Limit theorems in \((l)-\)groups with respect to 
\((D)-\)convergence

Abstract

Some Schur, Vitali-Hahn-Saks and Nikodým convergence theorems for \((l)-\)group-valued measures are given in the context of \((D)-\)convergence. We consider both the \(\sigma\)-additive and the finitely additive case. Here the notions of strong boundedness, countable additivity and absolute continuity are formulated not necessarily with respect to a same regulator, while the pointwise convergence of the measures is intended relatively to a common \((D)-\)sequence. Among the tools, we use the Fremlin lemma, which allows us to replace a countable family of \((D)-\)sequence with one regulator, and the Maeda-Ogasawara-Vulikh representation theorem for Archimedean lattice groups.

1 Introduction

The limit theorems for absolutely continuous, \(\sigma\)-additive and strongly bounded set functions (Schur Lemma, Vitali-Hahn-Saks, Nikodým convergence and Brooks-Jewett theorems, see [22, 31, 37, 47, 49, 52]) are objects of several studies in the literature. A survey about these kinds of theorems and related topics, among which some applications to integration theory, can be found in [25].

Key Words: \((l)-\)group, \((D)-\)sequence, Fremlin lemma, Nikodým convergence theorem, Maeda-Ogasawara-Vulikh theorem, Schur lemma, Vitali-Hahn-Saks theorem.

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These theorems were extended to the cases of Banach space- and topological group-valued measures, defined even in some domains more general than $\sigma$-algebras. For the literature we quote, for instance, [26] and its bibliography, [2, 19, 23, 24, 30, 34, 43, 48]. These kinds of theorems have several functional analytic applications, and they are related with matrix theorems, which are powerful tools to give some results both in measure theory and in the context of operators (see for instance [1, 21, 40, 51]). Some other applications to integration, control measures and signal processing can be found, for example, in [5, 20, 33, 38].

In this paper we deal with $(l)$-group-valued measures. Among the literature existing for limit theorems in order structures, we quote [3, 41]. In [9, 12, 13] the authors proved some limit theorems for measures taking values in Riesz spaces and $(l)$-groups. In [9] they considered positive measures converging pointwise to a regular measure and positive measures taking values in a subspace of the space $L^0(X, \mathcal{B}, \mu)$ of all measurable functions (up to sets of measure zero) with respect to a $\sigma$-finite and $\sigma$-additive extended real-valued positive measure. In [12, 13] they proved some limit theorems when the $(l)$-group-valued measures converge pointwise with respect to a common regulator, and also the concepts of $(s)$-boundedness, $\sigma$-additivity and absolute continuity are defined analogously. Some other versions of limit theorems for Riesz space-valued measures, defined in abstract structures, can be found in [4].

In this paper we prove some limit theorems for $(l)$-group-valued $\sigma$-additive measures and finitely additive set functions. The main used tools are the Maeda-Ogasawara-Vulikh representation theorem for Archimedean $(l)$-groups as sets of suitable continuous extended real-valued functions (see [6]) and the Fremlin lemma (see [45]), which allows us to replace a countable family of regulators with one regulator in the proof of limit theorems, without doing too restrictive hypotheses on the involved $(l)$-group. For this aim and for technical reasons we often use the notion of $(D)$-convergence in $(l)$-groups (see also [45]). Here only the pointwise convergence of the measures (and not the notions of strongly boundedness, countable additivity and absolute continuity) is considered with respect to a common regulator. Similar results were proved in [17] for $(l)$-group-valued $\sigma$-additive measures with respect to the relatively uniform convergence (see [39, 53]). Some related results on matrix theorems for $(l)$-group-valued measures were given in [15], while in [16] some similar limit theorems and Drewnowski-type theorems on relations between finite and countable additivity were proved in slightly different settings and with other kinds of techniques.

The paper is structured as follows. In Section 2 we introduce the preliminary notions and the tools used in our setting. In Section 3 we prove the Schur
Lemma, the Vitali-Hahn-Saks and the Nikodým convergence theorem.

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2 Preliminaries

In this section we introduce the preliminary notions in $(l)$-groups and the fundamental concepts of $(s)$-boundedness, finite and countable additivity and absolute continuity of $(l)$-group-valued set functions. Furthermore, we give some examples and explain the tools and techniques, used to prove the main results of the paper.

We begin with introducing the order convergence and $(D)$-convergence in $(l)$-groups.

**Definitions 2.1.** An Abelian group $(R, +)$ is called $(l)$-group if it is a lattice and for any $a, b, c \in R$ we get $a + c \leq b + c$ whenever $a \leq b$.

From now on, we denote by $\vee$ and $\wedge$ the lattice supremum and infimum respectively.

An $(l)$-group $R$ is said to be Dedekind complete if every nonempty subset of $R$, bounded from above, has supremum in $R$.

For the basic properties of $(l)$-groups, see [8] and [45]. Given an element $r \in R$, we call absolute value of $r$ the element $|r|$ defined by setting $|r| = r \vee (-r)$. If we put $r^+ = r \vee 0$, $r^- = (-r) \vee 0$, it is not difficult to see that $r = r^+ - r^-$ and $|r| = r^+ + r^-$. Let $R$ be an $(l)$-group. We say that a sequence $(p_n)_n$ in $R$ is an $(o)$-sequence if it is decreasing and $\land_n p_n = 0$. A sequence $(r_n)_n$ in $R$ is said to be order-convergent (or $(o)$-convergent) to $r$ if there exists an $(o)$-sequence $(p_n)_n$ in $R$ with $|r_n - r| \leq p_n$ for all $n \in \mathbb{N}$, and in this case we write $(o) \lim_n r_n = r$. If $\Lambda$ is any nonempty set, $(r^{(\lambda)}_n)_n$ are sequences in $R$ and $r^{(\lambda)} \in R$ for all $\lambda \in \Lambda$, we say that $(o) \lim_n r^{(\lambda)}_n = r^{(\lambda)}$ uniformly with respect to $\lambda \in \Lambda$ if there exists an $(o)$-sequence $(q_n)_n$ in $R$ with $|r^{(\lambda)}_n - r^{(\lambda)}| \leq q_n$ for all $n \in \mathbb{N}$ and $\lambda \in \Lambda$. A sequence $(r_n)_n$ in $R$ is $(o)$-Cauchy if $(o) \lim_n (r_n - r_{n+p}) = 0$ uniformly with respect to $p \in \mathbb{N}$.

We now introduce $(D)$-convergence (for its main properties, see [18, 45]).

A bounded double sequence $(a_{i,l})_{i,l}$ in $R$ is called $(D)$-sequence or regulator if the sequence $(a_{i,l})_l$ is an $(o)$-sequence for all $i \in \mathbb{N}$. A sequence $(r_n)_n$ in $R$ is said to be $(D)$-convergent to $r \in R$ (and we write $(D) \lim_n r_n = r$) if there exists a $(D)$-sequence $(a_{i,l})_{i,l}$ in $R$, such that to every $\varphi \in \mathbb{N}^\mathbb{N}$ there corresponds $n_0 \in \mathbb{N}$ with $|r_n - r| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$ whenever $n \geq n_0$. If $\Lambda$ is as above, $(r^{(\lambda)}_n)_n$
are sequences in $R$ and $r^{(\lambda)} \in R$ for all $\lambda \in \Lambda$, we say that $(D) \lim_n r_n^{(\lambda)} = r^{(\lambda)}$ uniformly with respect to $\lambda \in \Lambda$ if there is a $(D)$-sequence $(a_{i,j})_{i,j}$ in $R$, such that for every $\varphi \in \mathbb{N}^\mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that $|r_n^{(\lambda)} - r^{(\lambda)}| \leq \bigvee_{i=1}^\infty a_{i,\varphi(i)}$ whenever $n \geq n_0$ and $\lambda \in \Lambda$. The sequence $(r_n)_n$ is said to be $(D)$-Cauchy if $(D) \lim_n (r_n - r_{n+p}) = 0$ uniformly with respect to $p \in \mathbb{N}$.

We say that an $(l)$-group is $(o)$-complete if every $(o)$-Cauchy sequence is $(o)$-convergent, and $(D)$-complete if every $(D)$-Cauchy sequence is $(D)$-convergent. We recall that every Dedekind complete $(l)$-group is $(o)$-complete and $(D)$-complete (see also [18, Chapter 2]).

An $(l)$-group $R$ is said to be weakly $\sigma$-distributive if for every $(D)$-sequence $(a_{i,j})_{i,j}$ we have:

$$\bigwedge_{\varphi \in \mathbb{N}^\mathbb{N}} \left( \bigvee_{i=1}^\infty a_{i,\varphi(i)} \right) = 0.$$  

**Remark 2.2.** In general, the $(D)$-limit of a sequence is not unique. However, $(o)$-convergence of sequences implies always $(D)$-convergence; moreover, if $R$ is weakly $\sigma$-distributive, then a sequence is $(D)$-convergent if and only if it is $(o)$-convergent, and in this case the limit is unique (see [10] and [32, Proposition 1]).

