Modular filter convergence theorems for abstract sampling type operators

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We study the problem of approximating a real-valued function $f$ by considering sequences of general operators of sampling type, which include both discrete and integral ones. This approach gives a unitary treatment of various kinds of classical operators, such as Urysohn integral operators, in particular convolution integrals, and generalized sampling series.

\textbf{Keywords:} filter convergence; filter exhaustiveness; filter convergence in measure; integral operator; sampling series; modular; modular convergence; filter singularity

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1. Introduction

In this article we continue the investigation given in [1,2], by considering a very general class of nonlinear integral operators of ‘sampling’ type, generated by kernels satisfying some suitable singularity assumptions. Given a real-valued function $f$, we approximate it by means of operators of the form

\[(T_n f)(s) = \int_{H_n} K_n(s, t, f(t))d\mu_n(t), \quad n \in \mathbb{N}, \ s \in G,\]

where $G$ is a locally compact topological space endowed with its Borel $\sigma$-algebra $\mathcal{B}$, $(H_n)_n$ is a sequence of nonempty closed subsets of $\mathcal{B}$ such that $G = \bigcup_{n=1}^{\infty} H_n$. Here, for every $n \in \mathbb{N}$, $\mu_n$ is a regular measure defined on the Borel $\sigma$-algebra $\mathcal{B}_n$ of $H_n$ and $f$ belongs to the domain of the operators $T_n$.

These kinds of operators, introduced in [3,4], give a unifying approach for the treatment of both integral and discrete operators, simply by specifying the subspaces $H_n$ and the measures $\mu_n$. So, they represent a powerful tool for a general study of the approximation properties in various functional spaces, and include several classical discrete operators and integral operators of Urysohn type. Here, we study the

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approximation properties of $T_n$ in modular spaces, a very wide class of function spaces, which includes $L^p$, Orlicz and Sobolev–Orlicz spaces [5–7], and we determine a subspace $\Upsilon$ of the modular space involved, for which the modular convergence holds for every $f \in \Upsilon$.

Our main results extend in various directions to those obtained in [3,4]. Different from [4] we consider here a more general notion of singularity for the kernel $(K_n)_n$ and also an abstract version of the modular convergence, dealing with filter pointwise convergence and filter convergence in measure, and using the concept of filter exhaustiveness of a function sequence $(f_n)_n$, formulated in [1,2]. In this way, we also include the case of the statistical convergence (see also [8–11]).

As main application, a particular emphasis is given to the generalized sampling type operators (see [12–15]), associated with real-valued functions (signals) defined on the real line or in a multidimensional Euclidean space. This corresponds to the choice $G = \mathbb{R}$ or $\mathbb{R}^d$ endowed with the Lebesgue measure $\mu$, $H_n = \frac{1}{n}\mathbb{Z}$ or $H_n = \frac{1}{n}\mathbb{Z}^d$ with the counting measure $\mu_n$. These operators represent fundamental tools in signal processing, images and video reconstruction [16,17]. In this particular case, we show that the modular approximation property holds for functions belonging to a suitable subspace, which contains the space of all functions of bounded variation in the one-dimensional case. This holds for example in the classical case of the $L^p$ spaces for the generalized (linear) sampling series (see also [18]).

This fact shows the main difference in the theory involving integral operators of Urysohn type (including convolution integral operators), which correspond to the choice $H_n = G$ and $\mu_n = \mu$ for all $n \in \mathbb{N}$, where $\mu$ is a fixed measure on the Borel $\sigma$-algebra in $G$. In this case, the modular approximation property holds for all functions of the modular space involved [1,2].

We also give examples which show that the general results presented here are proper extensions of the related previous theorems given in [4].

2. Preliminaries

Let $G$ be a locally compact topological space, endowed with a uniform structure $\mathcal{U} \subset \mathcal{P}(G \times G)$ which generates the topology of $G$ (see also [19]). For every $U \in \mathcal{U}$ and $s \in G$, we set $U_s = \{t \in G : (s, t) \in U\}$. The family $\{U_s : U \in \mathcal{U}\}$ represents the class of the neighbourhoods of $s \in G$ in the uniform topology. For the given topological space $G$, let $\mathcal{B}$ be the $\sigma$-algebra of all Borel subsets of $G$, and $\mu : \mathcal{B} \rightarrow \mathbb{R}$ be a positive $\sigma$-finite regular measure. Let us denote by $L^0(G, \mathcal{B}, \mu) = L^0(G)$ the space of all real-valued $\mu$-measurable functions with identification up to sets of measure $\mu$ zero, by $L^\infty(G, \mathcal{B}, \mu) = L^\infty(G)$ the space of all essentially bounded measurable real-valued functions on $G$ and by $C_c(G)$ the subspace of $L^\infty(G)$ of all continuous functions with compact support on $G$.

A nonempty family $\mathcal{F}$ of subsets of $\mathbb{N}$ is called a filter of $\mathbb{N}$ iff $\emptyset \notin \mathcal{F}$, $A \cap B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and $B \supset A$ we get $B \in \mathcal{F}$.

As classical examples of filters we quote the filter $\mathcal{F}_{\text{cofin}}$ of all subsets of $\mathbb{N}$ whose complement is finite and the filter $\mathcal{F}_d$ associated with the statistical convergence, that is the class of all subsets of $\mathbb{N}$ whose asymptotic density is 1. The asymptotic density
of a set \( A \subset \mathbb{N} \) is defined as

\[
d(A) = \lim_{n \to \infty} \frac{\#(A \cap \{1, \ldots, n\})}{n}
\]

(provided that this limit exists), where the symbol \( \# \) denotes the cardinality of the set in brackets (see also [8,9]). Throughout this article, in (23), we will see some other filters, constructed analogously as above.

A filter \( \mathcal{F} \) is said to be free iff it contains \( \mathcal{F}_{\text{cofin}} \). An example of free filter is \( \mathcal{F}_d \).

A sequence \((x_n)_n \) in \( \mathbb{R} \) is said to be \( \mathcal{F} \)-bounded iff there exists an \( M > 0 \) such that \( \{n \in \mathbb{N} : |x_n| \leq M\} \in \mathcal{F} \).

A sequence \((x_n)_n \) in \( G \) is \( \mathcal{F} \)-convergent to \( x \in G \) (and we write \( x = (\mathcal{F}) \lim_n x_n \) iff \( \{n \in \mathbb{N} : (x_n, x) \in U\} \in \mathcal{F} \) whenever \( U \in \mathcal{U} \).

If \( \mathcal{B} = (x_n)_n \) is a sequence in \( \mathbb{R} \), and

\[
A_{\mathcal{B}} = \{a \in \mathbb{R} : \{n \in \mathbb{N} : x_n \geq a\} \notin \mathcal{F}\}, \quad B_{\mathcal{B}} = \{b \in \mathbb{R} : \{n \in \mathbb{N} : x_n \leq b\} \notin \mathcal{F}\},
\]

then the \( \mathcal{F} \)-limit superior and the \( \mathcal{F} \)-limit inferior of \((x_n)_n\) are defined by

\[
(\mathcal{F}) \limsup_n x_n = \begin{cases} 
\sup B_{\mathcal{B}}, & \text{if } B_{\mathcal{B}} \neq \emptyset, \\
-\infty, & \text{if } B_{\mathcal{B}} = \emptyset,
\end{cases}
(\mathcal{F}) \liminf_n x_n = \begin{cases} 
\inf A_{\mathcal{B}}, & \text{if } A_{\mathcal{B}} \neq \emptyset, \\
+\infty, & \text{if } A_{\mathcal{B}} = \emptyset,
\end{cases}
\]

respectively (see also [20]).

A sequence \( f_n : G \to \mathbb{R}, \ n \in \mathbb{N}, \) is called \( \mathcal{F} \)-exhaustive at \( s \in G \) iff for every \( \varepsilon > 0 \) there exist a neighbourhood \( U_s \) of \( s \) and \( A \in \mathcal{F} \) with \( |f_n(z) - f_n(s)| \leq \varepsilon, \) whenever \( n \in A \) and \( z \in U_s \). A sequence \( f_n : G \to \mathbb{R}, \ n \in \mathbb{N}, \) is exhaustive at \( s \in G \) iff it is \( \mathcal{F}_{\text{cofin}} \)-exhaustive at \( s \).

A sequence \( f_n : G \to \mathbb{R}, \ n \in \mathbb{N}, \) is said to be \( \mathcal{F} \)-convergent uniformly to \( f \) on \( G \) iff

\[
(\mathcal{F}) \lim_n \left[ \sup_{t \in G} |f_n(t) - f(t)| \right] = 0,
\]

and \( \mathcal{F} \)-convergent in measure to \( f \) iff

\[
(\mathcal{F}) \lim_n \mu(\{t \in G : |f_n(t) - f(t)| > \varepsilon\}) = 0,
\]

for all \( \varepsilon > 0 \).

