Kondurar Theorem and Itô formula in Riesz spaces

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Abstract

In this paper we formulate a version of the Kondurar theorem and a generalization of the Itô formula for functions taking values in Riesz spaces with respect to a convergence, satisfying suitable axioms. When the involved space is the space of measurable functions, both convergence
almost everywhere and convergence in probability are included. Finally
we present some comments and possible applications of the Itô formula.

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1 Introduction

In this paper we establish the Kondurar theorem and the Itô formula in Riesz
spaces.

The classical (scalar) version of the Kondurar theorem ensures the ex-
istence of the Riemann-Stieltjes integral \( \int_a^b f \, dg \), as soon as \( f \) and \( g \) are
Hölder-continuous, of orders \( \alpha \) and \( \beta \) respectively, with \( \alpha + \beta > 1 \).

Here we chose \( [a, b] = [0, 1] \), and fixed the Riesz space setting as a product
triple \((R_1, R_2, R)\) such that \( f \) is \( R_1 \)-valued, \( g \) is \( R_2 \)-valued, and the integral
takes values in \( R \).

More generally Riemann-Stieltjes integration is investigated also with re-
spect to an interval function \( q \) in the place of \( g \): we obtain meaningful exten-
sions of the Kondurar theorem, and deduce a version of the Itô formula for in-
tegrands in a Riesz space. The classical stochastic integrals (Itô, Stratonovich,
Backward) and the classical Itô formula are included.

Finally, we present some examples in order to illustrate the possible appli-
cations.

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2 Preliminaries

Definition 2.1 Throughout the paper, \( R \) will denote a Dedekind complete Riesz space. An axiomatic notion of convergence in \( R \) is defined by choosing a suitable order ideal \( \mathcal{N} \) (in the sense of [6], p. 93) on the Riesz product space \( R^N \). More precisely, this ideal \( \mathcal{N} \) is assumed to satisfy the following additional conditions:

a) Every element of \( \mathcal{N} \) is a bounded sequence.

b) Every sequence in \( R \) whose terms eventually vanish belongs to \( \mathcal{N} \).

c) Every sequence of the form \( \left( \frac{u}{n} \right)_n \), with \( u \in R \) and \( u \geq 0 \), belongs to \( \mathcal{N} \).

By means of the ideal \( \mathcal{N} \), it is possible to define the concept of convergence for nets in the following manner. Let \( (T, \geq) \) be any directed set. Given any bounded net \( \Psi : T \to R \), we say that \( \mathcal{N} \)-converges to an element \( y \) of \( R \) (or that it admits \( y \) as \( \mathcal{N} \)-limit) if there exists a sequence \( (s_n)_n \) of elements of \( T \) such that the sequence \( (p_n)_n \) defined by \( p_n = \sup_{t \in T, t \geq s_n} |\Psi(t) - y| \) belongs to \( \mathcal{N} \). The ideal \( \mathcal{N} \) then coincides with the space of all bounded sequences in \( R \) which converge to zero. For \( \mathcal{N} \)-convergence all usual properties of limits hold; in particular, uniqueness of the limit and Cauchy criterion for convergence: a bounded net \( \Psi : T \to R \) converges if and only if there exists a sequence \( (s_n)_n \) of elements of \( T \) such that the sequence \( (q_n)_n \) defined by \( q_n = \sup_{s, t \in T, s \land t \geq s_n} |\Psi(s) - \Psi(t)| \) belongs to \( \mathcal{N} \). Moreover, for a fixed directed set \( T \), the space of all bounded nets of the form \( \Psi : T \to R \) for which the limit exists is a Riesz subspace of \( R^T \), and the operator which assigns to each
convergent net the corresponding limit is linear and order-preserving.

Now, let us denote by \( J \) the family of all (nontrivial) closed subintervals of the interval \([0, 1]\), and by \( D \) the family of all finite decompositions of \([0, 1]\),
\[
D = \{0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1\}, \quad \text{or also} \quad D = \{I_1, \ldots, I_n\},
\]
where \( I_i = [t_{i-1}, t_i] \) for each \( i = 1, \ldots, n \).

Given any decomposition \( D = \{I_1, \ldots, I_n\} \), the mesh of \( D \) is the number
\[
\delta(D) := \max\{|I_i| : i = 1, \ldots, n\}, \quad \text{where} \quad |I| \quad \text{as usual denotes the length of the interval} \quad I.
\]

Then the set \( D \) can be endowed with the mesh ordering:
\( D_1 \geq D_2 \iff \delta(D_1) \leq \delta(D_2) \); this makes \( D \) a directed set, and, unless otherwise specified, convergence of any net \( \Psi : D \to \mathbb{R} \) will be always related to this order relation. However, besides this order relation, a weaker ordering will be helpful: for any two decompositions, \( D_1, D_2 \), we say that \( D_2 \) is finer than \( D_1 \), and write \( D_2 \succ D_1 \), if every interval from \( D_2 \) is entirely contained in some interval from \( D_1 \). Moreover, we shall say that a decomposition \( D \) is rational if all its endpoints are rational, and that a decomposition \( D \) is equidistributed if all intervals in \( D \) have the same length. If \( D \) is equidistributed, and consists of \( 2^n \) intervals, we say that \( D \) is dyadic of order \( n \). Equidistributed and dyadic decompositions of any interval \([a, b]\) are similarly defined. In general, the family of all decompositions of any interval \([a, b]\) \( \subset [0, 1] \) will be denoted by \( D_{[a,b]} \).

From now on, we shall assume that an ideal \( \mathcal{N} \) has been fixed, according with Definition 2.1, and consequently all limits we shall introduce (unless otherwise specified) are related to \( \mathcal{N} \)-convergence, without a particular notation.
**Definition 2.2** Let \( q : \mathcal{J} \to \mathbb{R} \) be any interval function. We say that \( q \) is **integrable**, if the net \( S(D) = \sum_{I \in D} q(I) \) is convergent to some element \( Y := \int q \).