If $R$ is a Dedekind complete not weakly $\sigma$-distributive $(l)$-group (such groups do exist, see [35, 54]), then there exist a $(D)$-sequence $(a_{i,j})_{i,j}$ and a positive element $b \in R$, for which

$$\bigwedge_{\varphi \in \mathbb{N}^\mathbb{N}} \left( \bigvee_{i=1}^\infty a_{i,\varphi(i)} \right) = b \neq 0.$$  

We now prove that every sequence $(r_n)_n$ with $-b \leq r_n \leq b$ for all $n \in \mathbb{N}$ $(D)$-converges to $0$, though it can be not $(o)$-convergent. Indeed, choose arbitrarily $\varphi \in \mathbb{N}^\mathbb{N}$: for each $n \in \mathbb{N}$ we get

$$|r_n| \leq b = \bigwedge_{\varphi \in \mathbb{N}^\mathbb{N}} \left( \bigvee_{i=1}^\infty a_{i,\varphi(i)} \right) \leq \bigvee_{i=1}^\infty a_{i,\varphi(i)};$$

that is $(D) \lim_n r_n = 0$.

From now on we assume that $R$ is a Dedekind complete weakly $\sigma$-distributive $(l)$-group. Note that weak $\sigma$-distributivity guarantees the uniqueness of the $(D)$-limit and is used to prove $\sigma$-additivity of the limit measure in Theorem 3.1.
As in the classical definition of series, given a sequence \((r_n)_n\) in \(R\), we denote by \(\sum_{n=1}^{\infty} r_n\) the limit (o) \(\lim_{n \to \infty} \sum_{i=1}^{n} r_i\) = \((D) \lim_{n \to \infty} \sum_{i=1}^{n} r_i\), if it exists in \(R\).

The following proposition will be useful in the sequel, in defining some suitable measures and proving some of its properties.

**Proposition 2.3.** If \((D) \lim_{n \to \infty} \sum_{i=1}^{n} |r_i|\) exists in \(R\), then the limit

\[(D) \lim_{n \to \infty} \sum_{i \in A, i=1, \ldots, n} r_i\]

exists in \(R\) uniformly with respect to \(A \subset \mathbb{N}\).

**Proof:** First of all note that \((D) \lim_{n \to \infty} \sum_{i=1}^{n} |r_i| = \bigvee_{n} \sum_{i=1}^{n} |r_i|\), since the corresponding sequence of partial sums is increasing. It is not difficult to deduce that the following equations hold in \(R\):

\[(D) \lim_{n \to \infty} \sum_{i \in A, i=1, \ldots, n} |r_i| = \bigvee_{n} \sum_{i \in A, i=1, \ldots, n} |r_i|,\]

\[(D) \lim_{n \to \infty} \sum_{i \in A, i=1, \ldots, n} r_i^+ = \bigvee_{n} \sum_{i \in A, i=1, \ldots, n} r_i^+ ,\]

\[(D) \lim_{n \to \infty} \sum_{i \in A, i=1, \ldots, n} r_i^- = \bigvee_{n} \sum_{i \in A, i=1, \ldots, n} r_i^- ,\]

uniformly with respect to \(A \subset \mathbb{N}\). Since \(r_i = r_i^+ - r_i^-\) for all \(i \in \mathbb{N}\), then it follows that

\[(D) \lim_{n \to \infty} \sum_{i \in A, i=1, \ldots, n} r_i\]

exists in \(R\) uniformly with respect to \(A \subset \mathbb{N}\), that is the assertion.

We denote by \(\sum_{n \in A} r_n\) the \((D)\)-limit in (1), when it exists.

The following result (Fremlin lemma, [36, Lemma 1C], see also [45, Theorem 3.2.3, pp. 42–45], [46, Proposition 2.1]) allows us to replace a countable family of \((D)\)-sequences with one regulator, and its technique will be useful in the proof of the main results of the paper.
Lemma 2.4. Let \((a^{(n)}_{i,l})_{i,l}, n \in \mathbb{N}\) be a sequence of regulators in \(R\). Then for every \(u \in R, u \geq 0\) there exists a \((D)\)-sequence \((a_{i,l})_{i,l}\) in \(R\) such that:

\[
 u \bigwedge \left( \sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} a^{(n)}_{i,\varphi(i+n)} \right) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}
\]

for all \(q \in \mathbb{N}\) and for every \(\varphi \in \mathbb{N}^{\mathbb{N}}\).

We now recall the famous Maeda-Ogasawara-Vulikh representation theorem in its version for \((l)\)-groups (see [6, Theorem 6]). This theorem allows us to identify any Archimedean Riesz space with a suitable space of continuous extended real-valued functions. Note that every Dedekind complete \((l)\)-group is Archimedean (see also [8, Lemma XIII.5]). Here we denote by sup and inf the pointwise supremum and infimum respectively.

Theorem 2.5. Given a Dedekind complete \((l)\)-group \(R\), there exists a compact extremely disconnected topological space \(\Omega\), unique up to homeomorphisms, such that \(R\) can be embedded as a solid subgroup of \(C_{\infty}(\Omega) = \{ f \in \tilde{\mathbb{R}}^{\Omega} : f \) is continuous, and \(\{ \omega : |f(\omega)| = +\infty \} \) is nowhere dense in \(\Omega\}\). Moreover, if \((a_\lambda)_{\lambda \in \Lambda}\) is any family such that \(a_\lambda \in R\) for all \(\lambda \in \Lambda\), and \(a = \bigvee_\lambda a_\lambda \in R\) (where the supremum is taken with respect to \(R\)), then \(a = \bigvee_\lambda a_\lambda\) with respect to \(C_{\infty}(\Omega)\), and the set \(\{ \omega \in \Omega : (\bigvee_\lambda a_\lambda)(\omega) \neq \sup_\lambda |a_\lambda(\omega)| \}\) is meager in \(\Omega\).

We now introduce the finitely additive and \(\sigma\)-additive set functions and their main properties.

Definitions 2.6. Let \(G\) be any infinite set and \(\mathcal{A} \subset \mathcal{P}(G)\) be a \(\sigma\)-algebra. A set function \(m : \mathcal{A} \to R\) is bounded if there is \(w \in R, w \geq 0, \) with \(|m(A)| \leq w\) for all \(A \in \mathcal{A}\). The set functions \(m_j : \mathcal{A} \to R, j \in \mathbb{N}\), are equibounded there exists an element \(u \in R, u \geq 0, \) such that

\[
|m_j(A)| \leq u
\]

for every \(j \in \mathbb{N}\) and for all \(A \in \mathcal{A}\).

Given a finitely additive bounded set function \(m : \mathcal{A} \to R\), we define \(m^+, m^-, v(m), \|m\|_\mathcal{A} : \mathcal{A} \to R\) by setting, for every \(A \in \mathcal{A}\):

\[
m^+(A) = \bigvee\{m(B) : B \in \mathcal{A} \text{ with } B \subset A\},
\]

\[
m^-(A) = \bigvee\{-m(B) : B \in \mathcal{A} \text{ with } B \subset A\},
\]

\[
v(m)(A) = m^+(A) + m^-(A),
\]

\[
\|m\|_\mathcal{A}(A) = \bigvee\{|m(B)| : B \in \mathcal{A} \text{ with } B \subset A\}.
\]
The quantities $m^+, m^-, v(m), \|m\|_\mathcal{A}$ are called positive part, negative part, variation and semivariation on $\mathcal{A}$ of $m$ respectively. Analogously as in the real case, we have clearly

$$\|m\|_\mathcal{A}(A) \leq v(m)(A) \leq 2\|m\|_\mathcal{A}(A), \quad \text{for all } A \in \mathcal{A}$$

(3)

(see also [7]).

A set function $m : \mathcal{A} \to \mathbb{R}$ or $m : \mathcal{A} \to [0, +\infty]$ is finitely additive if $m(A \cup B) = m(A) + m(B)$ whenever $A, B$ are two disjoint elements of $\mathcal{A}$ (with the convention that $+\infty + a = +\infty$ for all $a \in \tilde{\mathbb{R}}$). A finitely additive set function is said to be $(s)$-bounded if for every disjoint sequence $(H_n)_n$ in $\mathcal{A}$ we have:

$$(D) \lim_n \|m\|_\mathcal{A}(H_n) = 0.\)

We say that the finitely additive set functions $m_j : \mathcal{A} \to \mathbb{R}$, $j \in \mathbb{N}$, are uniformly $(s)$-bounded if $(D) \lim_n [\bigvee_j \|m_j\|_\mathcal{A}(H_n)] = 0$ whenever $(H_n)_n$ is a sequence of pairwise disjoint elements of $\mathcal{A}$.

We now prove that, in the context of $(l)$-groups, every bounded finitely additive set function is $(s)$-bounded too. Differently than in the real case, the converse is in general not true (see [50, Example 3]).

