Enhancing constraints manipulation in semiring-based formalisms[1]

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Abstract. Many "semiring-like" structures are used in Soft Constraint Satisfaction Problems (SCSPs). We review a few properties of semirings that are useful for dealing with soft constraints, highlighting the differences between alternative proposals in the literature.

We then extend the semiring structure by adding the notion of *division* as a weak inverse operation of product. In particular, division is needed to apply constraint relaxation when the product operation of the semiring is not idempotent. The division operator is introduced via *residuation* and it is also able to deal with partial orders, generalizing the approach given for Valued CSPs.

1 Introduction

Several formalizations of the concept of *soft constraints* are currently proposed in the literature. In general terms, a soft constraint may be seen as a function associating to each assignment (i.e., instantiation of the variables occurring in it) a value in a partially ordered set A, which can be interpreted as a set of preference values or costs. Combining constraints then has to take into account such additional values, and thus the formalism must provide suitable operations for the combination and the comparison of tuples of values and constraints.

The paper focuses on semiring-based frameworks: Assignments take values in a semiring, the order is the one associated to the $+$ operator of the semiring, and the combination is the $\times$ operator. We review the basics of semiring theory and identify a few properties that are needed to deal with constraints. These properties boil down to the notion of commutative, absorptive semiring, an instance of well-known tropical semirings [23]. Based on this characterization, a comparison between several proposals is performed, namely valuation structures [26], c-semirings [4, 7] and semiring valuations [27].

Soft constraint satisfaction problems can be solved by extending and adapting the techniques used for their classical version. For example, a branch and bound search algorithm can be used, instead of backtracking, for finding the best solution. A pivotal component of this generalization is the search for algorithms of constraint relaxation, in order to obtain some form of local consistency (i.e., such that changing the semiring values associated to an assignment does not change the overall solution). More to the point, in this paper we define a technique to move valuation (cost) information from constraints involving multiple variables to simpler, possible unary ones, resulting in an upper bound on the problem valuation.

Elaborating on a proposal by Cooper and Schiex [11, 12, 25], we propose improvements on classical local consistency techniques, in order to apply the framework also whenever the constraint composition operator is not idempotent. To this end, we extend the semiring structure by adding the notion of *division* as a weak inverse operation of product. The solution we pursue for characterizing that operator is based on *residuation theory* [8], a standard tool on so-called *tropical* arithmetics. It allows for obtaining a division operator *via* an approximate solution to the equation $b \times x = a$.

2 On semirings

Slightly different presentation for the notion of *semiring* occur in the literature. The less constrained definition we are aware of (compare e.g. the survey [24]) is given by a set A equipped with two binary operators, the sum $+$ and the product $\times$, such that $+$ is associative and commutative (that is, the pair $\langle A, + \rangle$ is a commutative semigroup), and the product operator $\times$ distributes over $+$.

In soft constraints literature the cost/preference associated to each variable instance is modeled as an element of a semiring, and the constraint combination is defined via the semiring product operator [4, 5, 7, 27]. Most often, the sum operator is used just for the induced pseudo order[4], given by $a \leq b$ iff $a + b = b$.

Which properties should be required for constraint combination?

- Since satisfaction problems are defined by "sets" of constraints, the order of the constraints combination has to be irrelevant. This leads to the *associativity* and *commutativity* for the $\times$ operator;
- since adding constraints decreases the number (and the quality) of the solutions, the combination of constraints has to worsen the value of the operands. This means that the ordering has to be *absorptive* ($a + (a \times b) = a + (b \times a) = a$);
- when dealing with soft constraints there might be the need of representing crisp features. That is, there must be an element in the semiring that has the crisp meaning of total dislike of any solution that involve a specific assignment. Thus, an element $\mathbf{0} \in A$ called *zero* or *annihilator* element ($a \times \mathbf{0} = \mathbf{0} \times a = \mathbf{0}$);
- similarly, there must be an element that has the crisp meaning of "indifference", i.e., the satisfaction of the specified constraint does not change the overall level of preference for a given tuple. Such an element $\mathbf{1} \in A$ is called *unit* ($a \times \mathbf{1} = \mathbf{1} \times a = a$).