A very useful tool to prove integrability is the Cauchy property. We first introduce a notation: for every interval function \( q : \mathcal{J} \to \mathbb{R} \), we set

\[
OB(q)(I) = \sup \left\{ \left| \sum_{J \in D} q(J) - \sum_{H \in D'} q(H) \right| : D, D' \in \mathcal{D}_I \right\}.
\]

Now we have the following:

**Theorem 2.3** Assume that \( q : \mathcal{J} \to \mathbb{R} \) is any interval function. The following three conditions are equivalent:

(i) \( q \) is integrable;

(ii) \( (r_n)_n \in \mathbb{N} \), where

\[
r_n := \sup \left\{ |S(D) - S(D_0)| : D, D_0 \in \mathcal{D}, \delta(D_0) \leq \frac{1}{n}, D \succ D_0 \right\};
\]

(iii) \( \int OB(q) = 0 \).

**Proof.** It is easy to prove that (i) implies (ii). To prove the implication (ii) ⇒ (iii), one can proceed in a similar fashion as in the proof of Theorem 1.4 in [4]. So we only prove that (iii) implies (i). By supposition, we have \( (\kappa_n)_n \in \mathbb{N} \), where \( \kappa_n = \sup \left\{ \sum_{I \in D} OB(q)(I) : \delta(D) \leq \frac{1}{n} \right\} \). Let us fix \( n \), and choose two decompositions, \( D_0 \) and \( D \), such that \( \delta(D_0) \leq \frac{1}{n} \) and \( D \succ D_0 \). From

\[
S(D_0) - S(D) = \sum_{I \in D_0} \left[ q(I) - \sum_{J \in D, J \subseteq I} q(J) \right]
\]

we obtain \( |S(D_0) - S(D)| \leq \kappa_n \). From this one can easily deduce that

\[
|S(D_1) - S(D_2)| \leq 2\kappa_n,
\]
as soon as $\delta(D_1)$ and $\delta(D_2)$ are less than $\frac{1}{n}$. Now define

$$M_n = \sup \left\{ S(D) : \delta(D) \leq \frac{1}{n} \right\}, \quad m_n = \inf \left\{ S(D) : \delta(D) \leq \frac{1}{n} \right\}. $$

Of course, $m_n \leq M_n$. Moreover, thanks to (1), it follows easily that $M_n - m_n \leq 2\kappa_n$. Then $\inf_{k \in \mathbb{N}} M_k = \sup_{k \in \mathbb{N}} m_k$, and now we prove that the common value $L$ is the integral: $L = \int q$. For every integer $n$, and each decomposition $D_0$ with $\delta(D_0) \leq \frac{1}{n}$, we have:

$$s(D_0) - L \leq s(D_0) - m_n \leq M_n - m_n \leq 2\kappa_n;$$

$$L - s(D_0) \leq M_n - s(D_0) \leq M_n - m_n \leq 2\kappa_n.$$ 

Therefore $|L - s(D_0)| \leq 2\kappa_n$, and this completes the proof, by arbitrariness of $D_0$. □

Our next goal is to prove that, if an interval function is integrable in $[a, b]$, then it is integrable in any subinterval, and the integral is an additive function.

**Theorem 2.4** Let $q : J \to \mathbb{R}$ be an integrable function. Then, for every subinterval $J \subset [0, 1]$, the function $q_J$ is integrable, where $q_J$ is defined as $q_J(I) = q(I \cap J)$, as soon as $I \cap J$ is nondegenerate, and 0 otherwise. Moreover, if $\{J_1, J_2\}$ is any decomposition of some interval $J \subset [0, 1]$, we have

$$\int_J q = \int_{J_1} q + \int_{J_2} q. \text{ (Here, as usual, } \int_J q \text{ means the integral of } q_J)$$

**Proof.** Fix any interval $J \subset [0, 1]$, and, for every positive integer $n$, define

$$\sigma_n(J) := \sup \left\{ \left| \sum_{I \in D'_J} q(I) - \sum_{H \in D''_J} q(H) \right| : D'_J, D''_J \in D_J, \delta(D'_J) \leq \frac{1}{n}, \delta(D''_J) \leq \frac{1}{n} \right\}. $$

It is easy to deduce that $\sigma_n(J) \leq \sigma_n([0, 1])$. As $(\sigma_n([0, 1]))_n \in \mathcal{N}$, the same holds for $(\sigma_n(J))_n$. Therefore, by Theorem 2.3, it follows that $q_J$ is integrable.
To prove additivity, fix an interval $J = [a,b]$, an integer $n > 0$, and a point $c \in ]a,b]$. Now, choose any decomposition $D_0$ of $[a,b]$, such that $\delta(D_0) \leq \frac{1}{n}$, and such that $c$ is one of its intermediate points; next, denote by $D'$ the decomposition consisting of the intervals from $D_0$ that are contained in $[a,c]$ and by $D''$ the decomposition consisting of the remaining intervals from $D_0$. Clearly we have

$$\left| \int_J q - \int_{[a,c]} q - \int_{[c,b]} q \right| =$$

$$\left| \int_J q - \sum_{I \in D_0} q(I) + \sum_{I \in D'} q(I) - \int_{[a,c]} q + \sum_{I \in D''} q(I) - \int_{[c,b]} q \right| \leq \gamma_n,$$

where $(\gamma_n)_n$ is a suitable element from $\mathcal{N}$. By arbitrariness of $n$, we can infer that

$$\left| \int_J q - \int_{[a,c]} q - \int_{[c,b]} q \right| = 0, \text{ i.e. additivity of the integral.}$$

A further consequence of the previous result is the following.

**Theorem 2.5** Let $q : J \rightarrow \mathbb{R}$ be any integrable function, and let us denote by $\psi$ its integral function, i.e. $\psi(I) = \int_I q$, $I \in J$. Then the function $|q - \psi|$ has null integral.

**Proof.** Let us denote by $Z$ the function $Z = |q - \psi|$. We must show that the sequence $(\beta_n)_n$ belongs to $\mathcal{N}$, where $\beta_n = \sup \left\{ \sum_{I \in D} Z(I) : \delta(D) \leq \frac{1}{n} \right\}$. For every $n > 0$, let us set $\sigma_n := \sup \left\{ \sum_{I \in D} OB(q)(I) : \delta(D) \leq \frac{1}{n} \right\}$. Fix now an integer $n > 0$, and a decomposition $D_0$, with $\delta(D_0) \leq \frac{1}{n}$. For every element $I \in D_0$, we have $q(I) - \psi(I) = \lim_{D \in D_{01}} (q(I) - S(D))$ (intended as $\mathcal{N}$-limit of
a net, depending on $I$, and therefore

$$\sum_{I \in D_0} Z(I) = \sum_{I \in D_0} \left| \lim_{D \in D_I} (q(I) - S(D)) \right| \leq \sum_{I \in D_0} OB(q)(I) \leq \sigma_n$$

according with all properties of convergence. But $(\sigma_n)_n \in \mathcal{N}$ by Theorem 2.3, hence also $(\beta_n)_n \in \mathcal{N}$, and the proof is finished. \qed

We now introduce some structural assumptions, which will be needed later.

