock-Stieltjes integral, integration by parts.
2 Preliminaries.

Let \( R \) be a Dedekind complete Riesz space. In some suitable cases, we add to \( R \) an extra element, \(+\infty\), extending ordering and operations, having the same role as the usual \(+\infty\) with the real numbers. We denote with the symbol \( \mathcal{R} \) the set \( R \cup \{+\infty\} \).

A Dedekind complete Riesz space is said to be super Dedekind complete if every subset \( R_1 \neq \emptyset \) bounded from above contains a countable subset having the same supremum as \( R_1 \).

Let \((\Lambda, \succeq)\) be an ordered directed set. A net \((r_\lambda)\) is said to be order convergent (or \((o)\)-convergent) to \( r \), if \( \sup_{\rho \in \Lambda} \inf_{\lambda \succeq \rho} r_\lambda = \inf_{\rho \in \Lambda} \sup_{\lambda \succeq \rho} r_\lambda = r \). An \((o)\)-net \((p_\lambda)\) is any decreasing net of positive elements of \( R \), convergent to zero.

From now on, denote by \( \Delta \) the set of all positive real-valued functions, defined on an interval \([a, b] \subset \mathbb{R} \).

**Definition 2.1** We say that \( f : [a, b] \to R \) is 
\((u)\)-continuous if there exists an \((o)\)-net \((p_\delta)\) such that \( \forall \delta \in \Delta \)
\[
|f(t) - f(x)| \leq p_\delta \quad \text{whenever } t, x \in [a, b] \text{ and } |t - x| \leq \delta(x).
\]

We note that the concept of \((u)\)-continuity is "continuity with respect to the same regulator", which is in general different both from continuity and from uniform continuity (see also [1]).

**Definition 2.2** A mapping \( f : [a, b] \to R \) is 
\((u)\)-differentiable if there exists a function \( g : [a, b] \to R \) such that
\[
(o) - \lim_\delta \left( \sup \left\{ \left| \frac{f(v) - f(u)}{v - u} - g(x) \right| : u, v, x \in A_\delta \right\} \right) = 0,
\]
where \( A_\delta \equiv \{u, v, x \in [a, b] : u \neq v \text{ and } x - \delta(x) \leq u \leq x \leq v + \delta(x)\} \) for all \( \delta \in \Delta \).

In this case, the map \( g \) will be called the \((u)\)-derivative of \( f \), or simply derivative, when no confusion can arise. It is easy to see that every \((u)\)-differentiable map is differentiable too, and the involved derivatives coincide. In general, the converse implication is not true (see also the example in [1]), but it is clear that, when \( R = \mathcal{R} \), the concepts of differentiability and \((u)\)-differentiability coincide.

**Definition 2.3** A decomposition of \([a, b]\) is a set of the type \( \{(A_i, \xi_i) : i = 1, \ldots, n\} \), where \( \{A_i\}_{i=1}^n \) is a family of pairwise nonoverlapping intervals of \([a, b]\) and \( \xi_i \in A_i \) for all \( i = 1, \ldots, n \). If \( \bigcup_{i=1}^n A_i = [a, b] \), then the decomposition \( \{(A_i, \xi_i) : i = 1, \ldots, n\} \) is called partition.

Given a decomposition \( E = \{(x_{i-1}, x_i], \xi_i), i = 1, \ldots, n\} \) and a function \( \delta \in \Delta \), we say that \( E \) is \( \delta \)-fine if \( x_i - x_{i-1} \leq \delta(\xi_i) \) for all \( i = 1, \ldots, n \).
We now are ready to introduce a property, which can be related to absolute continuity (and the so-called "property N") and implied by it, but in general not completely equivalent.

**Definition 2.4** Let $J \subset [a, b] \subset \mathbb{R}$, and $\Delta_J = (\mathbb{R}^+)^J$. We say that a function $P : [a, b] \rightarrow R$ is of class (SL) or has property (SL) on $J$ if for every set $Z \subset [a, b]$ of Lebesgue measure zero there exists an $(o)$-net $(p^Z_\delta)_{\delta \in \Delta_J}$ (possibly in $\mathcal{R}$), such that $\forall \delta \in \Delta_J$

$$\sup \left\{ \sum_{i=1}^{n} |P(x_i) - P(x_{i-1})| : [x_{i-1}, x_i], \xi_i \right\}$$

is a $\delta$–fine decomposition of $[a, b]$, $x_i \in [a, b]$ and $\xi_i \in Z \cap J \forall i \leq p^Z_\delta$.

We say that $P$ is of class (SL) (or simply (SL)) if it is of class (SL) on $[a, b]$.

We note that, even in the case $R = \mathbb{R}$, property (SL) in general does not imply boundedness of variation (see also [12]). Moreover, for any Dedekind complete Riesz space $R$, every $R$-valued absolutely continuous map in $[a, b]$ is (SL) in $[a, b]$ (see also [1]).

We now recall the Maeda-Ogasawara-Vulikh representation theorem for Riesz spaces.

**Theorem 2.5** Given an Archimedean [ Dedekind complete ] Riesz space $R$, there exists a compact Stonian topological space $\Omega$, unique up to homeomorphisms, such that $R$ can be embedded order densely as a solid subspace of $\mathcal{C}_\infty(\Omega) \equiv \{ f \in \tilde{\mathbb{R}}^\Omega : f \text{ is continuous, and } \{ \omega : |f(\omega)| = +\infty \} \text{ is nowhere dense in } \Omega \}$. Moreover, if $(a_\lambda)_{\lambda \in \Lambda}$ is any family such that $a_\lambda \in R \forall \lambda$, and $a = \sup_\lambda a_\lambda \in R$ (where the supremum is taken with respect to $R$), then $a = \sup_\lambda a_\lambda$ with respect to $\mathcal{C}_\infty(\Omega)$, and the set $\{ \omega \in \Omega : (\sup_\lambda a_\lambda)(\omega) \neq \sup_\lambda a_\lambda(\omega) \}$ is meager in $\Omega$.