We adopt a terminology inspired by [14] and, in lesser degree, by [24], aiming at a smooth presentation of the main concepts to the reader.

Definition 1 (semirings) *A commutative semiring is a five-tuple* $\mathcal{K} = \langle A, +, \times, \mathbf{0}, \mathbf{1} \rangle$ *such that* A *is a set,* $\mathbf{1}, \mathbf{0} \in A$*, and* $+, \times : A \times A \to A$ *are binary operators making the triples* $\langle A, +, \mathbf{0} \rangle$ *and* $\langle A, \times, \mathbf{1} \rangle$ *commutative monoids (semigroups with identity), satisfying*

(distributivity) $\forall a, b, c \in A. a \times (b + c) = (a \times b) + (a \times c)$;
(annihilator) $\forall a \in A. a \times \mathbf{0} = \mathbf{0}$.

[1] Research partially supported by the EU IST 2004-16004 SENSORIA and by the MIUR PRIN 2005-015491.

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[4] A pseudo order is a transitive, antisymmetric, possibly not reflexive relation.

Proposition 1 (absorptive semirings [24, Corollary 2.1]) *Let $\mathcal{K}$ be a commutative semiring. Then, these properties are equivalent*

(absorptiveness) $\forall a, b \in A. a + (a \times b) = a$,
(top element) $\forall a \in A. a + \mathbf{1} = \mathbf{1}$.

Semirings verifying the above properties are known as *absorptive* (or *simple*) [10] and represent the structure we put at the base of our proposal since they satisfy the properties absorptiveness, zero and unit element that seem pivotal for any soft constraint framework.

We can now state a simple characterization result linking absorptiveness to idempotency and to the notion of top element.

Proposition 2 (tropical semirings) *Let $\mathcal{K}$ be a commutative semiring. If $\mathcal{K}$ is absorptive, then the sum operator is idempotent.*

The former result is well-know in the literature, and commutative semirings such that the sum operator is idempotent are called *dioids* or *tropical semirings*[5]. These structures are well-studied in the literature [1, 2, 23, 24], and we take advantage of some classical constructions in the following sections. Notice in fact that absorptive semirings are just tropical semirings with top element.

2.1 Alternative approaches

We now have a look at some order structures proposed in the literature for interpreting constraints, showing their similarity and highlighting absorptive semirings as a common algebraic structure. More precisely, the section provides a comparison with c-semiring [4, 6, 7], valuation structures [26], and semiring valuations [27].

2.1.1 c-semirings

The starting point of our excursus are c-semirings [4, 7]: according to our notation, they are tropical semirings with top element. Our analysis, as summed up in Proposition 2, suggests that the intrinsic quality of c-semirings is the the absorptiveness of the order, whilst idempotency of the sum is merely a side-effect, in the sense that it is implied by the other properties (see Proposition 2).

Note also that most results on local consistency for c-semirings require the idempotency of the $\times$ operator [4]. This assumption results in a stronger structure, since $a \times b$ coincides with the greatest lower bound of a and b. Indeed, the focus of this article is also the extension of some algorithms proposed in the c-semirings formalism, in order to deal also with non-idempotent product operators.

2.1.2 Valuation structures

Adopting our terminology, a *valuation structure* [26] is a five-tuple $\langle A, \leq, \times, \mathbf{0}, \mathbf{1} \rangle$ such that $\langle A, \times, \mathbf{1} \rangle$ is a commutative monoid, $\langle A, \leq \rangle$ is a total order (with $\mathbf{0}$ and $\mathbf{1}$ as minimun and maximum, respectively), $\times$ is monotonic and $\mathbf{0}$ is its annihilator element.

As noted in [5], a commutative semiring can be associated to each valuation structure by defining the sum operator as $a+b = max\{a, b\}$, obtaining an absorptive semiring. Moreover, the order in the resulting semiring coincides with the original order of the valuation structure. In fact, note that in a tropical semiring the induced ordering $\leq$ is total iff $a + b \in \{a, b\}$ for all $a, b \in A$, so that there is a one-to-one correspondence between valutation structures and those tropical semirings (known in the literature as *additively extremal* semirings [14, p.11]) such that the associated order $\leq$ is total. We further discuss the properties of these structures later on, when comparing our notion of inverse for the product operator with the proposal in [12].