### 3. The structural hypotheses on the operators

Let \( (H_n)_n \) be a sequence of nonempty closed sets of \( B \) with \( \bigcup_{n=1}^{\infty} H_n = G \), and for each \( n \in \mathbb{N} \) let \( \mu_n \) be a regular measure defined on the Borel \( \sigma \)-algebra \( B_n \) generated by the family \( \{A \cap H_n : A \text{ open subset of } G\} \). For all \( n \in \mathbb{N} \) let us denote by \( \mathcal{L}_n \) the set of all measurable non-negative functions \( L : G \times H_n \to \mathbb{R} \) such that the sections \( L(\cdot, t) \) and \( L(s, \cdot) \) belong to \( L^1(G), L^1(H_n) \) for all \( t, s \in G \), where \( L^1(G) \) and \( L^1(H_n) \) are the spaces of Lebesgue integrable functions with respect to \( \mu \) and \( \mu_n \), respectively.

Let \( \mathbb{R}_0^+ \) be the set of all non-negative real numbers and \( \Psi \) be the class of all functions \( \psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) such that \( \psi \) is continuous, nondecreasing, \( \psi(0) = 0 \) and \( \psi(u) > 0 \) for all \( u > 0 \). Let \( \Xi = (\psi_n)_n \subset \Psi \) be a sequence, \( \mathcal{F} \)-exhaustive at 0 and such that for every \( u > 0 \) the sequence \((\psi_n(u))_n\) is \( \mathcal{F} \)-bounded. Let us denote by \( \mathcal{K}_{\Xi} \) the
class of all sequences of functions $K_n$: $G \times H_n \times \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, satisfying the following properties:

- $K_n(\cdot, \cdot, u)$ is measurable on $G \times H_n$ for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$;
- $K_n(s, t, 0) = 0$ for every $n \in \mathbb{N}$, $s \in G$ and $t \in H_n$;
- for all $n \in \mathbb{N}$ there are $L_n \in \mathcal{L}$ and $\psi_n \in \Psi$, with

$$|K_n(s, t, u) - K_n(s, t, v)| \leq L_n(s, t)\psi_n(|u - v|)$$

whenever $s \in G$, $t \in H_n$ and $u, v \in \mathbb{R}$.

Let $\mathbb{K} = (K_n)_n \in \mathcal{K}$ and $T = (T_n)_n$ be a sequence of operators defined by

$$(T_n f)(s) = \int_{H_n} K_n(s, t, f(t))d\mu_n(t), \quad s \in G,$$

where $f \in \text{Dom } T = \bigcap_{n=1}^{\infty} \text{Dom } T_n$, and for each $n \in \mathbb{N}$, $\text{Dom } T_n$ is the subset of $L^0(G)$ on which $T_n f$ is well-defined as a $\mu$-measurable function of $s \in G$.

We now give the notions of singularity and of regular sequences with respect to filters (see also [1, 2, 4]).

**Definitions 3.1**

(a) Let $\mathbb{K} \in \mathcal{K}$. We say that $\mathbb{K}$ is $\mathcal{F}$-singular iff

(3.1.1) there is a $D_1 > 0$ with

$$\Lambda = \left\{ n \in \mathbb{N} : \int_{H_n} L_n(s, t)d\mu_n(t) \leq D_1 \text{ for all } s \in G \right\} \in \mathcal{F};$$

(3.1.2) for every $s \in G$ and for each neighbourhood $U_s \subset G$ we get

$$(\mathcal{F}) \lim_n \int_{H_n \setminus U_s} L_n(s, t)d\mu_n(t) = 0;$$

(3.1.3) for every $s \in G$ and $u \in \mathbb{R}$ we have

$$(\mathcal{F}) \lim_n \int_{H_n} K_n(s, t, u)d\mu_n(t) = u.$$  

(b) We say that $\mathbb{K}$ is strongly $\mathcal{F}$-singular iff it fulfils (3.1.1) and

(3.1.4) for any $s \in G$ and for every neighbourhood $U_s \subset G$ there is a neighbourhood $Z_s \subset G$, with

$$(\mathcal{F}) \lim_n \left[ \sup_{z \in Z_s} \int_{H_n \setminus U_s} L_n(z, t)d\mu_n(t) \right] = 0;$$

(3.1.5) for any $s \in G$ and $u \in \mathbb{R}$ there are two neighbourhoods $U_s$ of $s$ and $W$ of $u$, respectively, with

$$(\mathcal{F}) \lim_n \left[ \sup_{z \in U_s, v \in W} \left( \int_{H_n} K_n(z, t, v)d\mu_n(t) - v \right) \right] = 0.$$
(c) We say that $K$ is $F$-singular in measure iff it satisfies (3.1.1) and the following conditions:

(3.1.6) for each $\varepsilon, \tau > 0$ and $U \subseteq \mathcal{U}$ there exists $\Lambda_0 \subseteq \mathcal{F}$ such that for every $n \in \Lambda_0$ there is $A_n^1 \subseteq B$ with $\mu(A_n^1) \leq \tau$ and

$$\int_{H_n \setminus U_n} L_n(s, t) \, d\mu_n(t) \leq \varepsilon,$$

for all $s \in G \setminus A_n^1$.

(3.1.7) for every compact set $C \subseteq \mathbb{R} \setminus \{0\}$ and for any $\varepsilon, \tau > 0$ there is $\Lambda_1 \subseteq \mathcal{F}$ such that for each $n \in \Lambda_1$ there exists $B_n^1 \subseteq B$ with $\mu(B_n^1) \leq \tau$ and

$$\left| \int_{H_n} K_n(s, t, u) \, d\mu_n(t) - u \right| \leq \varepsilon,$$

for every $s \in G \setminus B_n^1$ and $u \in C$.

(d) We say that $K$ is $F$-uniformly singular iff it fulfils (3.1.1), and

(3.1.8) for every $U \subseteq \mathcal{U}$ we get

$$(F) \lim_n \sup_{s \in G} \int_{H_n \setminus U_n} L_n(s, t) \, d\mu_n(t) = 0;$$

(3.1.9) for every compact set $C \subseteq \mathbb{R} \setminus \{0\}$,

$$(F) \lim_n \sup_{s \in G, u \in C} \int_{H_n} K_n(s, t, u) \, d\mu_n(t) - u = 0.$$

Let $\nu_n : G \times B_n \to \mathbb{R}^+_0$ be such that for all $n \in \mathbb{N}$, $\nu_n(\cdot, A)$ is measurable for any $A \subseteq B_n$ and $\nu_n(s, \cdot)$ is a measure on $B_n$ for every $s \in G$. We say that $(\nu_n)_n$ is an $F$-regular sequence iff it satisfies the following properties:

- $\nu_n^s(\cdot) = \nu_n(s, \cdot)$ is absolutely continuous with respect to $\mu_n$ for all $n \in \mathbb{N}$ and $s \in G$.
- There is a $D_1 > 0$ such that, if $b_n(s) = \nu_n(s, H_n)$ for all $n \in \mathbb{N}$ and $s \in G$, then

$$\left\{ n \in \mathbb{N} : 0 < b_n(s) \leq D_1 \text{ for all } s \in G \right\} \in \mathcal{F}.$$

- If $\xi_n(s, t) = \frac{dn}{d\mu_n}$ for all $n \in \mathbb{N}$, $s \in G$, $t \in H_n$, then $\xi_n$ is a measurable function on $G \times H_n$ and there exists a sequence $(z_n)_n$ in $\mathbb{R}^+$ with $z_n \leq D_1$ for all $n \in \mathbb{N}$ and such that the set

$$\left\{ n \in \mathbb{N} : \int_G |\xi_n(s, t)| \, d\mu(s) \leq z_n \text{ for all } t \in H_n \right\} \in \mathcal{F}.$$

Remark 3.2 If $K = (K_n)_n \in \mathcal{C}_2$ is $F$-singular, then it is not difficult to see that the sequence $\mu_{\mathbb{L}, n}$, $n \in \mathbb{N}$, defined by setting

$$\mu_{\mathbb{L}, n}(A, s) = \int_A L_n(s, t) \, d\mu_n(t), \text{ for all } n \in \mathbb{N}, s \in G, A \subseteq B_n,$$

is $F$-regular (see also [4,21]). In [4], some examples of regular sequences are given, in the classical setting.
4. The modular spaces

We now give the fundamental properties of modular spaces (see, e.g. [5–7]).