A Dedekind complete Riesz space is called algebra if there exists a "product" map $\cdot : R \times R \rightarrow R$, compatible with respect to the operations of sum, order and order limits. Some examples of algebras are $C(W)$ (the space of all continuous real-valued functions defined in $W$), $C_\infty(W)$ (the symbol $C_\infty$ has the same rôle as in 2.5), where $W$ is a locally compact extremely disconnected topological space, and $L^0(X, \mathcal{B}, \mu)$, where $(X, \mathcal{B}, \mu)$ is a measure space, with $\mu : \mathcal{B} \rightarrow \tilde{\mathbb{R}}_0^+ \sigma$-additive and $\sigma$-finite.

### 3 Integration by parts for the (H)-integral.

**Definition 3.1** A function $\delta : [a, b] \rightarrow \mathbb{R}_0^+$ is called gage if the set $Z_\delta \equiv \{ x \in [a, b] : \delta(x) = 0 \}$ has Lebesgue measure zero.
We now endow the set $\Delta^{(1)}$ of all gages with the following ordering. Given two gages $\delta_1$ and $\delta_2$, we say that $\delta_1 \geq \delta_2$ if and only if $\delta_1(x) \leq \delta_2(x)$ for all $x \in [a, b]$.

**Definition 3.2** If $\delta$ is a gage, we say that a decomposition $E = \{ (A_i, \xi_i) : i = 1, \ldots, n \}$ is $\delta$-fine if $\xi_i \not\in Z_\delta$ and $|A_i| \leq \delta(\xi_i)$ for all $i$.

We now recall the definition of ($H$)-integral (Henstock-Kurzweil integral) given in [1].

**Definition 3.3** A map $f : [a, b] \to R$ is ($H$)-integrable if there exists an element $Y \in R$ such that

$$
(o) - \lim_{\delta \in \Delta} \sup \{ |S(f, E) - Y| : E \text{ is a } \delta \text{-fine partition of } [a, b] \} = 0,
$$

where $E = ([x_{i-1}, x_i], \xi_i)_{i=1}^n$ and $S(f, E) = \sum_{i=1}^n f(\xi_i) (x_i - x_{i-1})$.

**Definition 3.4** Given an ($H$)-integrable function $f : [a, b] \to R$, set

$$
F(x) = \begin{cases} 
(H) \int_a^x f & \text{if } a < x \leq b, \\
0 & \text{if } x = a.
\end{cases}
$$

The map $F$ will be called the ($H$)-integral function (or integral function, when no confusion can arise) associated to $f$.

We begin with the following

**Proposition 3.5** Let $R$ be a super Dedekind complete Riesz space, $R \subset C_\infty(\Omega)$, where $\Omega$ is as in Theorem 2.5, and assume that $f : [a, b] \to R$ is ($H$)-integrable. Then there exists a meager set $N \subset \Omega$ such that

$$
f(x)(\omega) \in R \quad \forall x \in [a, b] \text{ and } \forall \omega \notin N.
$$

**Proof.** Let $R$ and $\Omega$ be as in the hypotheses, and $F$ be as in 3.4. By [1], Proposition 4.37, we know that $F$ is bounded. Thus there exists an element $M \in R$ such that

$$
|F(x)(\omega)| \leq M(\omega) \quad \forall \omega \in \Omega, \forall x \in [a, b];
$$

$$
M(\omega) \in R^+ \forall \omega \in \Omega \setminus N^*.
$$
where $N^* \subset \Omega$ is nowhere dense in $\Omega$. Now for each $x \in [a, b]$ and $\delta \in \Delta$, denote by $E^{(x)}_\delta$ the set of all $\delta$-fine partitions of $[a, x]$. By virtue of the Henstock Lemma, we get

$$(o)-\lim_{\delta \in \Delta} \left[ \sup_{x \in [a, b]} \left( \sup \left\{ \sum_{i=1}^{n} |f(\xi_i)(x_i - x_{i-1}) - |F(x_i) - F(x_{i-1})| : E = ([x_{i-1}, x_i], \xi)_{i=1}^{n} \in E^{(x)}_\delta \right\} \right) \right] = 0.$$ 

As $R$ is super Dedekind complete, there exists a sequence $(\delta_k)_k$ of elements of $\Delta$, such that $\delta_k \downarrow 0$ and

$$(o) - \lim_k \left[ \sup_{x \in [a, b]} \left( \sup \left\{ \sum_{i=1}^{n} |f(\xi_i)(x_i - x_{i-1}) - |F(x_i) - F(x_{i-1})| : E \in E^{(x)}_{\delta_k} \right\} \right) \right] = 0.$$ 

By virtue of Theorem 2.5, there exists a meager set $N \subset \Omega$, $N \supset N^*$ (where $N^*$ is as in (4)) such that

$$\lim_k \left[ \sup_{x \in [a, b]} \left( \sup \left\{ \sum_{i=1}^{n} |f(\xi_i)(\omega)(x_i - x_{i-1}) - |F(x_i)(\omega) - F(x_{i-1})(\omega)| : E \in E^{(x)}_{\delta_k} \right\} \right) \right] = 0$$

for all $\omega \in \Omega \setminus N$. From this we get

$$\lim_k \left[ \sup_{x \in [a, b]} \left( \sup \left\{ \sum_{i=1}^{n} |f(\xi_i)(\omega)(x_i - x_{i-1}) - F(x)(\omega)| : E \in E^{(x)}_{\delta_k} \right\} \right) \right] = 0 \quad (5)$$

$\forall x \in [a, b]$, $\forall \omega \notin N$.