**Assumptions 2.6** Let $R_1$, $R_2$, $R$ be three Dedekind complete Riesz spaces. We say that $(R_1, R_2, R)$ is a *product triple* if there exists a map $\cdot : R_1 \times R_2 \to R$, which we will call *product*, such that

2.6.1) $(r_1 + s_1) \cdot r_2 = r_1 \cdot r_2 + s_1 \cdot r_2$,

2.6.2) $r_1 \cdot (r_2 + s_2) = r_1 \cdot r_2 + r_1 \cdot s_2$,

2.6.3) $[r_1 \geq s_1, r_2 \geq 0] \Rightarrow [r_1 \cdot r_2 \geq s_1 \cdot r_2]$,

2.6.4) $[r_1 \geq 0, r_2 \geq s_2] \Rightarrow [r_1 \cdot r_2 \geq r_1 \cdot s_2]$ for all $r_j, s_j \in R_j, j = 1, 2$;

2.6.5) if $(a_\lambda)_{\lambda \in \Lambda}$ is any net in $R_2$ and $b \in R_1$, then $[a_\lambda \downarrow 0, b \geq 0] \Rightarrow [b \cdot a_\lambda \downarrow 0]$;

2.6.6) if $(a_\lambda)_{\lambda \in \Lambda}$ is any net in $R_1$ and $b \in R_2$, then $[a_\lambda \downarrow 0, b \geq 0] \Rightarrow [a_\lambda \cdot b \downarrow 0]$.

A Dedekind complete Riesz space $R$ is called an *algebra* if $(R, R, R)$ is a product triple.

Let now $(R_1, R_2, R)$ be a *product triple* of Riesz spaces. Given a bounded function $g : [0, 1] \to R_2$, we can associate with $g$ its *jump function* $\Delta(g)$:
\[ \Delta(g)([a, b]) := g(b) - g(a). \]

Moreover, the interval function

\[ \omega(g)(I) = \sup\{ |\Delta(g)([u, v])| : [u, v] \subset I \} \]

is called the oscillation of \( g \) in the interval \( I \).

**Definition 2.7** Let \( f : [0, 1] \rightarrow \mathbb{R}_1 \) and \( q : \mathcal{J} \rightarrow \mathbb{R}_2 \) be two functions. In order to properly define the Riemann-Stieltjes integral of \( f \) with respect to \( q \), we need a slight complication of the directed set \( D \). In fact, we shall denote by \( \tilde{D} \) the set of all pairs \((D, \tau)\), where \( D \) runs in \( D \) and \( \tau \) is a mapping which associates to every element \( I \in D \) an arbitrary point \( \tau_I \in I \). The set \( \tilde{D} \) will be ordered according with the usual mesh ordering on \( D \), i.e. \((D, \tau) \leq (D_1, \tau_1)\) iff \( \delta(D_1) \leq \delta(D) \). Given an element \((D, \tau) \in \tilde{D} \), we call Riemann-Stieltjes sum of \( f \) with respect to \( q \) (and denote it by \( S(D, \tau) \)) the following quantity:

\[ S(D, \tau) = \sum_{I \in D} f(\tau_I) q(I). \]

**Definition 2.8** Assume that \( f : [0, 1] \rightarrow \mathbb{R}_1 \) and \( q : \mathcal{J} \rightarrow \mathbb{R}_2 \) are two functions. We say that \( f \) is Riemann-Stieltjes integrable with respect to \( q \), if the net \((S(D, \tau))_{(D, \tau) \in \tilde{D}}\) is \( \mathcal{N} \)-convergent, i.e. if there exists an element \( Y \in R \), such that the sequence \((\pi_n)_n\) belongs to \( \mathcal{N} \), where

\[ \pi_n = \sup \left\{ |Y - S(D, \tau)| : (D, \tau) \in \tilde{D}, \delta(D) \leq \frac{1}{n} \right\}, \quad n \in \mathbb{N}. \]

If this is the case, the element \( Y \in R \) is called the Riemann-Stieltjes integral and denoted by \( Y := (RS) \int_0^1 f \, dq \). When \( q = \Delta(g) \), for some suitable function \( g : [0, 1] \rightarrow \mathbb{R}_2 \), integrability of \( f \) with respect to \( q \) will be expressed by saying that \( f \) is Riemann-Stieltjes integrable w.r.t. \( g \), and its integral will be denoted by \((RS) \int_0^1 f \, dg\).
3 The Kondurar theorem in Riesz spaces

Definition 3.1 Let $f : [0, 1] \rightarrow R_1$ and $q : J \rightarrow R_2$ be two fixed functions. We say that $f$ and $q$ satisfy assumption $(C)$ if the interval function $\phi : J \rightarrow R,$ defined as $\phi(I) = \omega(f)(I)|q(I)|,$ is integrable, and its integral is 0.

The next result is easy.

Proposition 3.2 Let $f : [0, 1] \rightarrow R_1$ and $q : J \rightarrow R_2$ be two bounded functions, satisfying assumption $(C).$ Then the Riemann-Stieltjes integral $(RS) \int_0^1 f \ dq$ exists in $R$ if and only if the interval function $Q(I) = f(a_I)q(I)$ is integrable, where $a_I$ denotes the left endpoint of $I$: if this is the case, the two integrals coincide.

In the next theorems, the concept of Hölder-continuous function will be crucial:

Definition 3.3 Let $R$ be an arbitrary Riesz space. We say that an interval function $q$ is Hölder-continuous (of order $\gamma$) if there exist a unit $u \in R,$ that is an element $R \ni u \geq 0,$ $u \neq 0,$ and a real constant $\gamma > 0,$ such that $|q(I)| \leq |I|^\gamma u$ for all $I \subset [0, 1].$ As usual, a function $g : [0, 1] \rightarrow R$ is said to be Hölder-continuous of order $\gamma$ if the function $\Delta(g)$ is.