[5] The adjective "tropical" was coined by French mathematicians [23] in honor of the Brazilian Imre Simon. The terminology "dioid" is adopted in [15, 17] to highlight that the structure can not be a ring, but it is "twice a monoid".

2.1.3 Semiring valuations

Semiring valuations [27] are constraint satisfaction problems taking values in a commutative semiring, where the ordering is the transitive relation $a \leq' b$ iff $\exists c. a + c = b$. The two alternative definitions of orderings coincide for tropical semirings, in the sense that $a \leq b$ iff $a \leq' b$ for all idempotent elements a, b.

The lack of idempotency for the sum operator results in a weaker structure than absorptive semirings, that has proved useful whenever counting the number of solutions is of interest, as for special computations in Bayesian networks. However, the associated reflexive and transitive relation $\leq'$ satisfies relatively few properties, since adding constraints does not lead to worsen the solution, thus resulting in a non-monotonic framework (because of the absence of the absorptive property). These remarks are summed up by the result below.

Proposition 3 *Let $\mathcal{K}$ be a commutative semiring. If $\leq'$ is a partial order and $\forall a, b \in A. a \times b \leq' a$, then $\mathcal{K}$ is absorptive.*

Semirings such that the relation $\leq'$ is a partial order are known in the literature as *uniquely difference ordered* [14, Section 2]. For these structures, whenever the $\times$ operator worsens the solution, the sum operator is idempotent and thus $\leq'$ equals $\leq$.

3 Adding division

The search for optimal solutions on constraint satisfaction problems has been mostly based on the idempotency of the $\times$ operator. While the resulting heuristics may have a neat presentation and behavior, many relevant examples fall outside its scope. As we shall see, the viability of local consistency algorithms can be recovered by requiring the existence of a suitable inverse operator.

Among the possible solutions for ensuring the relevant structure, *symmetrisation* embeds the semiring in a larger structure containing an inverse for each element. The approach is quite standard in tropical arithmetics (see e.g. [2, Section 3.4.1.1] and [13, Section 3.8]): the derived semiring has pairs (a, b) as elements, accordingly derived operators, and a minus operator, defined as $-(a, b) = (b, a)$. A related approach is suggested in [11] for obtaining an inverse operator for $\times$, starting from a *strictly monotonic* valuation structure (i.e., such that $\forall a, b, c \in A. (a < b) \wedge (c \neq \mathbf{0}) \implies a \times c < b \times c$).

These constructions are similar and the properties of the derived inverse operator basically hold due to the totality of the order on valuation structures. This proposal is used in [12, Example 2] as a mechanism for recovering what the authors call *fair* evaluation structures.

The solution we pursue here is based on *residuation theory* [8], a standard tool on tropical arithmetics (see e.g. [2, Section 4.4.2] and [14, Chapter 4]), which allows for obtaining a division operator *via* an approximate solution to the equation $b \times x = a$. Differently with respect to the use of a completion procedure, no new element is added, leaving the same set of preferences to the user. It is noteworthy that by using the newly defined division operators, suitable consistency algorithms are devised also for non idempotent products.

3.1 Basic definitions and results

This section introduces our notion of invertibility for absorptive semirings: three alternative properties are identified, and they are listed below in the same order of their strength.

Definition 2 *Let $\mathcal{K}$ be an absorptive semiring. Then*

- *$\mathcal{K}$ is* invertible *if there exists element $c \in A$ such that $b \times c = a$ for all elements $a, b \in A$ such that $a \leq b$;*
- *it is* weakly uniquely *invertible if c is unique whenever $a < b$;*
- *it is* uniquely *invertible if c is unique whenever $b \neq \mathbf{0}$.*

Note that the former definitions do not require the existence for each element a of an inverse, i.e., of an element a^{-1} verifying $a \times a^{-1} = \mathbf{1}$. In fact, the absorptiveness condition $a \times b \leq a$ guarantees that no element, except $\mathbf{1}$ itself, has an inverse.

We now present two results that establish conditions for ensuring invertibility of an absoprtive semiring. The first proposition concerns multiplicative idempotent semirings and their order structure.