A modular \( \rho \) is a functional \( \rho : L^0(G) \to \mathbb{R}_+^+ \), satisfying the following properties:

- \( \rho(f) = 0 \iff f = 0 \) \( \mu \)-almost everywhere on \( G \);
- \( \rho(-f) = \rho(f) \) for every \( f \in L^0(G) \);
- \( \rho(a f + b g) \leq a \rho(f) + b \rho(g) \) whenever \( f, g \in L^0(G) \) and \( a \geq 0, b \geq 0 \) with \( a + b = 1 \);
- \( \rho(F(t, \cdot)) \) is a measurable function of \( t \in G \) for each measurable function \( F: G \times G \to \mathbb{R}_+^+ \).

A modular \( \rho \) is convex iff \( \rho(a f + b g) \leq a \rho(f) + b \rho(g) \) for all \( f, g \in L^0(G) \) and for every \( a, b \geq 0 \) with \( a + b = 1 \).

Put

\[
L^\rho(G) = \left\{ f \in L^0(G) : \lim_{\lambda \to 0^+} \rho(\lambda f) = 0 \right\},
\]

\[
E^\rho(G) = \{ f \in L^\rho(G) : \rho(\lambda f) < +\infty \text{ for all } \lambda > 0 \}.
\]

The spaces \( L^\rho(G) \) and \( E^\rho(G) \) are the modular space associated with \( \rho \) and the space of the finite elements of \( L^\rho(G) \), respectively.

A modular \( \rho \) is monotone iff \( \rho(f) \leq \rho(g) \) for any \( f, g \in L^0(G) \) with \( |f| \leq |g| \).

We say that \( \rho \) is finite iff \( \chi_A \) (the characteristic function associated with \( A \)) belongs to \( L^\rho(G) \) whenever \( A \in \mathcal{B} \) with \( \mu(A) < +\infty \).

A finite modular \( \rho \) is absolutely finite iff for any \( \varepsilon, \lambda > 0 \) there is a \( \delta > 0 \) such that \( \rho(\lambda \chi_B) < \varepsilon \) for every \( B \in \mathcal{B} \) with \( \mu(B) < \delta \).

We say that \( \rho \) is strongly finite iff \( \chi_A \in E^\rho(G) \) for all \( A \in \mathcal{B} \) with \( \mu(A) < +\infty \).

A modular \( \rho \) is absolutely continuous iff there is an \( a > 0 \) with the property that, for all \( f \in L^0(G) \) with \( \rho(f) < +\infty \),

- for every \( \varepsilon > 0 \) there is a set \( A \in \mathcal{B} \) with \( \mu(A) < +\infty \) and \( \rho(af) \chi_{G \setminus A} \leq \varepsilon \),
- for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) with \( \rho(af) \chi_B \leq \varepsilon \) whenever \( B \in \mathcal{B} \) with \( \mu(B) < \delta \).

Given a modular \( \rho \) and a free filter \( \mathcal{F} \), we say that a sequence \( f_n : G \to \mathbb{R} \), \( n \in \mathbb{N} \), is \( \rho-\mathcal{F}\)-equi-absolutely continuous iff there is an \( a > 0 \) satisfying the following conditions:

- for every \( \varepsilon > 0 \) there are \( A \in \mathcal{B} \) with \( \mu(A) < +\infty \) and \( \Lambda_0 \in \mathcal{F} \) with \( \rho(af_n) \chi_{G \setminus A} \leq \varepsilon \) whenever \( n \in \Lambda_0 \);
- for every \( \varepsilon > 0 \) there are \( \delta > 0 \) and \( \Lambda \in \mathcal{F} \) with \( \rho(af_n) \chi_B \leq \varepsilon \) for every \( n \in \Lambda \) and whenever \( B \in \mathcal{B} \) with \( \mu(B) < \delta \).

We describe a fundamental example of modular space. Let \( \Phi \) (resp. \( \tilde{\Phi} \)) be the set of all continuous nondecreasing (resp. convex) functions \( \varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) with \( \varphi(0) = 0 \), \( \varphi(u) > 0 \) for any \( u > 0 \) and \( \lim_{u \to +\infty} \varphi(u) = +\infty \).

For all \( \varphi \in \Phi \) (resp. \( \tilde{\Phi} \)), set

\[
\rho^\varphi(f) = \int_G \varphi(|f(s)|) d\mu(s), \quad f \in L^0(G).
\]
The functional $\rho^\varphi$ is a (resp. convex) modular on $L^0(G)$, satisfying the given properties of the modulars (see also [7]). The subspace $L^\rho(G) = \{ f \in L^0(G) : \rho^\varphi(\lambda f) < +\infty \text{ for some } \lambda > 0 \}$ is the Orlicz space generated by $\varphi$ (see also [7,22]).

A sequence $(f_n)_n$ of functions in $L^\rho(G)$ is $\mathcal{F}$-modularly convergent to $f \in L^\rho(G)$ iff there is a $\lambda > 0$ with

$$\lim_n \rho[\lambda(f_n - f)] = 0.$$ 

The sequence $(f_n)_n$ in $L^\rho(G)$ is $\mathcal{F}$-strongly convergent to $f \in L^\rho(G)$ iff

$$\lim_n \rho[\lambda(f_n - f)] = 0$$

for any $\lambda > 0$. Note that the $\mathcal{F}_{\text{cofin}}$-modular (resp. $\mathcal{F}_{\text{cofin}}$-strong) convergence is equivalent to usual modular (resp. strong) convergence (see also [5,7]).

Given a subset $A \subset L^\rho(G)$ and $f \in L^\rho(G)$, we say that $f$ is in the modular closure of $A$ (and we write $f \in \overline{A}$) iff there is a sequence $(f_n)_n$ in $A$, modularly convergent in the usual sense.

The following result will be useful in the sequel (see also [23, Theorem 1]).

**Proposition 4.1** If $\rho$ is a monotone, absolutely finite and absolutely continuous modular on $L^0(G)$, then $\overline{C}(\mathcal{F}) = L^\rho(G)$.

For $n \in \mathbb{N}$, let $\rho_n, \eta_n$ be modulars on $L^0(H_n, B_n, \mu_n) = L^\rho(H_n)$. By $L^\rho_n(H_n)$ and $L^\rho(H_n)$ we denote the spaces of all functions $f \in L^0(G)$, such that the restriction $f|_{H_n}$ is an element of the modular spaces generated by $\rho_n$ and $\eta_n$, respectively.

Let $\mathcal{E} = (\psi_n)_n \subset \Psi$ be the family given in condition (1). We say that the triple $(\rho_n, \psi_n, \eta_n)$ is $\mathcal{F}$-properly directed iff there is a sequence $(c_n)_n$ in $\mathbb{R}$ with $\lim_n c_n = 0$ and such that for all $\lambda \in (0, 1)$ there exists $C_\lambda \in (0, 1)$ with $\rho_n(C_\lambda \psi_n \circ g) \leq \eta_n(\lambda g) + c_n$ whenever $n \in \mathbb{N}$ and $g \in L^0(G)$, $g \geq 0$.

The family $(L_n(\cdot, t))_{n \in \mathbb{N}, t \in H_n}$ satisfies property $(\ast)$ iff for every compact set $C \subset G$ there exist a compact set $B \subset G$ and $\beta, M > 0$ with

$$\left\{ n \in \mathbb{N} : \int_{G \setminus B} L_n(s, t) \, d\mu(s) \leq \frac{M}{n^\beta} \text{ for all } t \in H_n \cap C \right\} \in \mathcal{F}$$

and

$$\lim_n \frac{\mu_n(H_n \cap C)}{n^\beta} = 0.$$ 

The family $(L_n(\cdot, t))_{n \in \mathbb{N}, t \in H_n}$ is said to fulfill property $\rho(\ast)$ (with respect to the modular $\rho$) iff for every $\varepsilon, \lambda > 0$ and for each compact set $C \subset G$ there exists a compact set $B \subset G$ such that

$$\Lambda_* := \left\{ n \in \mathbb{N} : \rho \left[ \lambda \int_{H_n \cap C} L_n(\cdot, t) \, d\mu_n(t) \chi_{G \setminus B}(\cdot) \right] \leq \varepsilon \right\} \in \mathcal{F}.$$ 

**Remark 4.2** Note that, when the modular $\rho = \rho^\varphi$ is as in (4) and $\varphi$ is convex, we get that property $(\ast)$ implies $\rho(\ast)$.
Indeed, proceeding analogously as in [2, Remark 3.8], let $\Lambda \in \mathcal{F}, \ D_1 > 0$ be as in (3.1.1) and $n \in \Lambda$. Pick arbitrarily $\varepsilon, \lambda > 0$ and a compact set $C \subset G$.