We shall assume in addition that the interval function $q$ is additive, i.e. $q([a, b]) = q([a, c]) + q([c, b])$ as soon as $0 \leq a < c < b \leq 1.$ This means also that $q = \Delta(g)$ for some suitable bounded function $g : [0, 1] \rightarrow R_2,$ but we shall maintain our notation, without mentioning $g.$

The next step is the following result.
Proposition 3.4 Assume that $f : [0,1] \rightarrow R_1$ and $q : J \rightarrow R_2$ are two Hölder-continuous functions, for which assumption (C) is satisfied. Suppose also that $q$ is additive, and that $(\rho_n)_n \in N$, where
\[
\rho_n = \sup \left\{ \left| \sum_{J \in D} f(a_J)q(J) - \sum_{I \in D_0} f(a_I)q(I) \right| : D \ni D_0, D \text{ rational, } \delta(D_0) \leq \frac{1}{n}, D \succ D_0 \right\}
\]
for every $n \in N$. Then, there exists in $R$ the (RS)-integral of $f$ w.r.t. $q$.

Proof. Since assumption (C) is satisfied, from Proposition 3.2 it’s enough to prove that the $R$-valued interval function $Q(I) := f(a_I)q(I)$ is integrable. As usual, for every decomposition $D$ of $[0,1]$, we write $S(D) = \sum_{I \in D} Q(I) = \sum_{I \in D} f(a_I)q(I)$. By using Hölder-continuity and boundedness of $f$ and $q$ (which is a simple consequence), one can obtain the following key tool, which is rather technical, but not difficult.

Key Tool: There exists a unit $w \in R$ such that, for every $n \in N$ and every $D \in D$ with $\delta(D) < \frac{1}{n}$, one can find a rational decomposition $D_0$ such that $\delta(D_0) < \frac{1}{n}$ and
\[
|S(D) - S(D_0)| \leq \frac{w}{n}. \tag{2}
\]
Let us now turn to the proof that $Q$ is integrable. We first observe that
\[
\sup \left\{ |S(D_1) - S(D_2)| : D_1 \text{ and } D_2 \text{ rational, } \delta(D_1) < \frac{1}{n}, \delta(D_2) < \frac{1}{n} \right\} \leq 2\rho_n
\]
for every $n$; hence, using the key tool above, we see also that
\[
\sup \left\{ |S(D) - S(D')| : D \in D, D' \in D, \delta(D) < \frac{1}{n}, \delta(D') < \frac{1}{n} \right\} \leq 2\rho_n + 2\frac{w}{n}
\]
for every $n$. Now, the assertion follows from Theorem 2.3. \qed

Unless otherwise specified, we always assume that $q$ is additive, that $f$ and $q$ are Hölder-continuous of order $\alpha$ and $\beta$ respectively, and related units $u_1$ and $u_2$ respectively. Moreover, we shall require that $\gamma := \alpha + \beta > 1$.

**Lemma 3.5** Let $[a, b]$ be any sub-interval of $[0, 1]$, and let us denote by $c$ its midpoint. Then

$$|Q([a, b]) - (Q([a, c]) + Q([c, b]))| \leq u_1 u_2 \left(\frac{b - a}{2}\right)^\gamma.$$  

**Proof.** We have:

$$|Q([a, b]) - (Q([a, c]) + Q([c, b]))| = |f(a)q([a, b]) - f(a)q([a, c]) - f(c)q([c, b])]$$

$$\leq |f(c) - f(a)|q([c, b])| \leq u_1 u_2 \left(\frac{b - a}{2}\right)^{\alpha + \beta}. \qed$$

**Proposition 3.6** Let $[a, b]$ be any subinterval of $[0, 1]$, and assume that $D$ is a dyadic decomposition of $[a, b]$, of order $n$. Then there exists a positive element $v \in \mathbb{R}$, independent of $a, b$ and $n$, such that $|Q([a, b]) - \sum_{I \in D} Q(I)| \leq v |b - a|^{\gamma}$.

**Proof.** Let us consider the dyadic decompositions of $[a, b]$ of order $1, 2, \ldots, n$ and denote them by $D_1, D_2, \ldots, D_n = D$ respectively. Then we have

$$|Q([a, b]) - \sum_{I \in D} Q(I)| \leq |Q([a, b]) - \sum_{I \in D_1} Q(I)| + \sum_{i=2}^{n} |\sum_{J \in D_i} Q(J) - \sum_{I \in D_{i-1}} Q(I)|.$$  

Thanks to Lemma 3.5, the first summand in the right-hand side is less than $u_1 u_2 \left(\frac{b - a}{2}\right)^\gamma$. By the same Lemma, for every index $i$ from 2 to $n$, we have
Let us define, for suitable element of the decomposition of $[a,b]$ whose intermediate points are $t_0, t_1, \ldots, t_{h+1}$. From Proposition 3.6, it follows:

$$ |S(D_0) - S(D)| \leq \sum_{I \in D_0} |Q(I) - \sum_{J \in D, J \subset I} Q(J)| \leq \sum_{I \in D_0} v|I|^\gamma \leq ...$$
\[
\leq v \sum_{i=0}^{h} \left( 2^i \frac{b-a}{N} \right)^\gamma \leq v (b-a)^\gamma \frac{2^{(h+1)\gamma}}{N^\gamma} \leq v (b-a)^\gamma 2^\gamma.
\]

Thus, setting \( v_1 := 2^\gamma v \), we get \( |S(D_0) - S(D)| \leq v_1 (b-a)^\gamma \).

Let us now evaluate \( |Q([a,b]) - S(D_0)| \). We find

\[
|Q([a,b]) - S(D_0)| \leq |(f(a) - f(t_1))q([t_1, t_2]) + (f(a) - f(t_2))q([t_2, t_3]) + \ldots + (f(a) - f(t_h))q([t_h, b])|
\]

\[
\leq \sum_{j=1}^{h} |f(a) - f(t_j)| |q([t_j, t_{j+1}])| \leq \sum_{j=1}^{h} u_1 u_2 (t_j - a)^\alpha (t_{j+1} - t_j)^\beta.
\]

For all \( j = 1, 2, \ldots, h \) we have easily \( t_j - a \leq 2^j \frac{b-a}{N} \), hence

\[
|Q([a,b]) - S(D_0)| \leq \sum_{j=1}^{h} u_1 u_2 \left( 2^j \frac{b-a}{N} \right)^\alpha \left( 2^j \frac{b-a}{N} \right)^\beta \leq u_1 u_2 2^\gamma (b-a)^\gamma.
\]

Thus, setting \( v_2 := u_1 u_2 2^\gamma \), we get \( |Q([a,b]) - S(D_0)| \leq v_2 (b-a)^\gamma \), and finally we obtain the assertion by setting \( r = v_1 + v_2 \).