Fix now arbitrarily $\omega \notin N$. From (5) it follows that $\forall \varepsilon > 0$ there exists $\delta_\varepsilon \in \Delta$ such that

$$|f(\xi)(\omega)(y - x) - F(x)(\omega)| \leq \varepsilon \quad (6)$$

for all $\xi, x, y \in [a, b]$ with $\xi - \delta_\varepsilon(\xi) \leq x \leq \xi \leq y \leq \xi + \delta_\varepsilon(\xi)$. Let $\delta_1$ be the element of $\Delta$ associated with $\varepsilon = 1$, fix arbitrarily $\xi \in [a, b]$ and pick $x, y \in [\xi - \delta_1(\xi), \xi + \delta_1(\xi)] \cap [a, b]$ with $x < y$. By virtue of (6), we have

$$|f(\xi)(\omega)(y - x)| \leq M(\omega) + 1,$$

where $M$ is as in (4). Thus we get

$$f(\xi)(\omega) \in B \quad \forall \omega \notin N,$$

that is the assertion. □
Theorem 3.6 If $R = L^0(X, \mathcal{B}, \mu)$, where $(X, \mathcal{B}, \mu)$ is a measure space, with $\mu : \mathcal{B} \to \bar{\mathbb{R}}_0^+$ $\sigma$-additive and $\sigma$-finite, and $f : [a, b] \to R$ is $(H)$-integrable, then the function $F$ defined in 3.4 is of class $(SL)$ in $[a, b]$. Moreover, if $R$ is a super Dedekind complete Riesz space, then $F$ defined in 3.4 is $(u)$-continuous in $[a, b]$.

**Proof.** Proceeding analogously as in (5) it follows that there exist a $\mu$-null set $N \subset X$ and a sequence $(\delta_k)_k$ of elements of $\Delta$, such that $\delta_k \downarrow 0$ and such that $\forall x \in [a, b]$ and $\forall \omega \notin N$ there exists in $\mathcal{R}$ the quantity

$$F^{(\omega)}_{(\delta_k)_k} (x) = \lim_k \left[ \inf \left\{ \sum_{i=1}^{n} f(\xi_i)(\omega) (x_i - x_{i-1}) : E \in \mathcal{E}^{(x)}_{\delta_k} \right\} \right]$$

$$= \lim_k \left[ \sup \left\{ \sum_{i=1}^{n} f(\xi_i)(\omega) (x_i - x_{i-1}) : E \in \mathcal{E}^{(x)}_{\delta_k} \right\} \right]$$

(where the involved limits exist uniformly with respect to $x \in [a, b]$), and

$$F^{(\omega)}_{(\delta_k)_k} (x) = F(x)(\omega) \quad \forall x \in [a, b], \forall \omega \notin N. \quad (7)$$

By proceeding analogously as in [13], it is easy to check that for all $\omega \notin N$, the real-valued function

$$x \mapsto F^{(\omega)}_{(\delta_k)_k} (x)$$

is of class $(SL)$ in $[a, b]$. \quad (8)

Fix now arbitrarily a subset $Z \subset [a, b]$ of Lebesgue measure zero. For all $\omega \notin N$ we get:

$$\lim_k \left( \sup \left\{ \sum_{i=1}^{n} |F(\xi_i)(\omega) - F(\xi_{i-1})(\omega)| : E = (\{x_{i-1}, x_i\}, \xi_{i=1}^{n} \in \mathcal{F}^{(Z)}_{\delta_k} \right\} \right) = 0, \quad (9)$$

where $\mathcal{F}^{(Z)}_{\delta_k}$ is the set of all $\delta_k$-fine decompositions of $[a, b]$ of type $E \equiv \{(x_{i-1}, x_i), \xi_{i=1}^{n}\}$, such that $\xi_i \in Z \forall i$. From this we get

$$\lim_k \left( \sup \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : E \in \mathcal{F}^{(Z)}_{\delta_k} \right\} \right) = 0, \quad (10)$$

since in $L^0(X, \mathcal{B}, \mu)$ order convergence coincides with almost everywhere convergence. Thus the assertion follows. \qed

Remark 3.7 We observe that, by proceeding analogously as in 3.6, it is possible to prove $(u)$-continuity of the integral function, when $R \subset C_{\infty}(\Omega)$ is any super Dedekind complete Riesz space: indeed, it is enough to replace, in (8), ”(SL)” with ”continuous”, ”$\mu$-null” with ”meager”, and to replace the expressions in (9) and (10) with the quantities

$$\lim_k \left| \sup \{ |F(y)(\omega) - F(x)(\omega)| : |y - x| \leq \delta_k(x) \} \right|$$

6
and

\[(o) - \lim \limits_{k \to \infty} \sup \{|F(y) - F(x)| : |y - x| \leq \delta_k(x)\}\]

respectively (We note that the above sequence is bounded in \(R\), because of boundedness of the function \(F\)).

We now give a definition of Henstock-Stieltjes-type integral in Riesz spaces (see also [10], [18]).

**Definition 3.8** Let \(R\) be an algebra, \(f, \alpha : [a, b] \to R\) be two functions. We say that \(f\) is \((HS)\)-integrable with respect to \(\alpha\) if there exists an element \(Y \in R\) such that

\[(o) - \lim \limits_{\delta \to 0} \sup \{|S_\alpha(f, E) - Y| : E \text{ is a } \delta - \text{fine partition of } [a, b]\} = 0,

where \(E = ([x_{i-1}, x_i], \xi_i)_{i=1}^n, S_\alpha(f, E) \equiv \sum_{i=1}^n f(\xi_i) [\alpha(x_i) - \alpha(x_{i-1})].\)

Similarly as above, we can define the integral function relatively to the Henstock-Stieltjes integral.

**Definition 3.9** Given two functions \(f, \alpha : [a, b] \to R\) such that \(f\) is \((HS)\)-integrable with respect to \(\alpha\), set

\[F_\alpha(x) = \begin{cases} \text{(HS)} \int_a^x f \, d\alpha & \text{if } a < x \leq b, \\ 0 & \text{if } x = a. \end{cases}\]

The map \(F_\alpha\) will be called the \((HS)\)-integral function associated to \(f\) (with respect to \(\alpha\)).