Note that $\int_{H_n \cap C} \frac{L_n(s,t)}{D_1} \ d\mu_n(t) \leq 1$, and hence, by the Jensen inequality, we have

$$
\varphi \left( \lambda \int_{H_n \cap C} L_n(s,t) \ d\mu_n(t) \right) = \varphi \left( \lambda, D_1 \int_{H_n \cap C} \frac{L_n(s,t)}{D_1} \ d\mu_n(t) \right) \leq \frac{1}{D_1} \int_{H_n \cap C} \varphi(\lambda, D_1) L_n(s,t) \ d\mu_n(t). \quad (5)
$$

Set $\tilde{\Lambda} = \Lambda \cap \Lambda_\varepsilon$. By virtue of the Fubini theorem, from property (*) and (5) we obtain the existence of a compact set $B \subset G$ and $\beta, \ M > 0$ with

$$
\rho^\beta \left[ \lambda, \int_{H_n \cap C} L_n(s,t) \ d\mu_n(t) \chi_{G \setminus B}(\cdot) \right] = \int_{G \setminus B} \varphi \left( \lambda, \int_{H_n \cap C} L_n(s,t) \ d\mu_n(t) \right) \ d\mu(s) \leq \frac{1}{D_1} \int_{G \setminus B} \left[ \int_{H_n \cap C} \varphi(\lambda, D_1) L_n(s,t) \ d\mu_n(t) \right] \ d\mu(s) \leq \frac{1}{D_1} \int_{H_n \cap C} \left[ \int_{G \setminus B} L_n(s,t) \ d\mu(s) \right] \ d\mu_n(t) \leq \frac{1}{D_1} \int_{H_n \cap C} \frac{M \mu_n(H_n \cap C)}{n^\beta},
$$

for every $n \in \tilde{\Lambda}$. So we get property $\rho^\beta(\cdot)$.

An $\mathcal{F}$-regular sequence $(v_n)_n$ is $\mathcal{F}$-compatible with the pair $(\rho, \rho_n)$ iff there are two positive real numbers $D', \ M$ and a sequence $(b_n)_n$ of non-negative real numbers with $(\mathcal{F})\lim_n b_n = 0$ and

$$
\rho \left[ \int_{H_n} g(t) v_n(t) \right] \leq M z_n \rho_n(D'g) + b_n, \quad (6)
$$

for all $g \in L^0(G), \ g \geq 0$, where $(z_n)_n$ is as in the definition of $\mathcal{F}$-regularity.

Remark 4.3 If $\varphi \in \Phi$, then, by proceeding analogously as in [4, Example 5 (a)], it is possible to check that every $\mathcal{F}$-regular sequence is $\mathcal{F}$-compatible with the pair $(\rho, \rho_n)$, where $\rho, \rho_n$ are the modulars which generate the Orlicz spaces $L^\varphi(G)$ and $L^\varphi(H_n)$, respectively.

Some other examples of compatibility, in the classical setting, can be found in [4].

Let $\eta$ be a modular on $L^0(G)$, and $(\eta_n)_n$ be a sequence of modulars on $L^0(H_n)$. By proceeding analogously as in [3,4], let us denote by $\gamma_\eta$ the set of all functions $f \in L^\eta(G)$ with the property that there are a $P > 0$ and a bounded sequence $(\gamma_n)_n$ of positive real numbers with

$$(\mathcal{F}) \lim \sup_n \gamma_n \eta_n(\lambda f) \leq P \eta(\lambda f) \ \text{for every} \ \lambda > 0. \quad (7)$$

We assume that $\gamma_\eta$ contains a subspace $G \subset C_c(G)$, and denote by $\overline{G}_\eta$ the modular closure of $G$ in the space $L^\eta(G)$. 
Observe that this assumption allows us to consider several types of abstract integral operators, in which the $H_n$'s are proper subspaces of $G$: this is the case of the discrete operators, for example sampling, Bernstein, Szász-Mirak'jan and Baskakov operators (see, e.g. [4]). Note that even the nonlinear Urysohn operators [21] can be viewed as particular cases of these kinds of operators, setting $H_n = G$ for each $n \in \mathbb{N}$. In this case we have $\Upsilon_n = L^n(G)$.

5. The main results
In this section we prove general convergence results for our operators $T_n$.

5.1. Filter pointwise, in measure and uniform convergence
We begin with the following technical result, whose proof is analogous to that of [2, Theorem 4.1].

**Theorem 5.1** Let $\mathcal{F}$ be a free filter, and assume that $K \in \mathcal{K}_\Sigma$ is strongly $\mathcal{F}$-singular. Then, for every $s \in G$ and $f \in C(G)$, the sequence $(T_n f)_n$ is $\mathcal{F}$-exhaustive at $s$.

The following result concerns filter in measure and uniform convergence of the sequence $(T_n f)_n$.

**Theorem 5.2** Let $\mathcal{F}$ be any free filter, and $f : G \to \mathbb{R}$ be uniformly continuous and bounded on $G$. If $K \in \mathcal{K}_\Sigma$ is $\mathcal{F}$-singular in measure (resp. $\mathcal{F}$-uniformly singular), then the sequence $(T_n f)_n$ $\mathcal{F}$-converges in measure (resp. uniformly) to $f$.

**Proof** Let $f : G \to \mathbb{R}$ be as in the hypotheses. For every $s \in G$ and $n \in \mathbb{N}$ we have:

$$|(T_n f)(s) - f(s)| \leq \int_{H_n} |K_n(s, t, f(t)) - K_n(s, t, f(s))| \, d\mu_n(t)$$

$$+ \left| \int_{H_n} K_n(s, t, f(s)) \, d\mu_n(t) - f(s) \right| = I_1 + I_2.$$

Let $D_1 > 0$ be as in (3.1.1), and fix arbitrarily $\varepsilon > 0$. By $\mathcal{F}$-exhaustiveness at $0$ of $(\psi_n)_n$, there are $\sigma > 0$, $\bar{K} \in \mathbb{N}$ and a set $\Pi \in \mathcal{F}$ such that

$$D_1 \cdot \psi_n(1/k) + 1/k \leq \varepsilon \quad \text{and} \quad \psi_n(u) \leq \varepsilon,$$

whenever $k \geq \bar{K}$, $|u| \leq \sigma$ and $n \in \Pi$. Since $f$ is uniformly continuous, there is a set $U \in \mathcal{U}$ with $|f(t) - f(s)| < \sigma$ whenever $s \in G$ and $t \in U$. Moreover there exists a set $\Lambda \in \mathcal{F}$ such that, for all $n \in \Lambda \cap \Pi$ and $s \in G$, we get

$$I_1 \leq \int_{H_n} L_n(s, t)\psi_n(|f(t) - f(s)|) \, d\mu_n(t)$$

$$= \int_{H_n \cap U_s} L_n(s, t)\psi_n(|f(t) - f(s)|) \, d\mu_n(t) + \int_{H_n \setminus U_s} L_n(s, t)\psi_n(|f(t) - f(s)|) \, d\mu_n(t)$$

$$\leq \psi_n(\sigma) \cdot D_1 + \psi_n(2\|f\|_\infty) \cdot \int_{H_n \setminus U_s} L_n(s, t) \, d\mu_n(t)$$

$$\leq \varepsilon \cdot D_1 + \psi_n(2\|f\|_\infty) \cdot \int_{H_n \setminus U_s} L_n(s, t) \, d\mu_n(t).$$

(8)
Pick now \( \tau > 0 \). By (3.1.6) there is \( \Lambda_0 \in \mathcal{F} \) such that for every \( n \in \Lambda_0 \) there exists a set \( A^n_k \in \mathcal{B} \) with \( \mu(A^n_k) \leq \tau \) and

\[
\int_{H_n \Delta U_s} L_n(s, t) \, d\mu_n(t) \leq \varepsilon
\]

whenever \( s \in G \setminus A^n_k \). Let \( \Lambda^* = \Lambda_0 \cap \Pi \cap \Lambda \). Then, \( \Lambda^* \in \mathcal{F} \) and

\[
I_1 \leq \varepsilon D_1 + \varepsilon \psi_n(2\|f\|_\infty),
\]

for all \( n \in \Lambda^* \). Since, by hypothesis, the sequence \( (\psi_n(2\|f\|_\infty))_n \) is \( \mathcal{F} \)-bounded, there exist a positive real number \( D' \) and a set \( Q \in \mathcal{F} \), depending only on \( f \), such that \( \psi_n(2\|f\|_\infty) \leq D' \) for each \( n \in Q \). Thus \( I_1 \mathcal{F} \)-converges in measure to 0.