Proposition 4 *Let $\mathcal{K}$ be an absorptive, multiplicative idempotent semiring. Then, $\mathcal{K}$ is invertible. Moreover, if $\leq$ is a total order, then $\mathcal{K}$ is weakly uniquely invertible.*

The second proposition is for *cancellative* semirings, i.e., such that $\forall a, b, c \in A.(a \times c = b \times c) \wedge (c \neq \mathbf{0}) \implies a = b$.

Proposition 5 *Let $\mathcal{K}$ be an absorptive, invertible semiring. Then, $\mathcal{K}$ is uniquely invertible iff it is cancellative.*

3.2 On division on residuated semirings

Residuation theory is concerned with the study of sub-solutions of the equation $b \times x = a$. Note that on tropical semirings the relaxed equation $b \times x \leq a$ always has a solution (it is enough to set x to $\mathbf{0}$).

Definition 3 *Let $\mathcal{K}$ be a tropical semiring. Then, $\mathcal{K}$ is* residuated *if the set $\{x \in A \mid b \times x \leq a\}$ admits a maximum for all elements $a, b \in A$, denoted $a \div b$.*

Note that the previous definition actually suggests an algorithmic heuristics for approximating such a maximal element, intuitively given by the (possibly infinite) sum of all the elements satisfying the inequation [2]. The key point is that the set of sub-solutions of an equation contains also the possible solutions, whenever they exist, and in that case the maximal element is also a solution.

More properties hold if the semiring is absorptive, such as that $b \leq a \implies a \div b = \mathbf{1}$. This fact leads to our notion of invertibility.

Definition 4 *Let $\mathcal{K}$ be an absorptive, invertible semiring. Then, $\mathcal{K}$ is* invertible by residuation *if the set $\{x \in A \mid b \times x = a\}$ admits a maximum for all elements $a, b \in A$ such that $a \leq b$.*

With an abuse of notation, the maximal element among solutions is denoted $a \div b$. This choice is not ambiguous: if an absorptive semiring is invertible and residuated, then it is also invertible by residuation, and the two definitions yield the same value.

3.3 On complete semirings

Being residuated is implied by a sometimes easier to check property, namely, the existence of elements representing infinite sums.

Definition 5 *Let $\mathcal{K}$ be a tropical semiring. Then, $\mathcal{K}$ is* complete *if it is closed with respect to infinite sums, and the distributivity law holds also for an infinite number of summands.*

Also associativity and commutativity need to be generalized, and we refer the reader to e.g. [14, Section 3].

Proposition 6 ([2, Theorem 4.50]) *Let $\mathcal{K}$ be a tropical semiring. If $\mathcal{K}$ is complete, then it is residuated.*

The above proposition ensures that all classical soft constraint instances (see Section 4.1) are residuated (because complete) and the notion of division can be applied to all of them.

3.4 Further comparison with valuation structures

We now compare our definition of division with the proposal by Cooper and Schiex in [12, Definition 3.1]. A valuation structure is called *fair* if the set $\{x \in A \mid b \times x = a\}$ has a minimum whenever $a \leq b$ [12, Definition 3.1]. Hence, the division operator is defined as

$$a \div' b = \begin{cases} min\{x \mid b \times x = a\} & \text{if } a \leq b, \\ undefined & \text{otherwise.} \end{cases}$$

Let us say that an absorptive semiring is *fair* if it is invertible and the operator $\div'$ above is well defined. How does fairness compare with residuation? By definition $\{x \mid b \times x = a\} \subseteq \{y \mid b \times y \leq a\}$, so that the operation $a \div' b$ returns a smaller value than $a \div b$. Nevertheless, the two notions sometimes coincides, e.g. if a semiring is uniquely invertible, hence whenever the $\times$ operator is cancellative (equivalently, whenever a valuation structure is strictly monotonic).