**Theorem 2.7.** Every bounded finitely additive set function $m : \mathcal{A} \to \mathbb{R}$ is $(s)$-bounded.

**Proof:** Let $R$ be a subgroup of $C_\infty(\Omega)$, where $\Omega$ is as in 2.5. By Theorem 2.5 there is a nowhere dense set $N_0 \subset \Omega$ such that the real-valued set functions $m(\cdot)(\omega)$, $\omega \in \Omega \setminus N_0$, are finitely additive and bounded. By virtue of the classical results (see [7, Corollary 2.1.7]), they are $(s)$-bounded on $\mathcal{A}$.

Fix now an arbitrary disjoint sequence $(H_n)_n$ in $\mathcal{A}$. Then by Theorem 2.5 there is a meager set $N$, depending on $(H_n)_n$, without loss of generality with $N \supset N_0$ and such that

$$[(D) \lim_n \|m\|_\mathcal{A}(H_n)](\omega) = \left[\bigwedge_n \bigvee_{s \geq n} \|m\|_\mathcal{A}(H_s)\right](\omega)$$

$$= \left[\bigwedge_n \bigvee_{s \geq n} \left(\|m(A)\| : A \in \mathcal{A} \text{ with } A \subset H_s\right)\right](\omega)$$

$$= \inf \sup \sup \left(\|m(\cdot)(\omega)\| : A \in \mathcal{A} \text{ with } A \subset H_s\right)$$

$$= \inf \sup \|m(\cdot)(\omega)\|_\mathcal{A}(H_s) = \lim_n \|m(\cdot)(\omega)\|_\mathcal{A}(H_n) = 0$$

for all $\omega \in \Omega \setminus N$. By a density argument we obtain $(D) \lim_n \|m\|_\mathcal{A}(H_n) = 0$, and hence we get the assertion. $\square$
Remark 2.8. Observe that, in the context of Banach spaces, the relations
between boundedness and \((s)\)-boundedness of finitely additive set functions
are substantially different than in \((l)\)-groups. First of all, note that every \((s)\)-
bounded Banach space-valued set function defined in an algebra is bounded
too (see [44]). Let \(l^\infty\) and \(c_0\) be the spaces of all real sequences and of the real
sequences convergent to 0 respectively, endowed with the supremum norm. It
is known that a Banach space \(X\) has the property that every finitely additive
and bounded \(X\)-valued set function defined in an algebra (resp. \(\sigma\)-algebra)
is \((s)\)-bounded if and only if \(X\) does not contain isomorphically the space \(c_0\)
(resp. \(l^\infty\)) (see [27, 28, 29]).

We now give the following example of a finitely additive bounded set func-
tion, which is not \((s)\)-bounded. Let \(\Sigma\) be the \(\sigma\)-algebra of all Lebesgue mea-
surable subsets of \([0,1]\), \(\nu\) be the Lebesgue measure, \(X = L^\infty([0,1],\Sigma,\nu)\) be
the space of all essentially bounded functions (with identification up to sets of
Lebesgue measure zero), endowed with the essential supremum norm \(\| \cdot \|_\infty\).
We define \(m : \Sigma \to X\) by setting \(m(A) = \chi_A\), for all \(A \in \Sigma\), where \(\chi_A\) is the
characteristic function associated to \(A\). Then \(m\) is obviously a finitely additive
and bounded set function, since \(\| m(A) \|_\infty = \| \chi_A \|_\infty \leq 1\) for every \(A \in \Sigma\). But
\(m\) is not \((s)\)-bounded, since for the disjoint sequence (in \(\Sigma\)) \(E_n = \left(\frac{1}{n+1}, \frac{1}{n}\right)\),
\(n \in \mathbb{N}\), we have that \(\| m(E_n) \|_\infty = 1\) for all \(n \in \mathbb{N}\).

We now introduce the concept of \(\sigma\)-additivity.

Definitions 2.9. Let \(A \subset \mathcal{P}(G)\) be a \(\sigma\)-algebra. A finitely additive set func-
tion \(m : A \to R\) is called \(\sigma\)-additive measure on \(A\) if, for every decreasing
sequence \((H_n)_n\) in \(A\) with \(\bigcap_1^{\infty} H_n = \emptyset,\)
\[(D) \lim_n \| m \|_{\mathcal{L}}(H_n) = 0,\]
where \(\mathcal{L}\) is the \(\sigma\)-algebra generated by the \(H_n\)'s in \(H_1\).

The \(\sigma\)-additive measures \(m_j : A \to R, j \in \mathbb{N}\), are uniformly \(\sigma\)-additive if
for each decreasing sequence \((H_n)_n\) in \(A\) with \(\bigcap_1^{\infty} H_n = \emptyset,\)
\[(D) \lim_n [\vee_j \| m_j \|_{\mathcal{L}}(H_n)] = 0.\]

Our definition looks weaker than the classical one, in which the semivari-
ation on \(A\) is considered rather than the semivariation on \(\mathcal{L}\). In general, it
is still an open problem to find conditions under which these two kinds of
\(\sigma\)-additivity are equivalent. However, when \(R = \mathbb{R}\), it is not difficult to see
that a measure is $\sigma$-additive according to 2.9 if and only if it is $\sigma$-additive in the classical sense (see also [7]).

It is natural to deal with the notion of $\sigma$-additivity given in 2.9, since in the sequel we will examine in detail the case $\mathcal{A} = \mathcal{P}(\mathbb{N})$. Indeed, the following result holds:

**Proposition 2.10.** A finitely additive measure $m : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ is $\sigma$-additive on $\mathcal{P}(\mathbb{N})$ if and only if

\[(D) \lim_n \|m\|_{\mathcal{P}(\mathbb{N})}(\{n, n+1, n+2, \ldots\}) = 0.\] (4)

**Proof:** The necessary part is straightforward.

We turn to the sufficient part. Let $(C_n)_n$ be any decreasing sequence in $\mathcal{P}(\mathbb{N})$ with $\bigcap_{n=1}^{\infty} C_n = \emptyset$, and $\mathcal{L}$ be the $\sigma$-algebra generated by the $C_n$’s in $C_1$. Without loss of generality we can assume that $C_n \subset \{n, n+1, n+2, \ldots\}$ for any $n \in \mathbb{N}$. Hence

\[v_{\mathcal{L}}(m)(C_n) \leq v_{\mathcal{P}(\mathbb{N})}(m)(\{n, n+1, n+2, \ldots\}),\]

and thus the sufficient part is proved. \qed

**Remark 2.11.** Note that an analogous version of Proposition 2.10 holds even for uniform $\sigma$-additivity of a sequence of measures $m_j : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$, $j \in \mathbb{N}$.

We now turn to the concept of absolute continuity.

**Definitions 2.12.** Let $\mathcal{A} \subset \mathcal{P}(G)$ be a $\sigma$-algebra and $\nu : \mathcal{A} \to [0, +\infty]$, $m : \mathcal{A} \to \mathbb{R}$ be two finitely additive set functions. We say that $m : \mathcal{A} \to \mathbb{R}$ is $\nu$-absolutely continuous, if for each decreasing sequence $(E_n)_n$ in $\mathcal{A}$, with $\lim_n \nu(E_n) = 0$, we get $(D) \lim_n \|m\|_{\mathcal{L}}(E_n) = 0$, where $\mathcal{L}$ is the $\sigma$-algebra generated by $(E_n)_n$ and

\[\|m\|_{\mathcal{L}}(E_n) = \bigvee \{|m(B)| : B \in \mathcal{L} \text{ with } B \subset E_n\}\]

for each $n \in \mathbb{N}$.

The finitely additive set functions $m_j : \mathcal{A} \to \mathbb{R}$, $j \in \mathbb{N}$, are said to be uniformly $\nu$-absolutely continuous if

\[(D) \lim_n [\sup_j \|m_j\|_{\mathcal{L}}(E_n)] = 0\]

whenever $(E_n)_n$ is a decreasing sequence in $\mathcal{A}$ such that $\lim_n \nu(E_n) = 0$. 