We now estimate the term \( I_2 \). As \( f \) is bounded, there is \( k_0 \in \mathbb{N} \) such that \( |f(s)| \leq k_0 \) for each \( s \in G \). Let \( k^* = \max(k_0, \overline{k}) \), where \( \overline{k} \) is as above. For all \( k \in \mathbb{N} \), set

\[
C_k := \{ s \in G : 0 \leq |f(s)| \leq 1/k \}.
\]

Since \( K_n(s, t, 0) = 0 \) for each \( n \in \mathbb{N} \) and \( s, t \in G \), then for all \( n \in \mathbb{N} \) and \( s \in G \) we have

\[
I_2 \leq \left| \int_{H_n} K_n(s, t, f(s)\chi_{C_k}(s)) \, d\mu_n(t) - f(s)\chi_{C_k}(s) \right| + \left| \int_{H_n} K_n(s, t, f(s)\chi_{G\setminus C_k}(s)) \, d\mu_n(t) - f(s)\chi_{G\setminus C_k}(s) \right| = J_1 + J_2.
\]

For each \( k \in \mathbb{N} \), set \( E_k = \{ u \in \mathbb{R} : 1/k \leq |u| \leq k \} \). Note that \( |f(s)|\chi_{G\setminus C_k}(s) \in E_{k^*} \) for all \( s \in G \).

Thanks to (3.1.7), in correspondence with the compact set \( E_{k^*} \subset \mathbb{R} \setminus \{0\} \), we get that \( J_2 \mathcal{F} \)-converges in measure to 0.

We now estimate the term \( J_1 \). By (3.1.1) and (8), for all \( s \in G \) we have

\[
J_1 \leq \int_{H_n} L_n(s, t) \cdot \psi_n(|f(s)|\chi_{C_k}(s)) \, d\mu_n(t) + |f(s)|\chi_{C_k}(s) \leq D_1 \psi_n(1/k^*) + 1/k^* \leq \varepsilon,
\]

and so the sequence \( (T_n f)_n \mathcal{F} \)-converges in measure with respect to \( \mu \).

For what concerns the \( \mathcal{F} \)-uniform convergence, for the term \( I_1 \), by (3.1.8) the inequality in (10) is satisfied with \( A^n_k = \emptyset \), and so \( I_1 \mathcal{F} \)-converges uniformly to 0 as \( n \) tends to \( +\infty \). As to \( I_2 \), assuming (3.1.9), \( J_1 \) and \( J_2 \mathcal{F} \)-converge uniformly to 0. Thus the theorem is completely proved.

The next result gives filter pointwise convergence at the continuity points of \( f \).

**Theorem 5.3** Let \( \mathcal{F} \) be a free filter, \( f \in L^\infty(G) \) and \( \mathbb{K} \in \mathcal{K}_2 \) be \( \mathcal{F} \)-singular. Then we get \( (\mathcal{F})\lim_n T_n(f(s)) = f(s) \), at every continuity point \( s \in G \) of \( f \).

**Proof** Let \( s \in G \) be a fixed continuity point of \( f \). For every \( n \in \mathbb{N} \) we have:

\[
|(T_n f)(s) - f(s)| \leq \int_{H_n} |K_n(s, t, f(t)) - K_n(s, t, f(s))| \, d\mu_n(t) + \int_{H_n} K_n(s, t, f(s)) \, d\mu_n(t) - f(s) \bigg| = I_1 + I_2.
\]

By virtue of condition (3.1.3) of \( \mathcal{F} \)-singularity, we get \( (\mathcal{F})\lim_n I_2 = 0 \). So, in order to prove the theorem, it is enough to estimate the quantity \( I_1 \).
Fix arbitrarily $\varepsilon > 0$. By $\mathcal{F}$-exhaustiveness at $0$ of $(\psi_n)_n$ there are a $\sigma > 0$ and a set $\Pi \in \mathcal{F}$ with $\psi_n(u) \leq \varepsilon$ whenever $|u| \leq \sigma$ and $n \in \Pi$.

By continuity of $f$ at the point $s$, there exists a neighbourhood $U_s$ of $s$ such that $|f(t) - f(s)| < \sigma$ whenever $t \in U_s$. From (1), (3.1.1) and $\mathcal{F}$-exhaustiveness at $0$ of $(\psi_n)_n$, proceeding analogously as in (9), we obtain the existence of a set $\Lambda \in \mathcal{F}$ such that for all $n \in \Lambda \cap \Pi$ we get

$$I_1 \leq \varepsilon D_1 + \psi_n(2\|f\|_{\infty}) \cdot \int_{H \setminus U_s} L_n(s, t) \, d\mu_n(t).$$

By (3.1.2) there is $\Pi_s \in \mathcal{F}$ (depending on $s$) such that for every $n \in \Pi_s$ we have

$$\int_{H \setminus U_s} L_n(s, t) \, d\mu_n(t) \leq \varepsilon.$$

Thus $\Pi_s \cap \Pi \in \mathcal{F}$. Let $\Lambda_s = \Pi_s \cap \Pi \cap \Lambda$. We get

$$I_1 \leq \varepsilon D_1 + \psi_n(2\|f\|_{\infty})$$

for all $n \in \Lambda_s$. As the sequence $(\psi_n(2\|f\|_{\infty}))_n$ is $\mathcal{F}$-bounded, there exist a positive real number $D'$ and a set $Q \in \mathcal{F}$, depending only on $f$, such that $\psi_n(2\|f\|_{\infty}) \leq D'$ whenever $n \in Q$. Thus $(\mathcal{F})\lim_n I_1 = 1$, and so $(\mathcal{F})\lim_n T_n f(s) = f(s)$.

5.2. An extension of the Vitali theorem

We state a modular version of the Vitali theorem, which extends [5, Theorem 2.2.1].

**Theorem 5.4** Let $\mathcal{F}$ be any free filter, and $\rho$ be a monotone and finite modular on $L^0(G)$. Let $(f_n)_n$ be a sequence of functions in $L^0(G)$, $\mathcal{F}$-convergent in measure to $0$ and $\rho$-$\mathcal{F}$-equi-absolutely continuous.

Then there exists a positive real number $a$ with $(\mathcal{F})\lim_n \rho(a f_n) = 0$.

**Proof** Fix arbitrarily $\varepsilon > 0$, and in correspondence with $\varepsilon$ choose $a > 0$, $\delta > 0$ and $A \in \mathcal{B}$ with $\mu(A) < +\infty$ according to $\rho$-$\mathcal{F}$-equi-absolute continuity, with $a$ and $\varepsilon$ replaced by $3a$ and $\varepsilon/3$, respectively. Since $\rho$ is finite, we have $\chi_A \in L^0(G)$. So there is $\lambda_\varepsilon \in (0, 1)$ with

$$\rho(\lambda_\varepsilon \chi_A) \leq \varepsilon /3.$$

By $\mathcal{F}$-convergence in measure of $(f_n)_n$ to $0$, for all $\sigma > 0$ we have

$$(\mathcal{F})\lim_n \mu(\{t \in G : |f_n(t)| > \sigma\}) = 0.$$ 

So, in correspondence with $\sigma = \frac{\lambda_\varepsilon}{3a}$ and $\delta$ there is a set $E \in \mathcal{F}$ such that, for all $n \in E$, $\mu(A^n_\varepsilon) < \delta$, where

$$A^n_\varepsilon = \left\{ t \in A : |f_n(t)| > \frac{\lambda_\varepsilon}{3a} \right\}.$$

Thus we get that

$$3a|f_n(t)| \leq \lambda_\varepsilon \quad \text{whenever } t \in A \setminus A^n_\varepsilon.$$

$(\mathcal{F})\lim_n I_1 = 1$.
For all $n \in \mathbb{N}$ and $t \in G$ we have:

$$|f_n(t)| = \frac{1}{3} (3|f_n(t)| \chi_{G \setminus A}(t) + 3|f_n(t)| \chi_{A \setminus A'}(t) + 3|f_n(t)| \chi_{A'}(t)).$$

By the properties of modulars, for all $n \in E$ we get:

$$\rho(af_n) \leq \rho(3af_n \chi_{G \setminus A}) + \rho(3af_n \chi_{A \setminus A'}) + \rho(3af_n \chi_{A'}) = I_1 + I_2 + I_3.$$

By $\rho$-$\mathcal{F}$-equi-absolute continuity there exist $\Lambda_0$, $\Lambda \in \mathcal{F}$ with $I_1 \leq \varepsilon/3$ for all $n \in \Lambda_0$ and $I_3 \leq \varepsilon/3$ for all $n \in \Lambda$. Moreover, from (11), (12) and monotonicity of $\rho$, we get $I_2 \leq \rho(\lambda \chi_{A}) \leq \varepsilon/3$. Thus for every $n \in \Lambda_0 \cap \Lambda$ we get $\rho(a f_n) \leq \varepsilon$.

So we obtain $(\mathcal{F})\lim_n \rho(af_n) = 0$. 

We now recall the following result [2, Theorem 4.5].