From now on, without loss of generality, when we consider the Henstock-Stieltjes integral with respect to real-valued functions \(\alpha\), we always suppose that \(\alpha\) satisfies the following property:

A1) For each \(\xi \in [a, b]\) and for every neighbourhood \(U\) of \(\xi\) there exist \(x, y \in U\) such that

\[\alpha(x) \neq \alpha(y),\]

and we will not say it explicitly.

We observe that Proposition 3.5 holds true even when we replace \((H)\)-integrability with \((HS)\)-integrability with respect to \(\alpha\).

The proof of the following theorem is analogous to the one of Theorem 3.6.
Theorem 3.10 Let $R$ be a super Dedekind complete Riesz space. If $\alpha : [a, b] \to \mathbb{R}$ is continuous and $f : [a, b] \to R$ is $(HS)$-integrable with respect to $\alpha$, then the function $F_\alpha$ defined in 3.9 is $(u)$-continuous in $[a, b]$. If $R = L^0(X, \mathcal{B}, \mu)$ and $\alpha$ is (SL), then $F_\alpha$ is (SL) too.

We are now ready to state a version of the formula of integration by parts with respect to the

\[ \text{Theorem 3.10} \]

Let $\alpha : [a, b] \to \mathbb{R}$ be continuous, $G : [a, b] \to R$ be of bounded variation, $f : [a, b] \to R$ be an $(HS)$-integrable function with respect to $\alpha$, and $P = F_\alpha$ be the associated $(HS)$ integral function.

Then $f \cdot G$ is $(HS)$-integrable with respect to $\alpha$, and

\[ \int_a^b fG \, d\alpha = P(b)G(b) - \int_a^b P \, dG. \]

**Proof.** (see also [16]) Without loss of generality, we can assume that $G$ is nondecreasing.

By virtue of the Henstock Lemma, there exists an $(o)$-net $(r_\delta)_{\delta \in \Delta}$ such that $\forall \delta \in \Delta$ and for every $\delta$-fine decomposition $E = \{([x_{i-1}, x_i], \xi_i), i = 1, \ldots, n\}$ we have

\[ \sum_{i=1}^n |\alpha(x_i) - \alpha(x_{i-1})| f(\xi_i) - |P(x_i) - P(x_{i-1})| \leq r_\delta. \]

Let $(s_\delta)_{\delta \in \Delta}$ and $(q_\delta)_{\delta \in \Delta}$ be two $(o)$-nets, related with $(u)$-continuity of $P$ and $(HS)$-integrability of $P$ with respect to $G$ respectively. For each $\delta \in \Delta$, and for every $\delta$-fine partition $E$ of $[a, b]$, $E = \{([x_{i-1}, x_i], \xi_i)_{i=1}^n\}$, we have:

\[ \begin{align*}
0 \leq & \ |S_n(fG, E) + S_G(P, E) - P(b)G(b)| \\
\leq & \ \sum_{i=1}^n |\alpha(x_i) - \alpha(x_{i-1})| f(\xi_i) G(\xi_i) + \sum_{i=1}^n P(\xi_i) [G(x_i) - G(x_{i-1})] \\
& - \ \sum_{i=1}^n [P(x_i) - P(x_{i-1})] G(x_{i-1}) - \sum_{i=1}^n P(x_i) [G(x_i) - G(x_{i-1})] \\
= & \ \sum_{i=1}^n |\alpha(x_i) - \alpha(x_{i-1})| f(\xi_i) G(\xi_i) - \sum_{i=1}^n [P(x_i) - P(x_{i-1})] G(\xi_i) \\
& + \ \sum_{i=1}^n [P(x_i) - P(x_{i-1})] [G(\xi_i) - G(x_{i-1})] + \sum_{i=1}^n |P(\xi_i) - P(x_i)| [G(x_i) - G(x_{i-1})] \\
\leq & \ \sum_{i=1}^n |\alpha(x_i) - \alpha(x_{i-1})| f(\xi_i) - |P(x_i) - P(x_{i-1})| |G(\xi_i)| \\
& + \ \sum_{i=1}^n |P(\xi_i) - P(x_{i-1})| [G(\xi_i) - G(x_{i-1})] \\
\end{align*} \]

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\[ + \sum_{i=1}^{n} |P(x_i) - P(\xi_i)| \left[ (G(\xi_i) - G(x_{i-1})) + \sum_{i=1}^{n} |P(\xi_i) - P(x_i)| \right] G(x_i) - G(x_{i-1})] \]
\[ \leq r_\delta G(b) + s_\delta \sum_{i=1}^{n} [G(x_i) - G(x_{i-1})] + 2 s_\delta \sum_{i=1}^{n} [G(x_i) - G(x_{i-1})] \]
\[ \leq r_\delta G(b) + 3 s_\delta \sum_{i=1}^{n} [G(x_i) - G(x_{i-1})] \]
\[ \leq r_\delta G(b) \]

So we get
\[ 0 \leq \left| S_\alpha(f_\delta, E) + \sum_{i=1}^{n} (HS) \int_{x_{i-1}}^{x_i} P \, dG - \sum_{i=1}^{n} [P(x_i) G(x_i) - P(x_{i-1}) G(x_{i-1})] \right| \]
\[ \leq S_G(P, E) - \sum_{i=1}^{n} (HS) \int_{x_{i-1}}^{x_i} P \, dG \]
\[ + \left| S_\alpha(f_\delta, E) + S_G(P, E) - \sum_{i=1}^{n} [P(x_i) G(x_i) - P(x_{i-1}) G(x_{i-1})] \right| \]
\[ \leq r_\delta G(b) + 3 s_\delta V(G, [a, b]) + q_\delta. \]

Thus the assertion follows. \( \square \)

**Remark 3.12** Obviously, in general it is not advisable to relate "classical" primitives with the (H)-integral, even in the real case. However, if one considers the concept of weak primitive formulated in [1], one finds that the (H)-integral function of any (H)-integrable function \( f \) is a weak primitive of \( f \), and that any two weak primitives of the same function \( f \) differ by a constant.