Remark 2.13. Observe that, in our definition of \( \nu \)-absolute continuity, we consider only decreasing sequences in \( \mathcal{A} \), while in the classical setting all sequences in \( \mathcal{A} \) \( \nu \)-convergent to zero are considered. If \( R \) is a topological group, the definition of \( \nu \)-absolute continuity given in 2.12 coincides with the classical one, but in general this is not true in Riesz spaces (see [9]). Indeed, let \( \Sigma \) the \( \sigma \)-algebra of all measurable subsets of \([0,1] \) and \( \nu : \Sigma \to \mathbb{R} \) be the Lebesgue measure. The space \( R = L^0([0,1], \Sigma, \nu) \) of all measurable functions on \([0,1] \), with identification up to subsets of Lebesgue measure zero, is Dedekind complete and has the Egorov property (see [39]). By [11, Theorem 3.1], \( R \) is weakly \( \sigma \)-distributive (see also [12]). Let us define \( m : \Sigma \to \mathbb{R} \) as follows:

\[
m(A) = \chi_A,
\]

for all \( A \in \Sigma \), where \( \chi_A \) is the characteristic function associated to \( A \). Then \( \nu(A_n) \to 0 \) if and only if \( \chi_{A_n} \to 0 \) in \( L^1 \), but \( \lim_n \chi_{A_n} = 0 \Leftrightarrow \chi_{A_n} \to 0 \) \( \nu \)-almost everywhere. So, in general, the implication

\[
\nu(A_n) \to 0 \Rightarrow \lim_n m(A_n) = 0
\]

does not hold, and \( m \) is not \( \nu \)-absolutely continuous in the classical sense. However, it is easy to see that (5) holds whenever \( (A_n)_n \) is any decreasing sequence in \( \Sigma \). Thus \( m \) is \( \nu \)-absolutely continuous according the definition given in 2.12 (see also [9, Remark 1.13.1]).

We now formulate the concept of pointwise \((D)\)-convergence for set functions with respect to a common regulator, which will be our hypothesis in all of our versions of limit theorems.

Definitions 2.14. Let \( \mathcal{A} \subset \mathcal{P}(G) \) be a \( \sigma \)-algebra. Given a sequence of finitely additive set functions \( m_j : \mathcal{A} \to R, \ j \in \mathbb{N} \), we say that the \( m_j \)'s \((RD)\)-converge (or converge pointwise with respect to a same regulator) to \( m_0 \), or shortly \( (RD) \lim m_j = m_0 \), if there exists a \((D)\)-sequence \((b_{i,l})_{i,l} \) such that for each \( \varphi \in \mathbb{N}^\mathbb{N} \) and \( A \in \mathcal{A} \) there is an integer \( j_0 \) such that

\[
|m_j(A) - m_0(A)| \leq \sum_{i=1}^{\infty} b_{i,\varphi(i)}, \quad \text{for all } j \geq j_0.
\]

We say that \((D)\lim m_j = m_0 \) uniformly, or in short \((U)\lim m_j = m_0 \), if there exists a \((D)\)-sequence \((c_{i,l})_{i,l} \) such that for every \( \varphi \in \mathbb{N}^\mathbb{N} \) there is \( j_0 \in \mathbb{N} \) with

\[
|m_j(A) - m_0(A)| \leq \sum_{i=1}^{\infty} c_{i,\varphi(i)}, \quad \text{for all } A \in \mathcal{A} \text{ and for every } j \geq j_0,
\]

that is if and only if \((D)\lim m_j(A) = m_0(A) \) uniformly with respect to \( A \in \mathcal{A} \).
The following definitions are useful in order to present some examples.

**Definitions 2.15.** Let $u \in R$, $u \geq 0$. We say that $u$ has the *Egorov property* if, for every regulator $(a_{i,l})_{i,l}$ bounded from above by $u$, there exist an $(o)$-sequence $(b_n)_n$ and a sequence $(\varphi_n)_n$ of elements of $\mathbb{N}^N$, such that

$$\bigvee_{i=1}^{\infty} a_{i,\varphi_n(i)} \leq b_n$$

for all $n \in \mathbb{N}$. We say that $R$ has the *Egorov property* if each positive element $u \in R$ has the Egorov property (see also [39, pp. 458, 467]).

**Remark 2.16.** In [12] the authors introduced some concepts of $\sigma$-additivity, $(s)$-boundedness and absolute continuity with respect to a common regulator. However, there are sequences of measures (even uniformly) $\sigma$-additive according to our definition, but not with respect to the same regulator (see [50, Example 5]).

In this paper we will present some versions of limit theorems with respect to $(D)$-convergence. The use of $(D)$-convergence could seem apparently quite difficult, but it utilizes only Dedekind completeness and weak $\sigma$-distributivity of the involved $(l)$-group, and often it simplifies the proofs and allows us to replace a countable family of regulators with one $(D)$-sequence without assuming further additional hypotheses on the involved $(l)$-group, differently than in the contexts of relatively uniform convergence ($(r)$-convergence), where we require also some suitable regularity property (see [39, 53]) or of order convergence, where we often require super Dedekind completeness (see also [14]).

Some similar versions of the Schur and Nikodým theorems were proved in [17] with respect to relatively uniform convergence (or $(r)$-convergence). Note that in general $(D)$-convergence is weaker than $(r)$-convergence, and there are some cases in which $(r)$-convergence is strictly stronger than $(D)$-convergence. For example, in the space $l^\infty$ of all bounded real sequence endowed with the usual coordinatewise ordering, $(o)$-convergence is strictly weaker than $(r)$-convergence (see [39, Theorem 16.3, p. 80 and p. 479]). Moreover, note that $l^\infty$ is an ideal in the space $\mathbb{R}^N$ of all real sequences: indeed, given $x \in l^\infty$ and $y \in \mathbb{R}^N$ with $|y| \leq |x|$, we get clearly $y \in l^\infty$ (see [53, Definition III.9.1]). Therefore, since $l^\infty$ is an ideal of $\mathbb{R}^N$, we get that $l^\infty$ is Dedekind complete (see also [53, p. 156 and Theorem VI.2.2, p. 157]). Furthermore, observe that $l^\infty$ has the Egorov property (see [39, p. 465]), and hence, by [11, Theorem 3.1], $l^\infty$ is weakly $\sigma$-distributive. Thus, order and $(D)$-convergences coincide, but they are different from $(r)$-convergence.
3 Limit theorems

In this section we prove some versions of the Schur lemma, Vitali-Hahn-Saks and Nikodym convergence theorem with respect to \((D)\)-convergence. Similar versions in the context of relatively uniform convergence were proved in [17, Theorems 3.4 and 3.5]. Note that in our context only the pointwise convergence of the involved measures, and not \(\sigma\)-additivity or absolute continuity, is intended with respect to a same regulator, while in [12, 13] all concepts are formulated relatively to a common \((D)\)-sequence.

We begin with stating the Schur lemma, the Vitali-Hahn-Saks theorem and Nikodym convergence theorem.

**Theorem 3.1.** (Schur lemma) Let \(m_j : \mathcal{P}(\mathbb{N}) \to \mathbb{R}, j \in \mathbb{N}\), be a sequence of equibounded \(\sigma\)-additive measures, and assume that there exists a set function \(m_0 : \mathcal{P}(\mathbb{N}) \to \mathbb{R}\) such that \((RD)\lim_j m_j = m_0\). Then

\[
(D)\lim_j \left( \sum_{n=1}^{\infty} |m_j(\{n\}) - m_0(\{n\})| \right) = 0.
\]

Moreover \(m_0\) is \(\sigma\)-additive, \((U)\lim_j m_j = m_0\) and the \(m_j\)'s are uniformly \(\sigma\)-additive.

**Theorem 3.2.** (Vitali-Hahn-Saks theorem) Let \(G\) be any infinite set, \(\mathcal{A} \subset \mathcal{P}(G)\) be a \(\sigma\)-algebra, \(\nu : \mathcal{A} \to [0, +\infty]\) be a finitely additive set function, \(m_j : \mathcal{A} \to \mathbb{R}, j \in \mathbb{N}\), be a sequence of equibounded \(\nu\)-absolutely continuous finitely additive set functions. Assume that there exists \(m_0 : \mathcal{A} \to \mathbb{R}\) with \((RD)\lim_j m_j = m_0\). Then \(m_0\) is \(\nu\)-absolutely continuous and the \(m_j\)'s are uniformly \(\nu\)-absolutely continuous.

**Theorem 3.3.** (Nikodym convergence theorem) Let \(G, \mathcal{A}\) be as in Theorem 3.2, assume that \(m_j : \mathcal{A} \to \mathbb{R}, j \in \mathbb{N}\), is a sequence of equibounded \(\sigma\)-additive measures, and suppose that there is a set function \(m_0 : \mathcal{A} \to \mathbb{R}\) such that \((RD)\lim_j m_j = m_0\). Then \(m_0\) is \(\sigma\)-additive and the \(m_j\)'s are uniformly \(\sigma\)-additive.

In order to prove Theorems 3.1, 3.2 and 3.3, let us introduce some preliminary definitions and results.

**Definitions 3.4.** We denote by \(l^1(\mathbb{R})\) the set of all sequences \((a_j)\) of \(\mathbb{R}\), such that \(\sum_{j=1}^{\infty} |a_j|\) exists in \(\mathbb{R}\).