We are now ready for the main result.

**Theorem 3.8 (The Kondurar Theorem)** Let \( f : [0,1] \to R_1 \) and \( q : J \to R_2 \) be H"older-continuous of order \( \alpha \) and \( \beta \) respectively, with \( \gamma = \alpha + \beta > 1 \), and suppose that \( q \) is additive. Then \( f \) is \((RS)\)-integrable w.r.t. \( q \).

**Proof.** We first observe that the real valued interval function \( W(I) := |I|^\gamma \) is integrable, and its integral is null. This immediately implies that \( f \) and \( q \) satisfy assumption \((C)\). Let \( r \) be the unit in \( R \) given by Proposition 3.7, and set \( b_n := \frac{2r}{n} \). Fix \( n \) and choose any rational decomposition \( D_0 \) such that \( \delta(D_0) \leq \frac{1}{n} \) and \( \sum_{J \in D} |J|^\gamma \leq \frac{1}{n} \) for every decomposition \( D \), finer than \( D_0 \). If \( D \) is any rational decomposition, finer than \( D_0 \), then there exists an integer \( N \)
such that the equidistributed decomposition $D^*$, consisting of $N$ subintervals, is finer than $D$. Of course, $D^*$ is also finer than $D_0$, so we have, by Proposition 3.7:

$$|S(D^*) - S(D_0)| \leq \sum_{I \in D_0} r|I|^\gamma \leq \frac{1}{n}r$$

Therefore, by axioms of 2.1, we obtain:

$$(\sup \{|S(D) - S(D_0)| : D, D_0 \text{ are rational}, \delta(D_0) \leq 1/n, D \succ D_0\})_n \in \mathcal{L},$$

and the theorem is proved, thanks to Proposition 3.4. □

More concretely, we have the following version of the previous theorem.

**Theorem 3.9** Let $f : [0,1] \rightarrow R_1$ and $g : [0,1] \rightarrow R_2$ be two Hölder-continuous functions, of order $\alpha$ and $\beta$ respectively, and assume that $\alpha + \beta > 1$. Then $f$ is (RS)-integrable with respect to $g$.

However, we can obtain a more general result.

**Theorem 3.10** Assume that $f : [0,1] \rightarrow R_1$ is any Hölder-continuous function of order $\alpha$, and $q : \mathcal{J} \rightarrow R_2$ is any integrable interval function (not necessarily additive). If the integral function $\psi(I) = \int_I q$ is Hölder-continuous of order $\beta$, and $\alpha + \beta > 1$, then $f$ is (RS)-integrable with respect to $q$ and to $\psi$, and the two integrals coincide.

**Proof.** In 2.4 we proved that $\psi$ is an additive function; hence, if it is Hölder-continuous of order $\beta$, and $\alpha + \beta > 1$, we can deduce that $f$ is (RS)-integrable with respect to $\psi$, thanks to 3.8. Moreover, we also know from 2.5 that $|q - \psi|$ has null integral. This fact, together with boundedness of $f$, implies...
that \( \int_J f \, d(q - \psi) = 0 \), for every interval \( J \subset [0, 1] \). Now the conclusion is obvious. \( \Box \)

**Remark 3.11** Assume that \( W \) is the standard Brownian Motion, on a probability space \((X, \mathcal{B}, P)\), and \( g : [0, 1] \to L^0(X, \mathcal{B}, P) \) is the function which associates to each \( t \in [0, 1] \) the random variable \( W_t \in L^0 \). We can endow the Riesz space \( L^0 \) with the convergence in probability (which satisfies all assumptions in Definition 2.1); thus the interval function \( q(I) = (\Delta(g)(I))^2 \) is well-known (from the literature) to be integrable, and its integral function is \( \psi(I) = |I| \). So, Theorem 3.10 implies integrability with respect to \( q = \Delta(g)^2 \) of every function \( f \) which is Hölder continuous (of any order), and also of more general continuous functions (but we shall not discuss this here).

**Remark 3.12** The previous existence theorems remain true, if the definition 2.8 is modified, replacing the value \( f(\tau_I) \) with any element \( \theta_I \) such that \( \inf \{ f(x) : x \in I \} \leq \theta_I \leq \sup \{ f(x) : x \in I \} \). This mainly rests on Proposition 3.2 and on the fact that \( |\theta_I - f(\alpha_I)| \leq \omega(f)(I) \).

### 4 The Itô formula in Riesz spaces

In [4] we can find some results about the Itô formula for Riesz space-valued functions of one variable. Here we deal with functions of two variables; from now on we shall assume that our Riesz space \( R \) is an algebra, hence the involved functions and their integrals will take values in \( R \).

**Definition 4.1** Let \( R \) be an algebra, and \( f : [0, 1] \times R \to R \) a fixed function.
We say that $f$ satisfies Taylor’s formula of order 1 if there exist two $R$-valued functions, defined on $[0,1] \times R$ and denoted by $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}$, such that for every $(t,x) \in [0,1] \times R$, $h \in IR$ and $k \in R$, such that $t + h \in [0,1]$, we have:

$$f(t + h, x + k) = f(t, x) + \left(h \frac{\partial f}{\partial t} + k \frac{\partial f}{\partial x}\right)(t, x) + (|h| + |k|) B(t, x, h, k),$$

where $B$ is a suitable $R$-valued function, defined on all points of the type $(t, x, h, k)$ with $t + h \in [0,1]$, and bounded on bounded sets.