**Theorem 5.5** Let $\mathcal{F}$ be any free filter, $\rho$ be a monotone and finite modular on $L^0(G)$, $f_n: G \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of functions in $L^0(G)$, $\mathcal{F}$-exhaustive at every $s \in G$, $\rho$-$\mathcal{F}$-equi-absolutely continuous and with $(\mathcal{F})\lim_n f_n(t) = 0$ for all $t \in G$.

Then there is an $a > 0$ such that $(\mathcal{F})\lim_n \rho(af_n) = 0$.

### 5.3. Modular filter convergence

The following result, whose proof is analogous to that of [2, Theorem 4.7], is a sufficient condition, under which we get $\rho$-$\mathcal{F}$-equi-absolute continuity of the sequence $(T_n f)_n$.

**Theorem 5.6** Let $\mathcal{F}$ be any free filter, $\rho$ be a finite, monotone and absolutely finite modular, and let $\mathcal{K}$ satisfy (3.1.1). Suppose that $f$ is a bounded function with compact support $C$. If the family $(L_n(\cdot, t))_{t \in G, n \in \mathbb{N}}$ satisfies property $\rho$-(*) then there is an $a > 0$, independent of $f$, such that:

(i) for every $\varepsilon > 0$ there is a compact set $B \subseteq G$ with 

$$\{n \in \mathbb{N} : \rho(a(T_n f) \chi_{C \setminus B}) \leq \varepsilon \} \in \mathcal{F};$$

(ii) there is a set $\Lambda \in \mathcal{F}$ (depending only on $f$) such that for every $\varepsilon > 0$ there exists a $\delta > 0$ with $\rho(a(T_n f) \chi_{B}) \leq \varepsilon$ whenever $n \in \Lambda$ and $B \in \mathcal{B}$ with $\mu(B) < \delta$.

A consequence of Theorem 5.6 is the following:

**Corollary 5.7** Let $\mathcal{F}$ be any free filter, $\rho$ be a monotone, strongly finite, absolutely finite and absolutely continuous modular, and suppose that $\mathcal{K}$ is strongly $\mathcal{F}$-singular, or $\mathcal{F}$-singular in measure. If the family $(L_n(\cdot, t))_{t \in G, n \in \mathbb{N}}$ satisfies property $\rho$-(*) then there is an $a > 0$ with $(\mathcal{F})\lim_n \rho(a(T_n f-f)) = 0$ for each $f \in C_c(G)$.

**Proof** By Theorem 5.6, the sequence $(T_n f)_n$ is $\rho$-$\mathcal{F}$-equi-absolutely continuous. From this, strong finiteness and absolute continuity of $\rho$ we get that the sequence $(T_n f-f)_n$ is $\rho$-$\mathcal{F}$-equi-absolutely continuous too. Now, if $\mathcal{K}$ is strongly $\mathcal{F}$-singular, it is enough to apply Theorems 5.1, 5.3 and 5.5 to the sequence $(T_n f-f)_n$, while when $\mathcal{K}$ is $\mathcal{F}$-singular in measure, we apply Theorems 5.2 and 5.4 to $(T_n f-f)_n$. The assertion follows taking into account that, thanks to Theorem 5.6, it is possible to choose $a > 0$ independently of $f \in C_c(G)$.

We now prove the following technical lemma.
Lemma 5.8 Let $\mathcal{F}$ be a free filter, $\Xi = (\psi_n)_n \subset \Psi$, $\rho$ be a monotone modular and $\eta$ be a modular on $L^0(G)$. Let $\rho_n, \eta_n$ be modulars on $L^0(H_n)$ such that the triple $(\rho_n, \psi_n, \eta_n)$ is $\mathcal{F}$-properly directed, and assume that $\mathcal{K}$ satisfies (3.1.1). Moreover, suppose that the sequence

$$
\nu_n^{(\xi)}(A) := \int_A L_n(\cdot, t) \, d\mu_n(t), \quad n \in \mathbb{N}, \ A \in \mathcal{B}_n
$$

is $\mathcal{F}$-compatible with $(\rho, \rho_n)$.

Then for every $\lambda \in (0, 1)$ there exists an $a = a_\lambda > 0$ (depending only on $\lambda$) such that, for every $f, g \in L^0(G) \cap \text{Dom} \ T$ with $f - g \in \Upsilon_n$, we have

$$
\left(\mathcal{F}\right) \lim \sup_n \rho[a(T_n f - T_n g)] \leq P M \eta[\lambda(f - g)],
$$

where $P$ is as in (7) and $M$ is as in the $\mathcal{F}$-compatibility condition.

Proof Let $(c_n)_n$ be as in the definition of $\mathcal{F}$-properly directed triple, $\lambda \in (0, 1)$ and $a > 0$ be such that $aD_1 \leq C_1$, where $D_1$ satisfies (3.1.1) and $C_1$ is related the condition of $\mathcal{F}$-properly directed triple. Let $M > 0$ and $(b_n)_n$ be a sequence in $\mathbb{R}^+$ associated with $\mathcal{F}$-compatibility condition, and let $P$, $(\gamma_n)_n$ be as in (7). Proceeding analogously as in [4, Lemma 2] for each $n \in \mathbb{N}$ and whenever $\rho, \eta, f, g$ satisfy the given hypotheses, we have:

$$
\rho[a(T_n f - T_n g)] \leq M \gamma_n \eta[\lambda(f - g)] + M c_n \gamma_n + b_n.
$$

Taking the $\mathcal{F}$-limit superior, as $f - g \in \Upsilon_n$ and the sequence $(\gamma_n)_n$ is bounded, we get

$$
\left(\mathcal{F}\right) \lim \sup_n \rho[a(T_n f - T_n g)] \leq P M \eta[\lambda(f - g)].
$$

The assertion follows from (14). \hfill \blacksquare

We now turn to our main theorem in the general modular case.

Theorem 5.9 Let $\mathcal{F}$ be a free filter, $\rho$ and $\eta$ be two monotone, absolutely finite and absolutely continuous modulars on $L^0(G)$. Let $\rho_n$ and $\eta_n$ be modulars on $L^0(H_n)$ and $\Xi = (\psi_n)_n \subset \Psi$ such that the triple $(\rho_n, \psi_n, \eta_n)$ is $\mathcal{F}$-properly directed. Suppose that $\mathcal{K}$ is strongly $\mathcal{F}$-singular, or $\mathcal{F}$-singular in measure. Furthermore, assume that the sequence

$$
\nu_n^{(\xi)}(A) := \int_A L_n(\cdot, t) \, d\mu_n(t), \quad n \in \mathbb{N}, \ A \in \mathcal{B}_n
$$

is $\mathcal{F}$-compatible with the modulars $(\rho, \rho_n)$, and the family $(L_n(\cdot, t))_{t \in \mathcal{G}, n \in \mathbb{N}}$ satisfies property $\rho$-$(\ast)$.

Then for every $f \in \mathcal{G}_{\rho + \eta} \cap \text{Dom} \ T$ with $f - \mathcal{G} \subset \Upsilon_n$ there is an $a > 0$ with

$$
\left(\mathcal{F}\right) \lim_{n} \rho[a(T_n f - f)] = 0.
$$

Proof Let $f \in \mathcal{G}_{\rho + \eta} \cap \text{Dom} \ T$ be with $f - \mathcal{G} \subset \Upsilon_n$, and $P$ be as in (7). By Proposition 4.1, there exist a constant $\lambda' \in (0, 1)$ and a sequence $(f_k)_k$ in $\mathcal{G}$ such that for every $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$ with

$$
(\rho + \eta)[\lambda'(f_k - f)] < \varepsilon,
$$

for every $k \geq k_0$. 


In correspondence with \( \lambda' > 0 \) let \( a_{\lambda'} > 0 \) be according to Lemma 5.8. By Corollary 5.7 applied with \( f_k, k \in \mathbb{N} \), there exists a \( \lambda'' > 0 \), independent of \( k \), such that for all \( \varepsilon > 0 \) and \( k \geq K \), we get \( A_k \in \mathcal{F} \), where \( A_k = \{ n \in \mathbb{N} : \rho(\lambda''(T_n f_k - f_k)) \leq \varepsilon \} \). Let \( \lambda > 0 \) be with \( 3\lambda \leq \min(\lambda', a_{\lambda'}, \lambda'') \). For each \( n \in \mathbb{N} \) we get:

\[
\rho(\lambda(T_n f - f)) \leq \rho(3\lambda(T_n f - T_n f_k)) + \rho(3\lambda(T_n f_k - f_k)) + \rho(3\lambda(f - f_k)) = I_1 + I_2 + I_3.
\]

By Lemma 5.8 we get \( (\mathcal{F}) \limsup_n I_1 \leq PM \eta[\lambda'(f - f_k)] \); moreover, by Corollary 5.7, we obtain \( (\mathcal{F}) \lim I_2 = 0 \). So we have:

\[
(\mathcal{F}) \limsup_n \rho(\alpha(T_n f - f)) \leq PM \eta[\lambda'(f - f_k)] + PM \rho(\lambda'(f - f_k)) \leq PM \varepsilon.
\]

The assertion follows from arbitrariness of \( \varepsilon \).