It is not difficult to check that \(l^1(\mathbb{R})\), endowed with the coordinatewise ordering, is an \((l)\)-group.
A sequence \((a^{(n)})_n\) of elements of \(l^1(R)\), where \(a^{(n)} = (a^{(n)}_j)_j\), \(n \in \mathbb{N}\), is said to be \emph{convergent in} \(l^1(R)\) to \(a = (a_j)_j \in l^1(R)\) if

\[
(D) \lim_n \left( \sum_{j=1}^{\infty} |a^{(n)}_j - a_j| \right) = 0. \tag{8}
\]

We say that the sequence \((a^{(n)})_n\) is \emph{Cauchy in} \(l^1(R)\) if

\[
(D) \lim_n \left( \sum_{j=1}^{\infty} |a^{(n)}_j - a^{(n+p)}_j| \right) = 0 \tag{9}
\]

uniformly with respect to \(p \in \mathbb{N}\).

The set \(l^1(R)\) satisfies the following completeness condition:

\textbf{Proposition 3.5.} Every Cauchy in \(l^1(R)\) sequence of elements of \(l^1(R)\) is convergent in \(l^1(R)\).

\textbf{Proof:} The proof is similar to [17, Proposition 2.3] and is a consequence of the fact that, in weakly \(\sigma\)-distributive Dedekind complete \((l)\)-groups, order convergence of sequences coincides with \((D)\)-convergence. \(\square\)

The following lemma will be useful in the sequel ([17, Lemma 3.2]).

\textbf{Lemma 3.6.} Let \(m : \mathcal{P}(\mathbb{N}) \to R\) be a finitely additive bounded set function and \(X\) be a finite subset of \(\mathbb{N}\). Then the following inequality holds:

\[
\sum_{l \in X} |m\{l\}| \leq 2 \sup_{A \subseteq X} |m(A)|.
\]

In order to prove our version of the Schur lemma, we will use the following technical result.

\textbf{Lemma 3.7.} Let \(m_j : \mathcal{P}(\mathbb{N}) \to R, j \in \mathbb{N}\), be a sequence of \(\sigma\)-additive equibounded measures, and \(m_0 : \mathcal{P}(\mathbb{N}) \to R\) be a set function, with the property that \((RD)\) \(\lim_j m_j = m_0\). Then there exists a regulator \((c_{i,l})_{i,l}\) in \(R\) such that for every \(\varphi \in \mathbb{N}^\mathbb{N}\) and for all sequences \((j_s)_s\), \((p_s)_s\) in \(\mathbb{N}\) with \(j_s \geq s\) for every \(s \in \mathbb{N}\) there is \(\overline{s} \in \mathbb{N}\) such that

\[
|m_{j_s}(A) - m_{j_s+p_s}(A)| \leq \sup_{i=1}^{\infty} c_{i,\varphi(i)} \tag{10}
\]

for any \(s \geq \overline{s}\) and for all \(A \subseteq \mathbb{N}\).
Proof: First of all, note that for all positive sequences \((a_i)_i\) and \((b_i)_i\) in \(R\) the following equations hold:

\[
2 \bigvee_{i=1}^{\infty} a_i = \bigvee_{i=1}^{\infty} 2a_i; \tag{11}
\]

\[
\bigvee_{i=1}^{\infty} a_i + \bigvee_{j=1}^{\infty} b_j \leq \bigvee_{i=1}^{\infty} 2(a_i + b_i). \tag{12}
\]

Observe that (12) is an easy consequence of (11).

Let now \(u\) be as in (2), and for each \(j \in \mathbb{N}\) let \((a^{(j)}_{i,l})_i\) be a \((D)\)-sequence related with the necessary and sufficient condition for \(\sigma\)-additivity of \(m_j\) given in Proposition 2.10. For every \(\varphi \in \mathbb{N}^N\) and \(j \in \mathbb{N}\), let us define \(\xi_j : \mathbb{N} \to \mathbb{N}\) as follows: \(\xi_j(n) = \varphi(n + j)\), for all \(n \in \mathbb{N}\). So, for every \(j \in \mathbb{N}\), the regulator \((a^{(j)}_{i,l})_i\) is such that in correspondence with \(\xi_j\) there exists \(\pi \in \mathbb{N}\) with

\[
|m_j(A)| \leq \bigvee_{i=1}^{\infty} a^{(j)}_i = \bigvee_{i=1}^{\infty} a^{(j)}_{i,\varphi(i+j)} \tag{13}
\]

for all \(A \subset \{\pi, \pi + 1, \pi + 2, \ldots\}\). By (2) and (13) it follows that

\[
|m_j(A)| \leq u \bigwedge \left( \sum_{j=1}^{q} \bigvee_{i=1}^{\infty} a^{(j)}_{i,\varphi(i+j)} \right) \tag{14}
\]

for all \(A \subset \{\pi, \pi + 1, \pi + 2, \ldots\}\) and for each \(q \in \mathbb{N}\).

By virtue of (14) and Lemma 2.4 there exists a \((D)\)-sequence \((a_{i,l})_i\) such that for every \(\varphi \in \mathbb{N}^N\) and \(j \in \mathbb{N}\) there is \(\pi \in \mathbb{N}\) with

\[
|m_j(A)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \tag{15}
\]

for each \(A \subset \{\pi, \pi + 1, \pi + 2, \ldots\}\), that is

\[
\|m_j\|_{p([\pi])}(\{\pi, \pi + 1, \pi + 2, \ldots\}) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}. \tag{16}
\]

Let \((b_{i,l})_i\) satisfy the condition of \((RD)\)-convergence of the \(m_j\)'s to \(m_0\). For each \(\varphi \in \mathbb{N}^N\), \(n \in \mathbb{N}\) and for all sequences \((j_s)_s\), \((p_s)_s\) in \(\mathbb{N}\) with \(j_s \geq s\) for every \(s\), there is \(s_0 \in \mathbb{N}\) with

\[
\bigvee_{A \subset \{1, \ldots, n\}} |\nu_s(A)| \leq 2 \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \tag{17}
\]
for all \( s \geq s_0 \), where \( \nu_s = m_{j_s} - m_{j_s + p_s} \). From (17) and Lemma 3.6 it follows that
\[
|\nu_s(\{1\})| + \ldots + |\nu_s(\{n\})| \leq 2 \sum_{i=1}^{\infty} b_{i,\varphi(i)} \tag{18}
\]
for all \( s \geq s_0 \). Set now
\[
c_{i,l} = 8 a_{i,l} + 12 b_{i,l} \tag{19}
\]
for all \( i, l \in \mathbb{N} \). We prove that the \((D)\)-sequence \((c_{i,l})_{i,l}\) defined in (19) satisfies condition (10). Otherwise there are an element \( \varphi \in \mathbb{N}^N \) and two sequences \((j_s)_s, (p_s)_s \) in \( \mathbb{N} \), with \( j_s \geq s \) for each \( s \in \mathbb{N} \), and such that for all \( s \in \mathbb{N} \) there is a set \( A_s \subset \mathbb{N} \) with
\[
|\nu_s(A_s)| \not\leq \infty \bigvee_{i=1}^{\infty} c_{i,\varphi(i)} \tag{20}
\]
Arguing analogously as in (13-16), since the \( \nu_s \)'s are equibounded and satisfy condition (4) of Proposition 2.10 with respect to the regulator \((2a_{i,l})_{i,l}\) and by virtue of Lemma 2.4, in correspondence with the function \( \varphi \) satisfying (20) and \( \nu_1 \) it is possible to find a natural number \( n_1 \) such that
\[
||\nu_1||_{p(N)}(\{n_1 + 1, n_1 + 2, \ldots\}) \leq 2 \sum_{i=1}^{\infty} a_{i,\varphi(i)} \tag{21}
\]
By \((RD)\)-convergence of the \( \nu_j \)'s to 0 with respect to the regulator \((2b_{i,l})_{i,l}\), which is an easy consequence of \((RD)\)-convergence of \((m_j)_j\) to \( m_0 \) with respect to the \((D)\)-sequence \((b_{i,l})_{i,l}\), proceeding analogously as in (17) and (18), in correspondence with \( n_1 \) there exists \( s_1 \in \mathbb{N} \) such that
\[
|\nu_s(\{1\})| + \ldots + |\nu_s(\{n_1\})| \leq 2 \sum_{i=1}^{\infty} b_{i,\varphi(i)} \tag{22}
\]
for all \( s \geq s_1 \). Proceeding similarly as above, it is possible to associate to \( \nu_1 \) a natural number \( n_2 > n_1 \) such that
\[
||\nu_{s_1}||_{p(N)}(\{n_2 + 1, n_2 + 2, \ldots\}) \leq 2 \sum_{i=1}^{\infty} a_{i,\varphi(i)} \tag{23}
\]
and to find \( s_2 > s_1 \) such that
\[
|\nu_s(\{1\})| + \ldots + |\nu_s(\{n_2\})| \leq 2 \sum_{i=1}^{\infty} b_{i,\varphi(i)} \tag{24}
\]
for all $s \geq s_2$. Proceeding by induction, we get the existence of two strictly increasing sequences $(n_h)_h$ and $(s_h)_h$ in $\mathbb{N}$ such that for all $h \in \mathbb{N}$ we have:

$$\|\nu_s\|_{\mathcal{P}(\mathbb{N})}(\{n_h + 1, n_h + 2, \ldots\}) \leq 2 \sum_{i=1}^{\infty} a_{i,\varphi(i)}$$