Moreover, we say that $f$ satisfies Taylor’s formula of order 2 if $f$ satisfies Taylor’s formula of the first order, and moreover there exist three more functions, denoted by $\frac{\partial^2 f}{\partial t^2}, \frac{\partial^2 f}{\partial t \partial x}, \frac{\partial^2 f}{\partial x^2}$, such that for every $(t, x)$ in $[0,1] \times R$, $h \in IR$ and $k \in R$, $(t + h \in [0,1])$, we have

$$f(t + h, x + k) - f(t, x) = h \frac{\partial f}{\partial t}(t, x) + k \frac{\partial f}{\partial x}(t, x) + \frac{1}{2} \left[h^2 \frac{\partial^2 f}{\partial t^2}(t, x) + 2hk \frac{\partial^2 f}{\partial t \partial x}(t, x) + k^2 \frac{\partial^2 f}{\partial x^2}(t, x)\right] + [h^2 + |hk| + |k|^2] B(t, x, h, k),$$

where $B$ is a suitable function, bounded on bounded sets.

We now introduce a "weaker" concept of integrability for Riesz space-valued functions. As above, $\mathcal{N}$ will always denote any fixed ideal as in 2.1.

**Definitions 4.2** Let $F : [0,1] \times R \to R$ and $g : [0,1] \to R$ be two functions.

Following the Stratonovich approach ([8]), for every $\lambda \in [0,1]$ we say that $F(\cdot, g(\cdot))$ is $\langle \lambda \rangle$-integrable with respect to $g$ (or to $q(I) := \Delta(g)(I)$) if there
exists an element $Y_\lambda \in R$ such that

$$
\left( \sup \left\{ \left| Y_\lambda - \sum_{I \in D} F(\lambda u_i + (1 - \lambda)v_i, \lambda g(u_i) + (1 - \lambda)g(v_i))\Delta(g)(I) \right| : \delta(D) \leq \frac{1}{n} \right\} \right)_n \in \mathcal{N},
$$

where $D = \{([u_i, v_i]) : i = 1, \ldots, N\}$ is the involved decomposition of $[0,1]$. We denote the $(\lambda)$-integral of a function $F$ with respect to $g$ by the symbol

$$\langle \lambda \rangle \int_0^1 Fdg.$$

Similarly, we say that the function $F(\cdot, g(\cdot))$ is $(\lambda)$-integrable with respect to $g(I) := (\Delta(g)(I))^2$ if there exists an element $J_\lambda \in R$ such that

$$
\left( \sup \left\{ \left| J_\lambda - \sum_{I \in D} F(\lambda u_i + (1 - \lambda)v_i, \lambda g(u_i) + (1 - \lambda)g(v_i)) (\Delta(g)(I))^2 \right| : \delta(D) \leq \frac{1}{n} \right\} \right)_n \in \mathcal{N},
$$

where $D = \{([u_i, v_i]) : i = 1, \ldots, N\}$ is the involved decomposition of $[0,1]$. In this case, we write $\langle \lambda \rangle \int_0^1 F(s, g(s))(dg)^2(s) := J_\lambda(F)$. We now prove the following:

**Proposition 4.3** Let $R$ be an algebra, and $f : [0,1] \times R \to R$, $(t, x) \mapsto f(t, x)$, satisfy Taylor’s formula of order 1; suppose also that $g : [0,1] \to R$ is Hölder-continuous of order $\beta$, where $\beta > \frac{1}{3}$. If $f$ is $(\lambda)$-integrable with respect to $g(I) := (\Delta(g)(I))^2$, i.e. there exists the integral $J_1(f)$, then for every $\lambda \in [0,1]$ the integral $J_\lambda(f)$ exists in $R$, and $J_\lambda(f) = J_1(f)$.

**Proof.** Let $D = \{[u_i, v_i], i = 1, \ldots, n\}$ be a decomposition of the interval $[0,1]$ and $\lambda \in [0,1]$. By hypotheses, we have:
\[\sum_{i=1}^n f (\lambda u_i + (1 - \lambda)v_i, \lambda g (u_i) + (1 - \lambda)g(v_i)) \cdot q([u_i, v_i]) = \sum_{i=1}^n f (u_i, g(u_i)) \cdot q([u_i, v_i]) + \sum_{i=1}^n [f (\lambda u_i + (1 - \lambda)v_i, \lambda g (u_i) + (1 - \lambda)g(v_i)) - f(u_i, g(u_i))] \cdot q([u_i, v_i]) =
\]

\[\sum_{i=1}^n \left[ f (u_i, g(u_i)) + (1 - \lambda) (v_i - u_i) \frac{\partial f}{\partial x} (u_i, g(u_i)) + (1 - \lambda) \left( \Delta(g)([u_i, v_i]) \right) \frac{\partial f}{\partial x} (u_i, g(u_i)) \right] \cdot q([u_i, v_i]) + \sum_{i=1}^n B([u_i, v_i]) |v_i - u_i|^\gamma = V_1 + V_2 + V_3 + V_4, \]

where \( B : \{J\} \rightarrow \mathbb{R} \) is a suitable interval function, bounded on bounded sets (with bound independent on \( \lambda \)), and \( \tau > 1 \) by virtue of Hölder-continuity of \( g \) (we denote the consecutive terms of the above sum by \( V_1, V_2, V_3, V_4 \)). The first part \( V_1 \) tends to \( J_1(f) \) as \( \delta(D) \) tends to 0. Now we can conclude, observing that the expressions \( V_2, V_3, V_4 \) are negligible because we can present them in the form \( \sum_{i=1}^n B_0([u_i, v_i]) |v_i - u_i|^\zeta \), where \( B_0 : \mathcal{J} \rightarrow \mathbb{R} \) is bounded on bounded sets and \( \zeta > 1 \).

We now prove our generalization of the Itô formula in the context of Riesz spaces:

**Theorem 4.4** Let \( R \) be an algebra, \( F : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \) satisfy Taylor’s formula of order 2 and \( \frac{\partial F}{\partial t} : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \), \( \frac{\partial^2 F}{\partial x^2} : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \), satisfy Taylor’s formula of order 1. Assume that \( g : [0, 1] \rightarrow \mathbb{R} \) is Hölder-continuous of order \( \beta > \frac{1}{2} \). Assume also that \( q(I) = (\Delta(g)(I))^2 \) is integrable and that its integral function is Hölder-continuous of order \( \gamma > 1 - \beta \). Then

1) for every \( \lambda \in [0, 1] \) the integral \( J_\lambda := \langle \lambda \rangle \int_0^1 \frac{\partial^2 F}{\partial x^2} (s, g(s)) (dg)^2(s) \) exists in \( \mathbb{R} \), and is independent on \( \lambda \);

2) for every \( \lambda \in [0, 1] \) the integral \( \langle \lambda \rangle \int_0^1 \frac{\partial F}{\partial x} (s, g(s)) \, dg(s) \) exists in \( \mathbb{R} \);
3) the following formula holds:

\[ F(1, g(1)) - F(0, g(0)) = \int_0^1 \frac{\partial F}{\partial t}(s, g(s)) \, ds + \langle \lambda \rangle \int_0^1 \frac{\partial F}{\partial x}(s, g(s)) \, dg(s) + \frac{1}{2} (2\lambda - 1) J, \]

where \( J \) denotes any of the integrals \( J_\lambda \) above.