**Definition 3.13** Let \( R \) be any Dedekind complete Riesz space, and \( \Delta^{(1)} \) be the set of all gages, defined on \([a, b]\). Let \( f, P : [a, b] \to R \). We say that \( P \) is a weak primitive of \( f \) if \( P \) is (SL) and if there exists an \((o)\)-net \((\eta_\delta)_{\delta \in \Delta^{(1)}}\) such that
\[
\sup \left\{ \left| S(f, E) - \sum_{i=1}^{n} [P(x_i) - P(x_{i-1})] \right| : E = \{[x_{i-1}, x_i], \xi_i, i = 1, \ldots, n\} \text{ is a } \delta-\text{fine decomposition of } [a, b] \right\} \leq p_\delta
\]
(Here we denote by \( S(f, E) \) the quantity \( \sum_{i=1}^{n} f(\xi_i) (x_i - x_{i-1}) \).

An immediate consequence of the Henstock Lemma and Theorem 3.6 is that for every (H)-integrable function \( f \), its (H)-integral function \( F \) is a weak primitive of \( f \). We note that in general there are some Dedekind complete Riesz spaces \( R \neq R \) and some functions \( f : [a, b] \to R \),
which have weak primitives but are not \((H)\)-integrable (more precisely, there exist no element \(Y\) satisfying (2), see also \([1]\)); however, if \(R = \mathbb{R}\), \(f\) is \((H)\)-integrable if and only if \(f\) has weak primitives (see also \([13]\)).

Analogously one can formulate a definition of weak primitive relatively to the Henstock-Stieltjes integral and check that, if \(\alpha\) is of class \((SL)\) and \(f\) is \((HS)\)-integrable with respect to \(\alpha\), then the \((HS)\)-integral function \(F_\alpha\) is a weak primitive of \(f\).

In \([1]\) we proved the following:

**Proposition 3.14** Let \(f : [a, b] \to R\) admit two weak primitives \(P_1\) and \(P_2\). There exists an element \(r \in \mathbb{R}\) such that \(P_2(x) = P_1(x) + r\) for all \(x \in [a, b]\).

Analogous versions of Definition 3.13 and Proposition 3.14 can be given if we consider the \((HS)\)-integral (with respect to an \((SL)\) map \(\alpha\)) instead of the \((H)\)-integral.

An easy consequence of Theorems 3.11 and 3.6 and Proposition 3.14 is the following

**Theorem 3.15** Let \(R\) be a super Dedekind complete algebra, \(\alpha : [a, b] \to \mathbb{R}\) be continuous and of class \((SL)\), \(G : [a, b] \to R\) be of bounded variation, \(f : [a, b] \to R\) be an \((HS)\)-integrable function with respect to \(\alpha\), and \(P\) be a weak primitive of \(f\).

Then \(f \cdot G\) is \((HS)\)-integrable with respect to \(\alpha\), and

\[
(HS) \int_a^b fG \, d\alpha = P(b)G(b) - P(a)G(a) - (HS) \int_a^b P \, dG.
\]

**Remark 3.16** We note that in general the \((H)\)-integral has no "immediate" multiplicative property: indeed, even in the real case, there are some \((H)\)-integrable functions \(f\) and some Riemann integrable maps \(G\) such that \(fG\) is not \((H)\)-integrable (see also \([16]\) p. 107).

We shall prove another formula of integration by parts relatively to \((HS)\)-integral in Riesz spaces. We begin with the following

**Lemma 3.17** Let \(R = L^0(X, \mathcal{B}, \mu)\), where \((X, \mathcal{B}, \mu)\) is as above, \(\alpha : [a, b] \to \mathbb{R}\), \(h_1 : [a, b] \to R\) be a \((HS)\)-integrable function with respect to \(\alpha\) and \(h_2 : [a, b] \to R\) be a \((u)\)-continuous map. Then

\[
(o) - \lim_{\delta \to 0} \sup_{\xi \in \Delta} \{|h_1(x)| |h_2(x) - h_2(\xi)| : x, \xi \in [a, b], |\xi - x| \leq \delta(\xi)\} = 0. \tag{12}
\]

**Proof.** For all \(\omega \in X\) define \(h_1^{(\omega)} : [a, b] \to \mathbb{R}\) by setting \(h_1^{(\omega)}(x) = h_1(x)(\omega)\) for all \(x \in [a, b]\). By \([1]\), Proposition 3.8, there exists a \(\mu\)-null set \(N_0 \subset X\) such that \(\forall \varepsilon > 0, \forall \omega \notin N_0\), there exists \(\delta \in \Delta\) such that

\[
|h_2(\xi)(\omega) - h_2(x)(\omega)| \leq \varepsilon \quad \forall \, x, \xi \in [a, b] \text{ with } |\xi - x| \leq \delta(\xi). \tag{13}
\]
Moreover, as \( h_1 \) is \((HS)\)-integrable, by virtue of Proposition 3.5 there exists a \( \mu \)-null set \( N \subset X \), \( N \supset N_0 \), such that

\[
h_1(\xi)(\omega) \in \mathbb{R} \quad \forall \xi \in [a, b] \text{ and } \forall \omega \not\in N.
\]

(14)

By (13) and (14) it follows that \( \forall \varepsilon > 0, \forall \omega \not\in N \), there exists \( \delta_0 \in \Delta \) such that

\[
|h_1(\xi)(\omega)| |h_2(\xi)(\omega) - h_2(x)(\omega)| \leq \varepsilon \quad \forall x, \xi \in [a, b] \text{ with } |\xi - x| \leq \delta_0(\xi).
\]

(15)

From this we obtain

\[
(o) - \lim_{\delta \to \Delta} \sup \{|h_1(\xi)| |h_2(\xi) - h_2(x)| : x, \xi \in [a, b], |\xi - x| \leq \delta(\xi)| = 0,
\]

that is the assertion. \( \square \)

**Theorem 3.18** Let \( R = L^0(X, \mathcal{B}, \mu) \), \( \alpha : [a, b] \to \mathbb{R} \) be continuous and of bounded variation, \( f, g : [a, b] \to R \) be two \((HS)\)-integrable functions with respect to \( \alpha \) and \( F, G \) the \((HS)\)-integral functions of \( f \) and \( g \) respectively.