(25)

and

$$|\nu_s(\{1\}) + \ldots + |\nu_s(\{n_h\})| \leq 2 \sum_{i=1}^{\infty} b_{i,\varphi(i)}$$

(26)

for every $s \geq s_h$. Set now $s_0 = 1, n_0 = 0$ and

$$A = \bigcup_{h \in \mathbb{N} \cup \{0\}} (A_{s_h} \cap \{n_h + 1, \ldots, n_h + 1\}).$$

Notice that $A \cap \{1, \ldots, n_1\} = A_1 \cap \{1, \ldots, n_1\}$. From this and taking into account the finite additivity of $\nu_1$ we have:

$$\nu_1(A) = \nu_1(A_1 \cap \{1, \ldots, n_1\}) + \nu_1(A \cap \{n_1 + 1, n_1 + 2, \ldots\}),$$

$$\nu_1(A_1) = \nu_1(A_1 \cap \{1, \ldots, n_1\}) + \nu_1(A_1 \cap \{n_1 + 1, n_1 + 2, \ldots\}).$$

Observe now that $A \cap \{n_h + 1, \ldots, n_h + 1\} = A_{s_h} \cap \{n_h + 1, \ldots, n_h + 1\}$ for all $h \in \mathbb{N}$. From this and the finite additivity of $\nu_{s_h}$ we get:

$$\nu_{s_h}(A) = \nu_{s_h}(A \cap \{1, \ldots, n_h\}) + \nu_{s_h}(A_{s_h} \cap \{n_h + 1, \ldots, n_h + 1\})$$

$$+ \nu_{s_h}(A \cap \{n_h + 1, n_h + 1 + 2, \ldots\}),$$

(27)

$$\nu_{s_h}(A_{s_h}) = \nu_{s_h}(A_{s_h} \cap \{1, \ldots, n_h\}) + \nu_{s_h}(A_{s_h} \cap \{n_h + 1, \ldots, n_h + 1\})$$

$$+ \nu_{s_h}(A_{s_h} \cap \{n_h + 1, n_h + 1 + 2, \ldots\})$$

(28)

for every $h \in \mathbb{N}$. From (25), (26), (27) and (28), for all $h \in \mathbb{N}$ we obtain:

$$|\nu_{s_h}(A) - \nu_{s_h}(A_{s_h})| \leq 4 \sum_{i=1}^{\infty} a_{i,\varphi(i)} + 4 \sum_{i=1}^{\infty} b_{i,\varphi(i)}.$$  

(29)

By $(RD)$-convergence of the $\nu_j$'s to 0 with respect to the regulator $(2b_{i,l})_{i,l}$, in correspondence with $A$ there exists an integer $h_0$ such that for every $h \geq h_0$ we get

$$|\nu_{s_h}(A)| \leq \sum_{i=1}^{\infty} 2b_{i,\varphi(i)}.$$ 

(30)
From (29) and (30), taking into account (11) and (12), for all $h \geq h_0$ we have:

$$|\nu_{s_h}(A_{s_h})| \leq |\mu_{s_h}(A)| + |\nu_{s_h}(A) - \nu_{s_h}(A_{s_h})|$$

$$\leq 8 \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + 12 \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} = \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}.$$  

This is a contradiction with (20). Thus the lemma is completely proved. □

We are now in position to prove our Schur lemma.

**Proof of Theorem 3.1:** First of all we know that, thanks to Lemma 3.7, there exists a $(D)$-sequence $(c_{i,l})_{i,l}$ such that, for each $\varphi \in \mathbb{N}$ and for all sequences $(j_s)_s$, $(p_s)_s$ in $\mathbb{N}$ with $j_s \geq s$ for any $s$, there is $\bar{s} \in \mathbb{N}$ with

$$|m_{j_s}(A) - m_{j_s+p_s}(A)| \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)} \quad (31)$$

for any $s \geq \bar{s}$ and $A \subset \mathbb{N}$. From (31) and Lemma 3.6 applied to the set functions $m_{j_s} - m_{j_s+p_s}, s \geq \bar{s}$, it follows that for all $\varphi \in \mathbb{N}$, for all sequences $(j_s)_s$, $(p_s)_s$ in $\mathbb{N}$ with $j_s \geq s$ for each $s$, there exists $\bar{s} \in \mathbb{N}$ such that

$$\sum_{n=1}^{q} |m_{j_s}({n}) - m_{j_s+p_s}({n})|$$

$$\leq 2 \bigvee_{S \in S_{1,q}} |m_{j_s}(S) - m_{j_s+p_s}(S)| \leq 2 \bigvee_{i=1}^{\infty} c_{i,\varphi(i)} \quad (32)$$

for every $s \geq \bar{s}$ and $q \in \mathbb{N}$ (here $S_{1,q}$ is the set of all subsets of $\{1, \ldots, q\}$ for all $q \in \mathbb{N}$). Taking in (32) the supremum as $q$ varies in $\mathbb{N}$, we obtain that for any $\varphi \in \mathbb{N}$, for all sequences of natural numbers $(j_s)_s$, $(p_s)_s$ with $j_s \geq s$ for every $s$, there is $\bar{s} \in \mathbb{N}$ such that, whenever $s \geq \bar{s}$,

$$\sum_{n=1}^{\infty} |m_{j_s}({n}) - m_{j_s+p_s}({n})| \leq 2 \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}.$$  

(33)

From (33) it follows that for any $\varphi \in \mathbb{N}$ there exists $\bar{j}$ with the property that, for each $j \geq \bar{j}$ and $p \in \mathbb{N},$

$$\sum_{n=1}^{\infty} |m_{j}({n}) - m_{j+p}({n})| \leq 2 \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}.$$  

(34)
Otherwise, there is $\varphi \in \mathbb{N}$ such that for any $s \in \mathbb{N}$ there are $j_s, p_s \in \mathbb{N}$ with $j_s \geq s$ and such that
\[
\sum_{n=1}^{\infty} |m_{j_s}(\{n\}) - m_{j_s+p_s}(\{n\})| \leq 2 \sqrt{\sum_{i=1}^{\infty} c_{i,\varphi(i)}},
\]
that is there exist $\varphi \in \mathbb{N}$ and two sequences $(j_s)_s, (p_s)_s$ in $\mathbb{N}$ with $j_s \geq s$ for each $s \in \mathbb{N}$ and such that
\[
\sum_{n=1}^{\infty} |m_{j_s}(\{n\}) - m_{j_s+p_s}(\{n\})| \leq 2 \sqrt{\sum_{i=1}^{\infty} c_{i,\varphi(i)}},
\]
whenever $s \in \mathbb{N}$. This contradicts (33), and thus (34) is proved.

Set now, for all $j, n \in \mathbb{N}$, $a^{(j)}(n) = m_j(\{n\})$, and for every $j \in \mathbb{N}$, put $a^{(0)}(n) = (a^{(0)}(n))_n$.

From (34) we get that the sequence $(a^{(j)}(n))_j$ is Cauchy in $l^1(R)$. By Proposition 3.5, $(a^{(j)}(n))_j$ is convergent in $l^1(R)$, and thus there exist an element $a \in l^1(R)$, $a = (a^{(0)}(n))_n$, and a $(D)$-sequence $(\beta_i)_i$ such that to every $\varphi \in \mathbb{N}$ there corresponds $j \in \mathbb{N}$ with
\[
\sum_{n=1}^{\infty} |m_j(\{n\}) - a^{(0)}(n)| \leq 2 \sqrt{\sum_{i=1}^{\infty} \beta_{i,\varphi(i)}},
\]
for every $j \geq j$. Note that convergence of $(a^{(j)}(n))_j$ in $l^1(R)$ implies $(D)$-convergence of $(m_j(\{n\}))_j$ to $a^{(0)}(n)$ for all $n \in \mathbb{N}$, and this limit is unique, thanks to weak $\sigma$-distributivity of $R$. Thus we get $a^{(0)}(n) = m_0(\{n\})$ for every $n \in \mathbb{N}$.

We now prove that $m_0$ is $\sigma$-additive. For all $A \subset \mathbb{N}$ set
\[
m^*(A) = (D) \lim_q \left( \sum_{n \in A, n=1,\ldots,q} m_0(\{n\}) \right) = \sum_{n \in A} m_0(\{n\}).
\]
(36)

By Proposition 2.3 the limit in (36) exists in $R$ uniformly with respect to $A \subset \mathbb{N}$. This also implies that
\[
\sum_{n \in A, n \geq q} m_0(\{n\}) = (D) \lim_l \left( \sum_{n \in A, n=q,\ldots,q+l} m_0(\{n\}) \right)
\]
exists in $R$ and that
\[
(D) \lim_q \sum_{n \in A, n \geq q} m_0(\{n\}) = 0
\]
(37)
uniformly with respect to $A \subset \mathbb{N}$.