**Proof.** The function \( t \mapsto \frac{\partial^2 F}{\partial x^2}(t, g(t)) \) is Hölder-continuous of order \( \beta \), because of Taylor’s formula and of Hölder-continuity of \( g \). Thus, using Theorem 3.10, we deduce that the function \( t \mapsto \frac{\partial^2 F}{\partial x^2}(t, g(t)) \) is \((RS)\)-integrable with respect to \( q \), and thus by virtue of Proposition 4.3 (where \( f = \frac{\partial^2 F}{\partial x^2} \)), for every \( \lambda \in [0, 1] \) the integral \( J := \langle \lambda \rangle \int_0^1 \frac{\partial^2 F}{\partial x^2}(s, g(s)) (dg)^2(s) \) exists in \( R \) and is independent on \( \lambda \). This shows the assertion 1).

Let now \( D = \{ [u_i, v_i], i = 1, \ldots, n \} \) be a decomposition of \([0, 1] \) and fix \( \lambda \in [0, 1] \). We have:

\[ F(1, g(1)) - F(0, g(0)) = \]

\[ \sum_{i=1}^n \left[ F(v_i, g(v_i)) - F(\lambda u_i + (1 - \lambda) v_i, \lambda g(u_i) + (1 - \lambda) g(v_i)) \right] - \]

\[ \sum_{i=1}^n \left[ F(u_i, g(u_i)) - F(\lambda u_i + (1 - \lambda) v_i, \lambda g(u_i) + (1 - \lambda) g(v_i)) \right]. \]

Now we shall apply Taylor’s formula of order 2:

\[ F(1, g(1)) - F(0, g(0)) = \]

\[ \left\{ \sum_{i=1}^n \frac{\partial F}{\partial t}(\lambda u_i + (1 - \lambda) v_i, \lambda g(u_i) + (1 - \lambda) g(v_i)) \cdot [\lambda (v_i - u_i)] \right. + \]

\[ \left. \sum_{i=1}^n \frac{\partial F}{\partial x}(\lambda u_i + (1 - \lambda) v_i, \lambda g(u_i) + (1 - \lambda) g(v_i)) \cdot [\lambda (\Delta(g)([u_i, v_i]))] + \right. \]

\[ \sum_{i=1}^n \frac{\partial^2 F}{\partial x^2}(\lambda u_i + (1 - \lambda) v_i, \lambda g(u_i) + (1 - \lambda) g(v_i)) \cdot [\lambda^2 (v_i - u_i)(\Delta(g)([u_i, v_i]))] + \]

\[ \sum_{i=1}^n \frac{\partial^2 F}{\partial x^2}(\lambda u_i + (1 - \lambda) v_i, \lambda g(u_i) + (1 - \lambda) g(v_i)) \cdot \left[ \frac{1}{2} \lambda^2 (v_i - u_i)^2 \right] + \]
\[ \sum_{i=1}^{n} \frac{\partial^{2} F}{\partial x_{i}^{2}} (\lambda u_{i} + (1 - \lambda) v_{i}, \lambda g (u_{i}) + (1 - \lambda) g (v_{i})) \cdot \left[ \frac{1}{2} \lambda^{2} q (u_{i}, v_{i}) \right] + \]

\[ \sum_{i=1}^{n} B_{1} ([u_{i}, v_{i}]) |v_{i} - u_{i}|^\gamma \}

\[ \{ \sum_{i=1}^{n} \frac{\partial F}{\partial t} (\lambda u_{i} + (1 - \lambda) v_{i}, \lambda g (u_{i}) + (1 - \lambda) g (v_{i}) \cdot [\lambda - 1] (v_{i} - u_{i}) ] + \]

\[ \sum_{i=1}^{n} \frac{\partial F}{\partial x_{i}} (\lambda u_{i} + (1 - \lambda) v_{i}, \lambda g (u_{i}) + (1 - \lambda) g (v_{i})) \cdot [(\lambda - 1) (\Delta (g) ([u_{i}, v_{i}]) ] + \]

\[ \sum_{i=1}^{n} \frac{\partial^{2} F}{\partial x_{i}^{2}} (\lambda u_{i} + (1 - \lambda) v_{i}, \lambda g (u_{i}) + (1 - \lambda) g (v_{i})) \cdot \left[ \frac{1}{2} (1 - \lambda)^{2} (v_{i} - u_{i})^{2} \right] + \]

\[ \sum_{i=1}^{n} \frac{\partial^{2} F}{\partial x_{i}^{2}} (\lambda u_{i} + (1 - \lambda) v_{i}, \lambda g (u_{i}) + (1 - \lambda) g (v_{i})) \cdot \left[ \frac{1}{2} (1 - \lambda)^{2} q ([u_{i}, v_{i}]) \right] + \]

\[ \sum_{i=1}^{n} B_{2} ([u_{i}, v_{i}]) |v_{i} - u_{i}|^\gamma, \]

where \( B_{1}, B_{2} : \{ I \} \to R \) are suitable interval functions, bounded on bounded sets (with bound independent on \( \lambda \)), and \( \gamma > 1 \). By collecting the similar terms we obtain:

\[ F(1, g(1)) - F(0, g(0)) = \]

\[ \sum_{i=1}^{n} \frac{\partial F}{\partial t} (\lambda u_{i} + (1 - \lambda) v_{i}, \lambda g (u_{i}) + (1 - \lambda) g (v_{i}) \cdot (v_{i} - u_{i}) + \]

\[ \sum_{i=1}^{n} \frac{\partial F}{\partial x_{i}} (\lambda u_{i} + (1 - \lambda) v_{i}, \lambda g (u_{i}) + (1 - \lambda) g (v_{i})) \cdot (\Delta (g) ([u_{i}, v_{i}]) + \]