Proposition 7 (cancellativeness) *Let $\mathcal{K}$ be an absorptive, cancellative semiring. Then, $\mathcal{K}$ is fair iff it is invertible by residuation, and in such a case $a \div' b = a \div b$ for all elements a, b such that $a \leq b$.*

A weakly uniquely invertible semiring is invertible by residuation but it is not necessarily fair: the operation $a \div' a$ might not be defined, while $a \div a = \mathbf{1}$ always holds. In fact, the two operators usually differ for the division of an element by itself, since e.g. $a \div' a = a$ holds for multiplicative idempotent semirings. Combined with Proposition 4, relating weakly invertibility and multiplicative idempotency, the remark above establishes the correspondence result below.

Proposition 8 (idempotency) *Let $\mathcal{K}$ be an absorptive, multiplicative idempotent semiring. Then, $\mathcal{K}$ is fair. Moreover, if $\leq$ is a total order, then $\mathcal{K}$ is invertible by residuation, and in such a case $a \div b = a \div' b = a$ for all elements a, b such that $a < b$.*

For those semirings where $\leq$ is total, the proposition also yields that $b \times (a \div' b) = b \times (a \div b) = a$ for all a, b such that $a \leq b$.

We provide no general correspondence result between the two properties. Note however that by definition a residuated fair semiring is invertible by residuation, hence the proposition below follows.

Proposition 9 (completeness) *Let $\mathcal{K}$ be an absorptive, complete semiring. If $\mathcal{K}$ is fair, then it is invertible by residuation.*

In general, the two operations returns different values, as it happens for the set-based CSPs presented in Section 4.1.4. Note that most case studies in the literature satisfy the completeness property, and are either multiplicative idempotent or cancellative: this fact hold for all the examples presented in Section 4.1.

4 Soft constraints and local consistency

Several formalizations of *soft constraints* are currently available. The first part of this section briefly introduces the semiring-based formalism, directly borrowing from the c-semirings approach [4, 7]. The second part of this section represents another technical contribution of our work: It presents an extension of local consistency techniques for invertible semirings, thus generalizing those previously proposed for those cases where the $\times$ operator is idempotent.

4.1 Constraint problems

Let $\mathcal{K} = \langle A, +, \times, \mathbf{0}, \mathbf{1} \rangle$ be an absorptive semiring; V a finite (possibly ordered) set of variables; and D a chosen domain of interpretation for V. Then, a *constraint* $(V \to D) \to A$ is a function associating a value in A to each assignment $\eta : V \to D$ of the variables.

We define $\mathcal{C}$ as the set of constraints that can be built starting from $\mathcal{K}$, V and D. Note that even if a constraint involves all the variables in V, it must depend on the assignment of a finite subset of them. For instance, a binary constraint $c_{x,y}$ over variables x, y is a function $c_{x,y} : (V \to D) \to A$ which depends only on the assignment of variables $\{x, y\} \subseteq V$. We call this subset the *support* of the constraint. Often, if V is ordered, an assignment (over a support of cardinality k) is concisely presented by a tuple in D^k.

More formally, let $c \in \mathcal{C}$ be a constraint. We define its support as $supp(c) = \{v \in V \mid \exists \eta, d_1, d_2.c\eta[v := d_1] \neq c\eta[v := d_2]\}$, where

$$\eta[v := d]v' = \begin{cases} d & \text{if } v = v' \\ \eta v' & \text{otherwise} \end{cases}$$

While $c\eta$ is the application of a constraint function $c : (V \to D) \to A$ to a function $\eta : V \to D$, obtaining a semiring value, $c\eta[v := d_1]$ means $c\eta'$ where η' is η modified with the assignment $v := d_1$.

Later on we implicitly restrict to finitely supported constraints.

4.1.1 Classical CSPs

Classical (crisp) satisfaction problems may be recast to deal with soft constraints by considering the semiring $\mathcal{K}_{CSP} = \langle \{0, 1\}, \vee, \wedge, 0, 1\rangle$.

The semiring is finite, hence complete (and residuated by Proposition 6). Moreover, the $\times$ operator is idempotent, then the semiring is invertible, and since the order is total, it is weakly uniquely invertible (by Proposition 4). By instantiating the definition of division we get

$$a \div b = max\{x \mid b \wedge x \leq a\} = (b \implies a)$$

where $\implies$ is the logic implication.