We now claim that

$$m^*(A) = m_0(A), \quad \text{for all } A \subset \mathbb{N}. \quad (38)$$

Indeed, we have:

$$m_0(A) - m^*(A) = (D) \lim_j [m_j(A) - m^*(A)]$$

$$= (D) \lim_j \left[ (D) \lim_q \left( \sum_{n \in A, n = 1, \ldots, q} (m_j(\{n\}) - m_0(\{n\})) \right) \right]$$

for each $A \subset \mathbb{N}$; moreover from (35) it follows that to every $\varphi \in \mathbb{N}^\mathbb{N}$ a natural number $\bar{J}$ can be associated, with the property that, for all $j \geq \bar{J}$ and $A \subset \mathbb{N}$,

$$\left| (D) \lim_q \left( \sum_{n \in A, n = 1, \ldots, q} (m_j(\{n\}) - m_0(\{n\})) \right) \right|$$

$$\leq \sum_{n \in A} |m_j(\{n\}) - m_0(\{n\})|$$

$$\leq \sum_{n=1}^\infty |m_j(\{n\}) - m_0(\{n\})| \leq \bigvee_{i=1}^\infty \beta_i(\varphi(i)). \quad (40)$$

From (39), (40) and weak $\sigma$-distributivity of $R$ we get:

$$0 \leq |m_0(A) - m^*(A)| \leq \bigwedge_{\varphi \in \mathbb{N}^\mathbb{N}} \left( \bigvee_{i=1}^\infty \beta_i(\varphi(i)) \right) = 0$$

for all $A \subset \mathbb{N}$; thus we get (38).

From (36), (37) and (38) we have, uniformly with respect to $A \subset \mathbb{N}$:

$$(D) \lim_q |m^*(A \cap \{q, q + 1, \ldots\})| = (D) \lim_q \left| \sum_{n \in A, n \geq q} m^*(\{n\}) \right| \quad (41)$$

$$= (D) \lim_q \left| \sum_{n \in A, n \geq q} m_0(\{n\}) \right| = 0.$$

From (41) it follows easily that $(D) \lim_q \bigvee \{|m^*(B)| : B \subset \{q, q + 1, \ldots\} \} = 0$, namely

$$(D) \lim_q \|m^*\|_{\mathcal{P}(\mathbb{N})}(\{q, q + 1, \ldots\}) = 0. \quad (42)$$

From (42) and Proposition 2.10 it follows that $m^*$ is $\sigma$-additive on $\mathcal{P}(\mathbb{N})$, and hence $m_0$ is $\sigma$-additive too, since $m_0$ coincides with $m^*$. 
From σ-additivity of $m_0$ and (40) it follows that to every $\varphi \in \mathbb{N}^\mathbb{N}$ there corresponds $\mathcal{J} \in \mathbb{N}$ such that, for any $j \geq \mathcal{J}$ and $A \subset \mathbb{N}$, we have
\[
|m_j(A) - m_0(A)| = \left| \lim_{q \to \infty} \left( \sum_{n \in A, n=1}^q (m_j \{n\}) - m_0 \{n\} \right) \right| \leq \bigvee_{i=1}^\infty \beta_{i, \varphi(i)},
\]
and hence
\[
(U) \lim_j m_j = m_0.
\] (43)
Uniform σ-additivity of the $m_j$’s is a consequence of (43) and σ-additivity of $m_0$. Indeed, by virtue of (43), there exists a $(D)$-sequence $(h_{i,l})_{i,l}$ such that to every $\varphi \in \mathbb{N}^\mathbb{N}$ a positive integer $j$ can be associated, with
\[
|m_j(A) - m_0(A)| \leq \bigvee_{i=1}^\infty h_{i, \varphi(i)}
\] (44)
whenever $j \geq \mathcal{J}$ and $A \subset \mathbb{N}$. Moreover, by virtue of σ-additivity of $m_0$ and the $m_j$’s, their equiboundedness and Lemma 2.4, arguing analogously as in (13-15), there exists a $(D)$-sequence $(f_{i,l})_{i,l}$ such that, for all $\varphi \in \mathbb{N}^\mathbb{N}$ and $j \in \mathbb{N} \cup \{0\}$, there exists $\mathfrak{n} = \mathfrak{n}(\varphi, j) \in \mathbb{N}$ with
\[
|m_j(A)| \leq \bigvee_{i=1}^\infty f_{i, \varphi(i)}, \text{ for all } A \subset \{\mathfrak{n}, \mathfrak{n}+1, \mathfrak{n}+2, \ldots\}.
\] (45)
Fix arbitrarily $\varphi \in \mathbb{N}^\mathbb{N}$, and let $\mathcal{J}$ be as in (44). In correspondence with $\varphi$ and $j = 0, 1, \ldots, \mathcal{J} - 1$, there exist $\mathfrak{n}_0, \mathfrak{n}_1, \ldots, \mathfrak{n}_{\mathcal{J}-1}$ as in (45). Set $n^* = \max(\mathfrak{n}_0, \mathfrak{n}_1, \ldots, \mathfrak{n}_{\mathcal{J}-1})$: we have
\[
|m_j(A)| \leq \bigvee_{i=1}^\infty f_{i, \varphi(i)}, \text{ for all } A \subset \{n^*, n^*+1, n^*+2, \ldots\}.
\] (46)
Moreover, for every $j \geq \mathcal{J}$ and $A \subset \{n^*, n^*+1, n^*+2, \ldots\}$, we get
\[
|m_j(A)| \leq |m_j(A) - m_0(A)| + |m_0(A)| \leq \bigvee_{i=1}^\infty h_{i, \varphi(i)} + \bigvee_{i=1}^\infty f_{i, \varphi(i)}.
\] (47)
Uniform σ-additivity of the $m_j$’s follows from (46) and (47). □

The next step is to prove our version of the Vitali-Hahn-Saks-type theorem. In order to do it, we first formulate the following technical lemma.
Lemma 3.8. Let $R$ be a Dedekind complete weakly $\sigma$-distributive $(l)$-group, $G$ be any infinite set, $\mathcal{A} \subset \mathcal{P}(G)$ be a $\sigma$-algebra, $m : \mathcal{A} \to R$ be a finitely additive set function. Let $(E_n)_n$ be any decreasing sequence of $\mathcal{A}$, put $F = \bigcap_{n=1}^{\infty} E_n$ and suppose that $m(F) = 0$. Set $B_n = E_n \setminus E_{n+1}$ for all $n \in \mathbb{N}$, and denote by $\mathcal{K}$ and $\mathcal{L}$ the $\sigma$-algebras generated by the $B_n$’s in $E_1$ and by the $E_n$’s in $E_1$ respectively. Then for all $n \in \mathbb{N}$ we get
\[
\|m\|_{\mathcal{L}(E_n)} = \|m\|_{\mathcal{K}(\bigcup_{l=n}^{\infty} B_l)}.
\] (48)

Proof: First of all notice that $\mathcal{L} = \mathcal{K} \cup \{X \cup F : X \in \mathcal{K}\}$. For all $n \in \mathbb{N}$ let
\[
\mathcal{K}_n := \{X \in \mathcal{K} : X \subset \bigcup_{l=n}^{\infty} B_l\}.
\]
The following equalities hold for all $n \in \mathbb{N}$:
\[
v_{\mathcal{L}}(m)(E_n) = v_{\mathcal{L}}(m)(F \cup \bigcup_{l=n}^{\infty} B_l)) = \bigvee_{X \in \mathcal{K}_n} [\|m(X)\| \lor |m(X) + m(F)|], \text{ because } m \text{ is finitely additive}
\]
\[
= \bigvee_{X \in \mathcal{K}_n} |m(X)|, \text{ as } m(F) = 0 \text{ by hypothesis}
\]
\[
= v_{\mathcal{K}}(m)(\bigcup_{l=n}^{\infty} B_l).
\]
This ends the proof. \(\square\)

Finally we are ready to give our version of the Vitali-Hahn-Saks theorem for $\nu$-absolutely continuous set functions (not necessarily with respect to a common regulator, differently than in [12]).

Proof of Theorem 3.2: Let $(H_n)_n$ be any decreasing sequence of elements of $\mathcal{A}$ such that $\lim_n \nu(H_n) = 0$, set $B_n = H_n \setminus H_{n+1}$ for every $n \in \mathbb{N}$, and let $\mathcal{L}, \mathcal{K}$ be the $\sigma$-algebras generated by the $H_n$’s and by the $B_n$’s in $H_1$ respectively. Put $F = \bigcap_{n=1}^{\infty} H_n$.