\[ \sum_{i=1}^{n} \frac{\partial^{2} F}{\partial x_{i}^{2}} (\lambda u_{i} + (1 - \lambda) v_{i}, \lambda g (u_{i}) + (1 - \lambda) g (v_{i})) \cdot \left[ (2\lambda - 1) (v_{i} - u_{i}) (\Delta (g) ([u_{i}, v_{i}]) \right] + \]

\[ \sum_{i=1}^{n} \frac{\partial^{2} F}{\partial x_{i}^{2}} (\lambda u_{i} + (1 - \lambda) v_{i}, \lambda g (u_{i}) + (1 - \lambda) g (v_{i})) \cdot \left[ \frac{1}{2} (2\lambda - 1) (v_{i} - u_{i})^{2} \right] + \]

\[ \sum_{i=1}^{n} \frac{\partial^{2} F}{\partial x_{i}^{2}} (\lambda u_{i} + (1 - \lambda) v_{i}, \lambda g (u_{i}) + (1 - \lambda) g (v_{i})) \cdot \left[ \frac{1}{2} (2\lambda - 1) q ([u_{i}, v_{i}]) \right] + \]

\[ \sum_{i=1}^{n} (B_{1} ([u_{i}, v_{i}]) - B_{2} ([u_{i}, v_{i}]) |v_{i} - u_{i}|^\gamma = \]

\[ S_{1} + S_{2} + S_{3} + S_{4} + S_{5} + S_{6} \] (where \( S_{1}, S_{2}, \ldots, S_{6} \) denote the consecutive terms of the above sum, and the expressions \( S_{3}, S_{4}, S_{6} \) are negligible, as previously observed).
Moreover, since $\frac{\partial F}{\partial t}$ satisfies Taylor’s formula of order 1, we have:

$$S_1 = \sum_{i=1}^{n} \frac{\partial F}{\partial t} (u_i, g(u_i)) \cdot (v_i - u_i) + W,$$

where $W$ as usual is negligible.

The first summand, and hence $S_1$, tends to $\int_0^1 \frac{\partial F}{\partial t} (s, g(s)) ds$ as $n$ tends to $+\infty$ (the function $\frac{\partial F}{\partial t} \cdot (g(\cdot))$ is Hölder-continuous, so $\frac{\partial F}{\partial t} \cdot (g(\cdot))$ is $(RS)$-integrable w.r.t. $dt$). Now,

$$S_5 = \frac{1}{2} (2\lambda - 1) \sum_{i=1}^{n} \frac{\partial^2 F}{\partial x^2} (\lambda u_i + (1 - \lambda) v_i, \lambda g(u_i) + (1 - \lambda) g(v_i)) q([u_i, v_i])$$

tends to $\frac{1}{2} (2\lambda - 1) J$, as already observed. Hence it follows that

$$S_2 = \sum_{i=1}^{n} \frac{\partial F}{\partial x} (\lambda u_i + (1 - \lambda) v_i, \lambda g(u_i) + (1 - \lambda) g(v_i)) \cdot (\Delta(g)([u_i, v_i]))$$

(4)

converges, and its limit is the $\langle \lambda \rangle$-integral of $\frac{\partial F}{\partial x} \cdot (g(\cdot))$ with respect to $g$, that is the assertion 2). The formula in assertion 3) follows now easily.

Remark 4.5

I) Sometimes, the $\langle \lambda \rangle$-integral exists, without assumptions on the function $g$.

For example, let $F(t, x) = x^2$, and $\lambda = \frac{1}{2}$; so we are looking for the integral $\langle \frac{1}{2} \rangle \int_0^1 2g(s)dg(s)$. By definition, this integral is the limit of

$$2 \sum_{I \in D} [g(u_{i+1}) - g(u_i)] \left[ \frac{g(u_{i+1}) + g(u_i)}{2} \right]$$

(5)

as $\delta(D) \to 0$, where $D = \{[u_i, u_{i+1}] : i = 0, ..., n\}$. But the quantity (5) always coincides with $g^2(1) - g^2(0)$, so we obtain $\langle \frac{1}{2} \rangle \int_0^1 2g(s)dg(s) = g^2(1) - g^2(0)$.

II) On the other hand, once we have a function $g$ satisfying the hypotheses of Theorem 4.4, at least in particular spaces $R$ it is possible to find many other functions $\tilde{g}$ with the same properties. For example, let $R = L^0(X, \mathcal{B}, \mu)$ be as in the Remark 3.11 and assume that $g : [0, 1] \to R$ is the standard Brownian
Motion. Now, choose any $C^2$-function $\phi : \mathbb{R} \to \mathbb{R}$ and define, for every $t \in [0,1]$: $\tilde{g}(t)(x) = \phi(g(t)(x))$ for almost all $x \in X$. We shall see that $\Delta(\tilde{g})^2$ is integrable, and its integral function is Lipschitz. Indeed, if $0 \leq u < v \leq 1$, we get (a.e.)

$$ (\tilde{g}(v) - \tilde{g}(u))(x) = \phi(g(v)(x)) - \phi(g(u)(x)) = (g(v)(x) - g(u)(x)) \phi'(g(\tau)(x)), $$

where $\tau$ is a suitable point in $[u,v]$, depending also on $x$. So we have

$$ [\tilde{g}(v) - \tilde{g}(u)]^2(x) = [g(v)(x) - g(u)(x)]^2 (\phi'(g(\tau)(x)))^2. $$

(6)

(We remark here that $(\phi'(g(\tau)(x)))^2$ is an element of $R$, between the extrema of the function $t \mapsto f(t) = \phi'(g(t))^2$ in the interval $[u,v]$). Now consider the function $t \mapsto f(t) = \phi'(g(t))^2$, for all $t \in [0,1]$. This is a H"older-continuous function, hence Riemann-Stieltjes integrable with respect to $q = \Delta(g)^2$ by Theorem 3.10. From this, (6) and the Remark 3.12, we deduce that $(\Delta(\tilde{g})(I))^2$ is integrable, and its integral function coincides with the Riemann integral

$$ \int_I \phi'(g(t))^2 \, dt \text{ (which can be computed pathwise), and this is clearly a Lipschitz function of interval.} $$

References