6. Applications

In this section, as applications of our main results, we consider (nonlinear) discrete operators of type

\[
(T_n f)(s) = \sum_{k=-\infty}^{+\infty} K_n \left( s, \frac{k}{n}, f \left( \frac{k}{n} \right) \right), \quad n \in \mathbb{N}, \ s \in \mathbb{R}, \tag{16}
\]

where \( f \in \text{Dom} \, T \). Here, we consider \( G = \mathbb{R} \) endowed with the Lebesgue measure \( \mu \), \( H_n = \frac{1}{n} \mathbb{Z} \), \( B_n \) the set of all subsets of \( H_n \), \( \mu_n \) the counting measure.

We assume that \( K = (K_n) \) belongs to \( \mathcal{K} \), for a fixed family \( \mathcal{K} = (\psi_n)_n \subset \Psi \), and we identify \( L^1(H_n) \) with \( l^1, n \in \mathbb{N} \).

For all \( n \in \mathbb{N}, t = \frac{k}{n} \in \frac{1}{n} \mathbb{Z} \) and \( s \in \mathbb{R} \) we get:

\[
\int_G L_n(s, t) \, d\mu(s) = \int_{\mathbb{R}} L_n \left( s, \frac{k}{n} \right) \, ds, \quad \int_{\mathbb{Z}} L_n(s, t) \, d\mu_n(t) = \sum_{k=-\infty}^{+\infty} L_n \left( s, \frac{k}{n} \right).
\]

In particular, note that the conditions (3.1.1)–(3.1.3) involving \( \mathcal{F} \)-singularity are expressed respectively as follows:

(j) there is a positive real number \( D_1 \) with

\[
\Lambda := \left\{ n \in \mathbb{N} : \sum_{k=-\infty}^{+\infty} L_n \left( s, \frac{k}{n} \right) \leq D_1 \text{ for all } s \in \mathbb{R} \right\} \in \mathcal{F};
\]

(jj) for every \( s \in \mathbb{R} \) and \( r > 0 \) we get

\[
(\mathcal{F}) \lim_n \sum_{k \in \mathbb{Z} : |an - k| > nr} L_n \left( s, \frac{k}{n} \right) = 0;
\]

(jjj) for every \( s, u \in \mathbb{R} \) we have

\[
(\mathcal{F}) \lim_n \sum_{k=-\infty}^{+\infty} K_n \left( s, \frac{k}{n}, u \right) = u.
\]
Moreover, observe that uniform $\mathcal{F}$-singularity is satisfied iff condition (j) is fulfilled together with the two following:

(jj') for every $r > 0$,

$$
(\mathcal{F}) \lim_{n} \left[ \sup_{s \in \mathbb{R}} \left( \sum_{k \in \mathbb{Z} : |n-k| > m} L_n(s, k/n) \right) \right] = 0;
$$

(jj'')

$$
(\mathcal{F}) \lim_{n} \left[ \sup_{s \in \mathbb{R}, u \in C} \left( \sum_{k=-\infty}^{+\infty} K_n(s, k/n, u) - u \right) \right] = 0
$$

for every compact set $C \subset \mathbb{R} \setminus \{0\}$.

Analogously it is possible to formulate the concepts of strong $\mathcal{F}$-singularity and of $\mathcal{F}$-singularity in measure.

Let us consider modulars $\eta = \eta_{\mathbb{R}}$ on $L^0(\mathbb{R})$ and $\eta_n = \eta_{\mathbb{Z}}$ on $L^0(\mathbb{Z})$. In this context, the class $\Upsilon_\eta$ is the subset of $L^\eta(\mathbb{R})$ whose elements $f$ satisfy the inequality

$$
(\mathcal{F}) \limsup_n \eta_n \eta_{\mathbb{Z}}(\lambda, f) \leq P \eta_{\mathbb{R}}(\lambda, f) \quad \text{for every } \lambda > 0 \tag{17}
$$

for a suitable positive real number $P$ and a bounded sequence $(\gamma_n)_n$ in $\mathbb{R}^+$ (independent of $f$). We consider the case when $\eta$ and $\eta_n$, $n \in \mathbb{N}$, generate an Orlicz space, that is

$$
\eta_{\mathbb{R}}(f) = \int_{-\infty}^{+\infty} \varphi(|f(s)|) \, ds, \quad \eta_n = \eta_{\mathbb{Z}} = \sum_{k=-\infty}^{+\infty} \varphi\left(\left| f\left(\frac{k}{n}\right) \right| \right),
$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ belongs to $\tilde{\Phi}$. If we suppose that our singularity conditions are satisfied and property (7) is fulfilled with $P = 1$ and $\gamma_n = \frac{1}{n}$, then $\Upsilon_\eta$ is the set of all functions $f \in L^\eta(\mathbb{R})$ with

$$
(\mathcal{F}) \limsup_n \frac{1}{n} \sum_{k=-\infty}^{+\infty} \varphi\left(\left| \lambda f\left(\frac{k}{n}\right) \right| \right) \leq \int_{-\infty}^{+\infty} \varphi(\lambda |f(s)|) \, ds. \tag{18}
$$

Note that, since ordinary convergence of sequences implies always their filter convergence for any free filter [24], we get that, if $\varphi$ is convex, then $\Upsilon_\eta$ contains the set of all real-valued functions which are of bounded variation on $\mathbb{R}$ (see also [4,25]). Indeed, on the left-hand side of (18) a Riemann sum appears of the generalized Riemann integral on the right-hand side. Thus, the general theory developed before can be applied to the operators in (16), by considering the modulars $\rho_n := \rho_{\mathbb{Z}}$, $\eta_n := \eta_{\mathbb{Z}}$. In this way we extend Corollary 4 in [4] to the context of filter convergence.

We now give an example showing that our results are proper extensions of the corresponding classical ones.

Let us consider the modulars $\rho$, $\eta$, $\eta_n$, $n \in \mathbb{N}$, generating the Lebesgue spaces $L^p(\mathbb{R})$, with $p \geq 1$, and consider $\Upsilon_\eta$ as in (18). Choose a function $f : \mathbb{R} \to \mathbb{R}$, belonging to $L^p(\mathbb{R})$ for some $p \geq 1$, with $f \in C^\infty_c(\mathbb{R}) \subset \Upsilon_\eta$ and such that, for $r$ sufficiently large, $f(t) \geq |t|^{-r}$ for $|t| \geq 1$, and $f(t) = 0$ for $|t| < 1$. Pick any free filter $\mathcal{F} \neq \mathcal{F}_{\text{cofin}}$ and an
infinite set $E$, with $\mathbb{N} \setminus E \in \mathcal{F}$. Since $\mathcal{F} \neq \mathcal{F}_{\text{cofin}}$, then $E$ does exist. For each $n \in \mathbb{N}$, let $L_n$ be as in the hypotheses of Theorem 5.11, and suppose that there exists a set $B^*$ of positive Lebesgue measure, independent of $n$, with

$$\sum_{k=n}^{2n} L_n(s, \frac{k}{n}) \geq v_n \quad \text{for all } s \in B^*,$$

where $(v_n)_n$ is an infinitesimal sequence of positive real numbers. Let $(w_n)_n$ a sequence in $\mathbb{R}^+$, such that $\lim_n v_n w_n = +\infty$. For each $n \in \mathbb{N}$, $s \in \mathbb{R}$ and $t \in H_n$ put

$$L^*_n(s, t) = \begin{cases} L_n(s, t), & \text{if } n \in \mathbb{N} \setminus E, \\ w_n L_n(s, t), & \text{if } n \in E, \end{cases}$$

and

$$K^*_n(s, \frac{k}{n}, f(\frac{k}{n})) = L^*_n(s, \frac{k}{n}) \cdot f(\frac{k}{n}).$$

Thanks to (19) and (20), it is not difficult to see that

$$\sum_{k=-\infty}^{+\infty} K^*_n(s, \frac{k}{n}, f(\frac{k}{n})) \geq w_n n^r \cdot \sum_{k=n}^{2n} L_n(s, \frac{k}{n}) \frac{1}{k^r} \geq v_n w_n \frac{1}{2^r},$$

for each $s \in B^*$ and $n \in E$, and hence we obtain $\lim_n (T_n f)(s) = +\infty$ for every $s \in B^*$. So, in this case, Theorem 5.11 is satisfied, but the corresponding classical result (i.e. when $\mathcal{F} = \mathcal{F}_{\text{cofin}}$, see also [4, Corollary 4]) does not hold.