Then \( f \cdot G + g \cdot F \) is \((HS)\)-integrable with respect to \( \alpha \), and

\[
\int_a^b (f \cdot G + g \cdot F) \, d\alpha = F(b)G(b).
\]

**Proof.** (See also [15], Theorem, pp. 230-231) First of all, we observe that \( F \) and \( G \) are bounded, and hence there exists \( M \in R \) such that \( |F(x)| \leq M, \ |G(x)| \leq M \ \forall \ x \in [a, b] \). Moreover, by \((HS)\) integrability of \( f \) and \( g \) and Henstock’s Lemma, there exist two \((o)\)-nets \( (p_\delta)_{\delta \in \Delta} \) and \( (q_\delta)_{\delta \in \Delta} \) such that \( \forall \delta \in \Delta \) and for all \( \delta \)-fine partitions \( E = \{([x_{i-1}, x_i], \xi_i), i = 1, \ldots, n\} \) we have

\[
\sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})| f(\xi_i) - [F(x_i) - F(x_{i-1})]| \leq p_\delta
\]

and

\[
\sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})| g(\xi_i) - [G(x_i) - G(x_{i-1})]| \leq q_\delta.
\]

Moreover, by Lemma 3.17, there exist two \((o)\)-nets \( (w_\delta)_{\delta \in \Delta} \) and \( (z_\delta)_{\delta \in \Delta} \) such that \( \forall \delta \in \Delta \) and for all \( x, \xi \in [a, b] \) with \( |\xi - x| \leq \delta(\xi) \) we get

\[
|f(\xi)| |G(\xi) - G(x)| \leq w_\delta, \quad |F(\xi) - F(x)| |g(\xi)| \leq z_\delta.
\]

For every \( \delta \in \Delta \) let \( E_\delta \) be the set of all \( \delta \)-fine partitions. For each \( E \in E_\delta \) we have

\[
0 \leq \left| \sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})|[f(\xi_i) G(\xi_i) + F(\xi_i) g(\xi_i)] - F(b)G(b) \right|
\]
By virtue of hypothesis, (o) of Proposition 3.19:

\[ \sum_{i=1}^{n} \{ [\alpha(x_i) - \alpha(x_{i-1})] f(\xi_i)G(\xi_i) + [\alpha(x_i) - \alpha(x_{i-1})] F(\xi_i)g(\xi_i) \} \]

\[ - \sum_{i=1}^{n} |F(x_i)G(x_i) - F(x_{i-1})G(x_{i-1})| \]

\[ \leq \sum_{i=1}^{n} |[\alpha(x_i) - \alpha(x_{i-1})] f(\xi_i)G(\xi_i) - G(x_i)[F(x_i) - F(x_{i-1})]| \]

\[ + \sum_{i=1}^{n} |[\alpha(x_i) - \alpha(x_{i-1})] F(\xi_i)g(\xi_i) - F(\xi_i)G(x_i) - G(x_{i-1})]| \]

\[ \leq \sum_{i=1}^{n} |[\alpha(x_i) - \alpha(x_{i-1})] f(\xi_i) - [F(x_i) - F(x_{i-1})]| |G(\xi_i)| \]

\[ + \sum_{i=1}^{n} |F(x_{i-1})| |[\alpha(x_i) - \alpha(x_{i-1})] g(\xi_i) - [G(x_i) - G(x_{i-1})]| \]

\[ + \sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})||f(\xi_i)||G(\xi_i) - G(x_i)| \]

\[ + \sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})| |F(\xi_i) - F(x_i)||g(\xi_i)| \]

\[ \leq M(p_\delta + q_\delta) + M V(\alpha, [a, b])(w_\delta + z_\delta). \]

So we obtain

\[ (o) - \lim_{\delta \to 0} [\sup \{ |S_\alpha(fG + Fg) - F(b)G(b) : E \in \mathcal{E}_\delta \}] = 0. \]

Thus the assertion follows. \( \Box \)

We now prove the following

**Proposition 3.19** Let \( R \) be any Dedekind complete Riesz space; \( \alpha : [a, b] \to R, g : [a, b] \to R; f : [a, b] \to R \) be \((H)\)-integrable, and \( F \) be the integral function associated to \( f \) with respect to \( \alpha \). Then \( gf \) is \((HS)\)-integrable with respect to \( \alpha \) if and only if \( g \) is \((HS)\)-integrable with respect to \( F \), and in this case we have

\[ (HS) \int_{a}^{b} gf d\alpha = (HS) \int_{a}^{b} g dF. \]

**Proof.** We begin with the ”only if” part. For all \( \delta \in \Delta \), let \( \mathcal{E}_\delta \) be the set of all \( \delta \)-fine partitions of \([a, b]\). Set \( J \equiv (HS) \int_{a}^{b} gf d\alpha \), and for all \( \delta \in \Delta \) put

\[ p_\delta \equiv \sup \{|S_\alpha(gf, E) - J| : E \in \mathcal{E}_\delta \}. \]

By virtue of hypothesis, \( (o) - \lim_{\delta \to 0} p_\delta = 0. \) Moreover, by Henstock ’s Lemma, we get
We note that the hypotheses in Corollary 3.20 are essential: indeed, even in the Remark 3.21
φ
Corollary 3.20 is an easy consequence of Theorem 3.18, Proposition 3.19 and the fact
be any weak primitives of
\( f \)

Thus the proposition is completely proved.