4.1.2 Fuzzy CSPs

Fuzzy CSPs (FCSPs) extend the standard notion by allowing non-crispness features, and can be modeled by the semiring $\mathcal{K}_{FCSP} = \langle [0, 1], max, min, 0, 1\rangle$.

The semiring is clearly complete (and residuated by Proposition 6). Moreover, the $\times$ operator is idempotent, then the semiring is invertible, and since the order is total, it is weakly uniquely invertible (by Proposition 4). By instantiating the definition of division we obtain

$$a \div b = max\{x \mid min\{b, x\} \leq a\} = \begin{cases} 1 & \text{if } b \leq a \\ a & \text{if } a < b \end{cases}$$

4.1.3 Weighted CSPs

While fuzzy CSPs associate a level of preference, in weighted CSPs (WCSPs) tuples come with an associated cost to be minimized. The associated semiring structure is in this case $\mathcal{K}_{WCSP} = \langle \mathcal{R}^+ \cup \{\infty\}, \text{min}, \hat{+}, \infty, 0\rangle$ for $\hat{+}$ the sum of reals.

The semiring is clearly complete (and residuated by Proposition 6). The $\times$ operator is not idempotent, but the semiring is nevertheless invertible, as it can be easily proved by checking the definition itself. Moreover, the semiring is cancellative, hence it is uniquely invertible by Proposition 5. By instantiating the definition of division we obtain

$$a \div b = min\{x \mid b \hat{+} x \geq a\} = \begin{cases} 0 & \text{if } b \geq a \\ a \hat{-} b & \text{if } a > b \end{cases}$$

where $\hat{-}$ is the arithmetic difference of a and b.

4.1.4 Set-based CSPs

An interesting class of instances of the soft constraint framework is based on set operations like union and intersection, using the semiring $\mathcal{K}_{set} = \langle \wp(A), \cup, \cap, \emptyset, A\rangle$, where A is any set: the order $\leq_{\mathcal{K}_{set}}$ reduces to set inclusion, and therefore it is partial.

The semiring is clearly complete (and residuated by Proposition 6). The $\times$ operator is idempotent, then it is invertible; but the order is not total, so that it is not uniquely invertible (Proposition 4 can not be applied). By instantiating the definition of division we obtain

$$a \div b = \bigcup\{x \mid b \cap x \subseteq a\} = (A \setminus b) \cup a$$

where $\setminus$ is the set difference operator.

4.2 Combining and projecting soft constraints

The *combination* function $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is defined as $(c_1 \otimes c_2)\eta = c_1\eta \times c_2\eta$. Thus, combining two constraints means building a new constraint whose support involves all the variables of the original pairs, and which associates with each tuple of domain values for such variables a semiring element which is obtained by multiplying the elements associated by the original constraints to the appropriate subtuples. Let $c \in \mathcal{C}$ be a constraint and $v \in V$ a variable. The *projection* of c over $V - \{v\}$, written $c \Downarrow_v$, is the constraint c' such that $c'\eta = \sum_{d \in D} c\eta[v := d]$. Informally, projecting means eliminating some variables from the support.

4.3 Local consistency

The main point in the generalization of local consistency techniques to soft CSPs concerns the fact that, instead of removing tuples, local consistency means changing the semiring values associated to them. In particular, the change always brings these values towards the worst element of the semiring, that is, $\mathbf{0}$.

Arc-consistency [3, 16] (AC) is an instance of local consistency where the information present in the constraints is propagated over the variables. In fact, an arc-consistency rule considers a constraint, say with support over variables $x, y_1, \ldots, y_n$, and all unitary constraints over these variables (that is, constraints whose support is one of the variables), and combines all these constraints to get some information (by projecting) over one of the variables, say x.[6]

Such a *local consistency rule* [7] involves a constraint c with support over the set of variables X and a unary constraint c_x with support over the variable $x \in X$, and consists of three phases

- the computation of the solution of the subproblem $\{c_x, c\}$ by computing $c_x \otimes c$,
- the projection of the computed solution over variable x by computing $c'_x = (c_x \otimes c) \Downarrow_x = c_x \otimes (c \Downarrow_x)$, and then
- substituting the original constraint c_x over x with the new one by the assignment $c_x \leftarrow c'_x$.