Choose arbitrarily $j \in \mathbb{N}$. Since $m_j$ is $\nu$-absolutely continuous, it follows that
\[
\bigwedge_n \|m_j\|_{\mathcal{L}(H_n)} = \bigwedge_n \bigvee \{ |m_j(C)| : C \in \mathcal{L} \text{ with } C \subset H_n \} = 0.
\] (49)
Since $0 \leq \|m_j\|_{\mathcal{L}(F)} \leq \|m_j\|_{\mathcal{L}(H_n)}$ for every $n \in \mathbb{N}$, from (49) it follows that $\|m_j\|_{\mathcal{L}(F)} = 0$, and a fortiori $m_j(F) = 0$.

Now, for every $A \subset \mathbb{N}$, set $\mu_j(A) = m_j \left( \bigcup_{n \in A} B_n \right)$. We claim that

$$\|\mu_j\|_{\mathcal{P}(\mathbb{N})} \left( \{a, a+1, a+2, \ldots \} \right) = \bigvee \left( |\mu_j(B)| : B \subset \{a, a+1, a+2, \ldots \} \right)$$

for all $a \in \mathbb{N}$. Indeed, in order to prove that

$$\|\mu_j\|_{\mathcal{P}(\mathbb{N})} \left( \{a, a+1, a+2, \ldots \} \right) \leq \|m_j\|_{\mathcal{L}(H_n)},$$

it is enough to associate to every $B \subset \{a, a+1, a+2, \ldots \}$ the set $C = \bigcup_{n \in B} B_n$, which is contained in $H_n$. Conversely, observe that by Lemma 3.8 we have

$$\|m_j\|_{\mathcal{L}(H_n)} = \|m_j\|_{\mathcal{K}} \left( \bigcup_{l=n}^\infty B_l \right)$$

(51)

for every $n \in \mathbb{N}$.

If $C \in \mathcal{K}$ and $C \subset \bigcup_{l=n}^\infty B_l$, then there is $B \subset \{a, a+1, a+2, \ldots \}$ such that $C = \bigcup_{n \in B} B_n$. The equality (50) follows from this and (51). From (49) and (50) we obtain

$$\bigwedge_n \|\mu_j\|_{\mathcal{P}(\mathbb{N})} \left( \{a, a+1, a+2, \ldots \} \right) = 0. \quad (52)$$

By virtue of (52) and Proposition 2.10 we get that $\mu_j$ is a $\sigma$-additive measure on $\mathcal{P}(\mathbb{N})$.

Now, for each $A \subset \mathbb{N}$, set $\mu_0(A) = m_0 \left( \bigcup_{n \in A} B_n \right)$. The equiboundedness of the $\mu_j$’s and $(RD)$-convergence of the $\mu_j$’s to $\mu_0$ follow easily from the equiboundedness of the $m_j$’s and $(RD)$-convergence of the $m_j$’s to $m_0$ respectively. By the Schur lemma 3.1, the measures $\mu_j$, $j \in \mathbb{N}$, are uniformly $\sigma$-additive on $\mathcal{P}(\mathbb{N})$. By Remark 2.11 we get

$$\bigwedge_n \bigvee_j \|\mu_j\|_{\mathcal{P}(\mathbb{N})} \left( \{a, a+1, a+2, \ldots \} \right) = 0. \quad (53)$$

From (50) and (53) we obtain that $\bigwedge_n \bigvee_j \|m_j\|_{\mathcal{L}(H_n)} = 0$ for all $j \in \mathbb{N}$, and thus the set functions $m_j$, $j \in \mathbb{N}$, are uniformly $\nu$-absolutely continuous.
From this it follows easily that $m_0$ is $\nu$-absolutely continuous too. This ends the proof. □

The proof of our version of the Nikodým convergence theorem is similar to the one of the Vitali-Hahn-Saks theorem and uses the Schur lemma 3.1 too.

**Proof of Theorem 3.3:** Let $(H_n)_n$ be any disjoint sequence of elements of $A$, and for every $n \in \mathbb{N}$ put $E_n = \bigcup_{i=n}^{\infty} H_i$. Let $\mathcal{L}$ be the $\sigma$-algebra generated by the $H_n$’s in the set $\bigcup_{n=1}^{\infty} H_n$.

Choose arbitrarily $j \in \mathbb{N}$. Since $m_j$ is $\sigma$-additive, we have

$$\bigwedge_n \|m_j\|_\mathcal{L}(E_n) = \bigwedge_n \bigvee \{\|m_j(C)\| : C \in \mathcal{L} \text{ with } C \subset E_n\} = 0. \quad (54)$$

Now, for every $A \subset \mathbb{N}$, set $\mu_j(A) = m_j\left(\bigcup_{n \in A} H_n\right)$. By arguing similarly as in (50), it follows that

$$\|\mu_j\|_{\mathcal{P}(\mathbb{N})}(\{n, n+1, n+2, \ldots\}) = \|m_j\|_\mathcal{L}(E_n) \quad (55)$$

for all $n \in \mathbb{N}$. From (54) and (55) we obtain

$$\bigwedge_n \|\mu_j\|_{\mathcal{P}(\mathbb{N})}(\{n, n+1, n+2, \ldots\}) = 0. \quad (56)$$

By virtue of (56) and Proposition 2.10 we obtain that $\mu_j$ is a $\sigma$-additive measure on $\mathcal{P}(\mathbb{N})$.

For each $A \subset \mathbb{N}$, put $\mu_0(A) = m_0\left(\bigcup_{n \in A} H_n\right)$. The equiboundedness of the $\mu_j$’s and $(RD)$-convergence of the $\mu_j$’s to $\mu_0$ are easy consequences of the equiboundedness of $m_j$’s and $(RD)$-convergence of the $m_j$’s to $m_0$ respectively. By the Schur lemma 3.1, the measures $\mu_j$, $j \in \mathbb{N}$, are uniformly $\sigma$-additive and $\mu_0$ is $\sigma$-additive. Now, proceeding analogously as in the final part of the proof of Theorem 3.2, we get that $\bigwedge_n \bigvee_j \|m_j\|_\mathcal{L}(E_n) = 0$ for all $j \in \mathbb{N}$. Thus the measures $m_j$, $j \in \mathbb{N}$, are uniformly $\sigma$-additive and $m_0$ is $\sigma$-additive too. □

Finally, we give the following result, which is a consequence of Lemma 3.8 and connects uniform $\sigma$-additivity and uniform absolute continuity. Observe that every $\nu$-absolutely continuous finitely additive set function $m$ defined on $A$ has the property that $m(S) = 0$ whenever $S \in A$ and $\nu(S) = 0$. 


Theorem 3.9. Let $R$, $G$ and $A$ be as in Lemma 3.8, $m_j : A \rightarrow R$, $j \in \mathbb{N}$, be a sequence of uniformly $\sigma$-additive measures. Let $\nu : A \rightarrow R \cup \{+\infty\}$ be a finitely additive set function. If $\nu(S) = 0$ implies $m_j(S) = 0$ for each $j \in \mathbb{N}$ and each $S \in A$, then the $m_j$’s are uniformly $\nu$-absolutely continuous.

Proof: Let $(E_n)_n$ be any decreasing sequence of elements of $A$ such that $\lim_n \nu(E_n) = 0$. Put $F = \bigcap_{n=1}^{\infty} E_n$. As $\nu(F) \leq \nu(E_n)$ for all $n \in \mathbb{N}$, it follows that $\nu(F) = 0$. Therefore $m_j(F) = 0$ for all $j \in \mathbb{N}$.

Put $B_n = E_n \setminus E_{n+1}$ for all $n \in \mathbb{N}$. Denote by $\mathcal{K}$ and $\mathcal{L}$ the $\sigma$-algebras generated by the $B_n$’s in $E_1$ and by the $E_n$’s in $E_1$ respectively. By Lemma 3.8 applied to $m_j$, for every $j$ and $n \in \mathbb{N}$ we get:

$$\|m_j\|_{\mathcal{L}(E_n)} = \|m_j\|_{\mathcal{K}\left(\bigcup_{l=n}^{\infty} B_l\right)}.$$  \hfill (57) \hfill 

The $m_j$’s are uniformly $\sigma$-additive, thus

$$(D) \lim_{n} \bigvee_{j} \|m_j\|_{\mathcal{K}\left(\bigcup_{l=n}^{\infty} B_l\right)} = \bigwedge_{n} \bigvee_{j} \|m_j\|_{\mathcal{K}\left(\bigcup_{l=n}^{\infty} B_l\right)} = 0.$$

Hence $(D) \lim_{n} \bigvee_{j} \|m_j\|_{\mathcal{L}(E_n)} = \bigwedge_{n} \bigvee_{j} \|m_j\|_{\mathcal{L}(E_n)} = 0$, therefore the $m_j$’s are uniformly $\nu$-absolutely continuous.  \hfill \Box

Open problem: Find some necessary and/or sufficient conditions for which the semivariation of $(l)$-group-valued measures introduced in the paper is equal to the one with respect to the $\sigma$-algebra where the measures are defined.

References


Limit theorems in $(l)$-groups w.r.t. $(D)$-convergence