By hypothesis, (o) \( \lim_{\delta \in \Delta} p^2_{\delta} = 0 \). For every \( \delta \in \Delta \), and for all \( E \in \mathcal{E}_{\delta} \), we have:

\[
\left| \sum_{i=1}^{n} g(\xi_i)[F(x_i) - F(x_{i-1})] - J \right| \\
\leq \sum_{i=1}^{n} g(\xi_i) \{|F(x_i) - F(x_{i-1})| - f(\xi_i)|\alpha(x_i) - \alpha(x_{i-1})|\} \\
+ \sum_{i=1}^{n} g(\xi_i)f(\xi_i)|\alpha(x_i) - \alpha(x_{i-1})| - J \leq q^1_{\delta} + p^2_{\delta}.
\]

From this the assertion of the necessary part follows.

Now we turn to the "if" part. Put \( I = (HS) \int_{a}^{b} g \, dF \), and \( \forall \delta \in \Delta \) let

\[
q^1_{\delta} = \sup \{|S_F(g, E) - I| : E \in \mathcal{E}_{\delta}\}.
\]

By hypothesis, \( (o) - \lim_{\delta \in \Delta} p^2_{\delta} = 0 \). For every \( \delta \in \Delta \), and for all \( E \in \mathcal{E}_{\delta} \), we have:

\[
\left| \sum_{i=1}^{n} g(\xi_i)f(\xi_i)|\alpha(x_i) - \alpha(x_{i-1})| - I \right| \\
\leq \sum_{i=1}^{n} g(\xi_i) \{|F(x_i) - F(x_{i-1})| - f(\xi_i)|\alpha(x_i) - \alpha(x_{i-1})|\} \\
+ \sum_{i=1}^{n} g(\xi_i)|F(x_i) - F(x_{i-1})| - I \leq q^1_{\delta} + p^2_{\delta}.
\]

Thus the proposition is completely proved. \( \square \)

**Corollary 3.20** Let \( R = L^0(X, \mathcal{B}, \mu), f, g : [a, b] \to R \) be two \((H)\)-integrable maps, and \( F, G \) be any weak primitives of \( f \) and \( g \) respectively. If there exists in \( R \) the integral \((HS) \int_{a}^{b} F \, dG\),

then there exists in \( R \) the integral \((HS) \int_{a}^{b} G \, dF\), and

\[
(HS) \int_{a}^{b} F \, dG = F(b)G(b) - F(a)G(a) - (HS) \int_{a}^{b} F \, dG.
\]

**Proof.** Corollary 3.20 is an easy consequence of Theorem 3.18, Proposition 3.19 and the fact that, if \( \phi : [a, b] \to R \) is any \((u)\)-differentiable map, then \( \phi' \) is \((H)\)-integrable and \((H) \int_{a}^{b} \phi' = \phi(x) - \phi(a) \) for all \( x \in [a, b] \) (see also [1], [23]). \( \square \)

**Remark 3.21** We note that the hypotheses in Corollary 3.20 are essential: indeed, even in the real case, there exist some maps \( F \) and \( G \), everywhere differentiable but such that the integrals \((HS) \int_{a}^{b} F \, dG\) and \((HS) \int_{a}^{b} G \, dF\) do not exist (see [15], Example 1, p. 232).
4 Applications.

Let $R$ be a Dedekind complete Riesz space. Analogously as in the classical case it is possible to give the definition of uniform and total convergence of series of functions. Let us denote by $R'$ the space $L^0(W, \mathcal{A}, \mu)$, where $(W, \mathcal{A}, \mu)$ is a measure space, with $\mu : \mathcal{A} \to \mathbb{R}_0^+$ $\sigma$-additive and $\sigma$-finite. A stochastic process $f$ is an element of $(R')^I$, where $I$ is a proper connected subset of $\mathbb{R}$. Throughout this section, we will deal with periodic stochastic processes of period $2\pi$: so we can assume, without loss of generality, that $I = [-\pi, \pi]$. We will consider the Fourier series associated with a stochastic process $f$, that is (formally) the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

(19)

where

$$a_n = \frac{1}{\pi} (H) \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} (H) \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

(20)

for all $n \in \mathbb{N}$, provided that the right members of (20) make sense, and we give some necessary and sufficient conditions, strictly weaker than the corresponding classical ones, for the convergence of the series

$$\sup_{x \in [-\pi, \pi]} \left\{ \left| \frac{a_0}{2} + \sum_{n=1}^{\infty} |a_n \cos nx + b_n \sin nx| \right| \right\}.$$ 

(21)

For a literature about relations between integration by parts and Fourier series, see also [4], [5] and [22].

**Definition 4.1** Let $f : I \to R'$ be a stochastic process. We say that $f$ satisfies property $(D)$ if there exist a subset $N \subset I$, with Lebesgue measure zero, a stochastic process $f_1$ and a net $(r_h)_{h \in R}$ in $R'$, (the $r_h$'s can be viewed as random variables, when $W$ is a probability space) such that $\lim_{h \to 0} r_h(w) = 0$ for almost all $w \in W$, and

$$\frac{|f(x+h)(w) - f(x)(w) - h f_1(x)(w)|}{|h|} \leq r_h(w)$$

(22)

for all $x, h$ with $x \notin N$, $x+h \in I$ and for almost all $w \in W$.

Of course, if $f$ satisfies condition $(D)$, then $f$ is $(u)$-differentiable in $I \setminus N$ and $f'(x) = f_1(x)$ $\forall x \in I \setminus N$.