The application of a local consistency rule leads to an equivalent problem if multiplicative idempotency holds [7]. We relax this condition by performing two assignments at each step of the new rule.

Definition 6 (local consistency rule) *A local consistency rule involving a constraint c and a unary constraint c_x with $supp(c_x) = \{x\} \subset supp(c)$ consists of the following phases*

- *substituting the original constraint c_x with c'_x, computed as usual [7]*

$$c'_x = c_x \otimes (c \Downarrow_x)$$

- *modifying the constraint c in a new constraint c' that takes into account the changes performed on c_x*[7]

$$c' = c \oplus\!\!\!\!\div (c \Downarrow_x),$$

where the constraint division function $\oplus\!\!\!\!\div : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is defined as $(c_1 \oplus\!\!\!\!\div c_2)\eta = c_1\eta \div c_2\eta$.

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[6] Note that this notion represents a generalized form of arc-consistency, since it was originally defined for binary constraints only [22].

[7] Since constraint c_x is combined with $c \Downarrow_x$, c' is divided by the same value.

Notice that the two steps of Definition 6 corresponds to the steps performed by the *project* operator used in fair valued CSPs (see e.g. Algorithm 1 in [12, p.207]). The *extend* operator is instead not considered here, but it is worth to notice that it can be emulated by using constraint removal and combination.

The main result of this section is that the application of the above local consistency rules does not change the solution of soft constraint satisfaction problems defined on invertible by residuation semirings.

Proposition 10 (preserving solutions) *Let $\mathcal{K}$ be an absorptive, invertible by residuation semiring, and let us consider a soft constraint satisfaction problem on it. Then, the application of the local consistency rules in Definition 6 does not change the solution of the problem, that is $c_x \otimes c = c'_x \otimes c'$.*

5 Conclusions and further work

Our work represents an investigation on the semiring-based approaches to soft constraints. After revising some basics of (tropical) semirings and of residuation theory, we show how to define a suitable division operator, proving how the latter can be used to generalize current algorithm for local consistency.

The papers that are most related to our study are those by Cooper and Schiex [11, 12, 25]. These authors propose a generalized version of arc-consistency for valued CSPs [5, 26], by defining a difference operation which is the inverse of the aggregation operation (corresponding to our $\times$ operator). As shown in Section 3.4, the two proposals share many similarities. However, our solution differs in two aspects. First of all, it has been explicitly devised for obtaining a total operation that can be applied to absorptive semirings where the induced ordering is partial, such as those arising in set-based CSPs.

More importantly, though, by relying on the notion of residuation, our solution may take advantage on the large family of studies on tropical arithmetics, in particular in finding general criteria establishing when a semiring is invertible by residuation, and for obtaining an algorithmic procedure for the calculus of the result of the division operation (considered as the resolution of a linear equation), as it is illustrated e.g. in [2, Section 4.5]. We consider mandatory a future, exhaustive comparison of our formalism with fair valued CSPs.

Further algorithmic considerations should then be taken into account before proposing an actual implementation of our local consistency rules. For example, note that if the value of a constraint c should coincide with that of its projection $c \Downarrow_x$, than the value of the constraint c' in Definition 10 would be $\mathbf{1}$, thus possibly increasing, hence the chaotic iteration applied to our rules could possibly not terminate. Using a set of *guarded* rules that never perform such (useless) divisions would however guarantee in this case termination.

The papers by Brown, Larrosa, Meseguer, Schiex and Verfaillie consider max and weighted CSPs and define local consistency algorithms also in the presence of a non-idempotent aggregation operator [9, 18, 19, 20, 21]. Our work shares the same aim, even if we do not consider one specific formalism but instead the whole semiring-based approach. It is a matter of further investigation if their proposals could be generalized and applied to our framework.

Current work is now devoted to the introduction of an inverse of the sum operator, and the definition of a framework for soft constraint databases, as well as a throughout analysis of the relationship between these operators in absorptive semirings.

ACKNOWLEDGEMENTS

We are greatly indebted to Nic Wilson for his suggestions, as well as to Paola Cellini, Martin Cooper, Javier Larrosa, and Thomas Schiex for their help in discussing and improving the contents of the paper.

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