An important special case of operators (16) is given by the linear generalized sampling series, defined by

$$\left(\overline{T}_n f\right)(s) := \sum_{k=-\infty}^{+\infty} f(\frac{k}{n}) \kappa(ns - k), \quad n \in \mathbb{N}, \ s \in \mathbb{R}. \quad (21)$$

Here $\kappa$ is a given kernel function in $L^1(\mathbb{R})$, $f: \mathbb{R} \to \mathbb{R}$. In this instance we set

$$K_n(s, \frac{k}{n}, f(\frac{k}{n})) = \kappa(ns - k) \cdot f(\frac{k}{n}) \quad \text{and} \quad L_n(s, \frac{k}{n}) = |\kappa(ns - k)|.$$

Under the classical assumptions

$$\sum_{k=-\infty}^{+\infty} \kappa(u - k) = 1$$

uniformly with respect $u \in \mathbb{R}$, and

$$\sup_u \sum_{k=-\infty}^{+\infty} |\kappa(u - k)| < +\infty,$$

it is possible to prove that the kernel $K_n$ is $\mathcal{F}_{\text{cofin}}$-uniformly singular (see [5, Theorem 8.1], [26, Lemma 3.1] and [27, Lemma 1]), and $a\ fortiori$ $\mathcal{F}$-uniformly singular for each free filter $\mathcal{F}$. 
These operators have several applications to the problem of reconstructing a signal from its samples \( f(\frac{k}{n}) , k \in \mathbb{Z} \), at the nodes \( \frac{k}{n} \) (see for instance [3,14,16,18,28–30]).

We now give an example of a kernel

\[
K_n \left( s, \frac{k}{n} , u \right) = L_n \left( s, \frac{k}{n} \right) \cdot u,
\]

which satisfies the conditions of filter singularity in measure, but not filter singularity. Let us define \( h_l : \mathbb{R} \rightarrow \mathbb{R} \), \( l \in \mathbb{N} \), as follows. If \( x \in [0,1] \), set \( h_l(x) = 0 \) for each \( l \), and set \( h_l(x) = 1 \) for any \( x \in ]0,1] \). For every \( n \geq 2 \) and

\[
\frac{n(n-1)}{2} < l \leq \frac{n(n+1)}{2} , \quad l = \frac{n(n-1)}{2} + j , \quad j = 1, \ldots , n ,
\]

set

\[
h_l(x) = \begin{cases} 
1, & \text{if } x \in \left[ \frac{j-1}{n}, \frac{j}{n} \right], \\
0, & \text{otherwise}.
\end{cases}
\]

Observe that \( (h_l)_l \) converges in measure to 0 in the usual sense, and hence \( \mathcal{F} \)-converges in measure to 0 for every free filter \( \mathcal{F} \).

We present an example of free filter \( \mathcal{F}_0 \neq \mathcal{F}_{c_{o}f_{i}n} \), such that the sequence \( (h_l(x))_l \) does not \( \mathcal{F}_0 \)-converge to 0 at any point \( x \in ]0,1] \). Let \( (\xi_n)_n \) be a fixed strictly increasing sequence of positive integers, with \( \lim_{n} \frac{n}{\log(\xi_n)} = 0 \), and set

\[
\mathcal{F}_0 := \left\{ A \subset \mathbb{N} : d^*(A) := \lim_{n} \frac{\#(A \cap \{1, \ldots , \xi_n\})}{\log(\xi_n)} = 1 \right\},
\]

where \( \# \) denotes the cardinality of the set in brackets. It is easy to see that \( \mathcal{F}_0 \) is a filter, containing \( \mathcal{F}_{c_{o}f_{i}n} \). Choose now an infinite set \( E := \{ \alpha_1 < \cdots < \alpha_n < \cdots \} \), with \( 1 \leq \alpha_1 \leq \xi_1 \) and \( \xi_{n-1} < \alpha_n \leq \xi_n \) for each \( n \geq 2 \). By construction, we get \( \#(E \cap \{1, \ldots , \xi_n\}) = n \). Hence \( d^*(E) = \lim_{n} \frac{n}{\log(\xi_n)} = 0 \), and \( \mathbb{N} \setminus E \in \mathcal{F}_0 \setminus \mathcal{F}_{c_{o}f_{i}n} \).

Fix arbitrarily \( x \in ]0,1] \). To prove that \( (h_l(x))_l \) does not \( \mathcal{F}_0 \)-converge to 0, it is enough to show that

\[
F := \{ n \in \mathbb{N} : h_n(x) = 0 \} \notin \mathcal{F}_0 .
\]

For each \( n , l \in \mathbb{N} \), \( \frac{n(n-1)}{2} < l \leq \frac{n(n+1)}{2} \), we get

\[
\frac{\#((\mathbb{N} \setminus F) \cap \{1, \ldots , l\})}{\log l} \geq \frac{n}{\log \left( n \left( \frac{n+1}{2} \right) \right)} \geq \frac{n}{\log(n^2)} = \frac{n}{2 \log n} .
\]

Hence \( \lim_{l} \frac{\#((\mathbb{N} \setminus F) \cap \{1, \ldots , l\})}{\log l} = +\infty \), and a fortiori \( \lim_{n} \frac{\#((\mathbb{N} \setminus F) \cap \{1, \ldots , \xi_n\})}{\log(\xi_n)} = +\infty \). Thus it is not true that \( d^*(\mathbb{N} \setminus F) = 0 \), or equivalently \( d^*(F) = 1 \). This proves (24).

Let us take a function \( \kappa : \mathbb{R} \rightarrow \mathbb{R} \) with the above assumptions, and set

\[
L_n \left( s , \frac{k}{n} \right) = \begin{cases} 
(1 - h_n(s)) \kappa(ns - k), & \text{if } s \in ]0,1], \\
\kappa(ns - k), & \text{if } s \notin ]0,1].
\end{cases}
\]

for \( n \in \mathbb{N} \) and \( k \in \mathbb{Z} \). It is not difficult to check that \( F_n(s) := \sum_{k=-\infty}^{+\infty} L_n \left( s , \frac{k}{n} \right) , n \in \mathbb{N} \), \( \mathcal{F} \)-converges to 1 in measure for each free filter \( \mathcal{F} \), while it does not \( \mathcal{F}_0 \)-converge (and a fortiori does not converge in the usual sense) for every \( s \in ]0,1] \). So, the
sequence $\mathcal{K} := (K_n)_n$ defined in (22), with $(L_n)_n$ as in (25), is $\mathcal{F}$-singular in measure for every free filter $\mathcal{F}$, but not $\mathcal{F}_0$-singular.

Our general approach on discrete operators enables us to treat also the multidimensional sampling series ([12,15,31]), which have several applications, for instance in the image and video reconstruction (see e.g. [16,17]). For example, in the two-dimensional case, $G = \mathbb{R}^2$ endowed with the Euclidean distance, $\mu$ is the two-dimensional Lebesgue measure, $H_n = \frac{1}{n} \mathbb{Z}^2$, $B_n$ is the set of all subsets of $\mathbb{Z}^2$ and $\mu_n$ is the counting measure.

In this frame the operators (16) and (21) are given by

$$(V_n f)(x, y) = \sum_{k=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} K_n \left( x, y, \frac{k}{n}, \frac{j}{n}, f \left( \frac{k}{n}, \frac{j}{n} \right) \right), \quad n \in \mathbb{N}, \ (x, y) \in \mathbb{R}^2$$

and

$$(\tilde{V}_n f)(x, y) := \sum_{k=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} f \left( \frac{k}{n}, \frac{j}{n} \right) \kappa (nx - k, ny - j), \quad n \in \mathbb{N}, \ (x, y) \in \mathbb{R}^2,$$

respectively, where $\kappa \in L^1(\mathbb{R}^2)$.

Similar applications can be given also to classical Bernstein polynomials, Szasz–Mirak’jan, Baskakov and many other operators (see also [4]).

7. Concluding remarks

The above abstract theory can also be applied to integral operators. In this case we take $H_n = G$ and $\mu_n = \mu$ for every $n \in \mathbb{N}$. This corresponds to the case of Urysohn-type operators which are described in [1,2,21]. In particular, we can obtain applications to convolution integral operators, including also the Mellin convolution operators (see, e.g. [32,33,34]), for which we set $G = H_n = \mathbb{R}^+ \cap \mathbb{R}$ and for any measurable set $S \subset \mathbb{R}^+$ we set $\mu(S) = \int_S \frac{dt}{t}$. Indeed, proceeding analogously as in [1,2], it is possible to prove that several Mellin-type kernels, like moment, Mellin–Poisson–Cauchy and Mellin–Gauss–Weierstrass kernels, satisfy the conditions of $\mathcal{F}$-uniform singularity, getting $\mathcal{F}$-uniform convergence for the sequence of the integral operators which approximate the given function $f$.

References