**Definition 4.2** A stochastic process $f$ is said to satisfy property $(L)$ if for every set $N \subset I$ with Lebesgue measure zero there exists an element $u \in R'$ such that
4.2.1) \( u(w) \geq 0 \) for almost all \( w \in \mathcal{W} \),

4.2.2) \( \forall \varepsilon > 0 \) there exists \( \delta^* \in \Delta \) such that

\[
\sum_{i=1}^{n} |f(x_i)(w) - f(x_{i-1})(w)| \leq \varepsilon u(w)
\]

for any \( \delta^* \)-fine decomposition \( E = \{([x_{i-1}, x_i], \xi_i) : i = 1, \ldots, n\} \) with \( \xi_i \in N \ \forall i \), and for almost all \( w \in \mathcal{W} \).

As \((r)\)-convergence and \((o)\)-convergence coincide in \( R' \), then every process \( f \) satisfies \((L)\) iff it is of class \((SL)\) (and hence \( f : I \to R' \) is also a bounded function).

The following definition can be formulated also for \emph{any} Dedekind complete Riesz space \( R \) (see [1]).

**Definition 4.3** We say that \( f : [a, b] \to R \) is \((SL)\)-integrable if it has a weak primitive \( P \). In this case we put (by definition) \((SL) \int_a^b f = P(b) - P(a)\).

In [1] we proved that Definition 4.3 makes sense. Moreover we observe that, thanks to theorem 3.6, \((H)\)-integrability \emph{always} implies \((SL)\)-integrability. The converse, in general, is not true (see also [1]).

We note that, if \( f \) is of class \((SL)\) and almost everywhere \((u)\)-differentiable, then the map

\[
f_1^*(x) \equiv \begin{cases} 
  f'(x) & \text{if } x \notin N, \\
  0 & \text{if } x \in N,
\end{cases}
\]

(here \( N \) is the set where \( f' \) is not defined) is \((SL)\)-integrable and

\[
(SL) \int_a^x f_1^* = f(x) - f(a) \quad \text{for all } x \in [a, b]
\]

(see [1]). Conversely, if there exists a function \( g : [a, b] \to R \) such that

\[
f(x) - f(a) = (SL) \int_a^x g \quad \forall x \in [a, b],
\]

then \( f \) is necessarily of class \((SL)\): this is an immediate consequence of the definition of \((SL)\)-integrability.

We now give an application of Theorem 3.11, which holds also when the \((H)\)-integral is replaced by the \((SL)\)-integral with respect to absolutely continuous functions. We note that, differently as in the classical cases, we do not require that the involved processes are absolutely
continuous, and we will assume only conditions \((D)\) and \((L)\). We now give an example of a not absolutely continuous real-valued functions, satisfying properties \((D)\) and \((L)\). Let us define 
\[ f : [0, 1] \to \mathbb{R} \] 
by setting 
\[ f(0) = 0 \quad \text{and} \quad f(x) = x^{-1} \sin x^{-1} \quad \text{for all} \quad x \in [0, 1]. \]
As \( f \) is integrable in the Riemann generalized sense, then \( f \) is Henstock integrable and hence its integral function \( F(x) = \int_{x}^{1} f(t)dt \) is of class \((SL)\) (see also \([11]\)). Moreover, thanks to the properties of the classical Henstock-Kurzweil integral, we have \( F'(x) = -f(x) \) for almost all \( x \in [0, 1] \). So, \( F \) satisfies conditions \((D)\) and \((L)\). Moreover \( F \) is not absolutely continuous: otherwise, \( f \) would be Lebesgue integrable, but this is not true.

**Theorem 4.4** Let \( f : \mathbb{R} \to \mathbb{R}' \) be a periodic stochastic process with period \( 2\pi \), satisfying conditions \((D)\) and \((L)\), and \( f_1 \) be as in \((23)\). Then the following are equivalent:

4.4.1) the series \((19)\) converges totally in \([-\pi, \pi]\) to \( f \).

4.4.2) the series \( \sum_{n=1}^{\infty} \left( \frac{a'_n}{n} \cos nx + \frac{b'_n}{n} \sin nx \right) \) converges totally in \([-\pi, \pi]\), where

\[
\begin{align*}
a'_n &= \frac{1}{\pi} (H) \int_{-\pi}^{\pi} f_1^*(x) \cos nx \, dx, \\
b'_n &= \frac{1}{\pi} (H) \int_{-\pi}^{\pi} f_1^*(x) \sin nx \, dx,
\end{align*}
\]

\( n \in \mathbb{N}. \) \( (24)\)

**Proof.** By properties \((D)\) and \((L)\) and by virtue of the above arguments, the map \( f_1^* : [-\pi, \pi] \to \mathbb{R}' \) is \((SL)\)-integrable and

\[
(SL) \int_{-\pi}^{x} f_1^* = f(x) - f(-\pi) \quad \text{for all} \quad x \in [-\pi, \pi].
\]

Moreover, the functions \( G_n, J_n : [-\pi, \pi] \to \mathbb{R} \), defined by setting \( G_n(x) = \cos nx \), \( J_n(x) = \sin nx \), \( x \in [-\pi, \pi] \), are of class \( C^\infty \): thus, in particular, they are absolutely continuous in \([-\pi, \pi]\). So, by Theorem 3.11, the terms in \((24)\) do exist in \( \mathbb{R}' \).

Let \( a_n \) and \( b_n \) be as in \((20)\); these quantities do exist in \( \mathbb{R}' \), by virtue of \((H)\)-integrability of \( f \) and Theorem 3.11. Thanks to it, for each \( n \in \mathbb{N} \) we have also \( b'_n = -n a_n \), \( a'_n = n b_n \) and thus \( |a_n| + |b_n| = \frac{|a'_n|}{n} + \frac{|b'_n|}{n} \). Furthermore, it is not difficult to check that, as \( f \) is \((u)\)-continuous, then total convergence of the series \((19)\) implies its uniform convergence just to \( f \).

From this the assertion follows. \( \Box \)

**References**


